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ON SPECTRUM OF STRINGS WITH δ'-LIKE PERTURBATIONS OF MASS DENSITY

To my Teacher Olga Arsenievna Oleinik

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We study the asymptotic behaviour of eigenvalues and eigenfunctions of a boundary value problem for the Sturm-Liouville operator with general boundary conditions and the weight function perturbed by the so-called δ'-like sequence $\varepsilon^{-2}h(x/\varepsilon)$. The eigenvalue problem is realized as a family of non-self-adjoint matrix operators acting on the same Hilbert space and the norm resolvent convergence of this family is established. We also prove the Hausdorff convergence of the perturbed spectra.

Key words: Sturm-Liouville operator, adjoint mass, concentrating mass, singular perturbation, spectral problem, norm resolvent convergence, Hausdorff convergence, non-self-adjoint operator.

1. Introduction

The vibrating systems with added masses have become the subject of research for mathematicians and physicists since the time of Poisson and Bessel [1, Ch.2], and an enormous number of studies have been devoted to these problems. Many authors have investigated properties of one-dimensional continua (strings and rods) with the mass density perturbed by the finite or infinite sum $\sum_k M_k \delta(x - x_k)$, where δ is the Dirac function (see for instance [2, 3, 4, 5] and the references given there). The mathematical models involving the δ-functions are in general non suitable for 2D and 3D elastic systems, because the formal partial differential expressions which appear in the models often have no mathematical meaning. Such models are also not adequate in the one-dimensional

case, when the added masses M_k are large enough. The large adjoint mass can lead to a strong local reaction which brings about a considerable change in the basic form of the oscillations. But this reaction cannot be described on the discrete set which is a support of singular distributions. It is natural that the geometry of a small part of the vibrating system where the large mass is loaded should also have an effect on eigenfrequencies and eigenvibrations. Since the works of E. Sánchez-Palencia [6, 7, 8], more adequate and more complicated mathematical models of media with the concentrated masses have gained popularity; the asymptotic analysis began to be applied to the spectral problems with the perturbed mass density having the form

$$\rho_\varepsilon(x) = \rho_0(x) + \sum_k \varepsilon^{-m_k} h_k\left(\frac{x - x_k}{\varepsilon}\right),$$

where h_k are functions of compact support and $m_k \in \mathbb{R}$. The most interesting cases of the limit behaviour of eigenvalues and eigenfunctions as $\varepsilon \to 0$ arise when the powers m_k are greater than or equal to the dimension of vibrating system.

These improved models have attracted considerable attention in the mathematical literature over three past decades (see review [9]). The classic elastic systems such as strings, rods, membranes, plates and bodies with the perturbed density

$$\rho_\varepsilon(x) = \rho_0(x) + \varepsilon^{-m} h(x/\varepsilon)$$

have been considered in [10, 11, 12, 13, 14, 15, 16, 17], where the convergence of spectra for each real m and the complete asymptotic expansions of eigenvalues and eigenfunctions for selected values of m have been obtained. The influence of the concentrated masses on the spectral characteristics and oscillations of junctions, the objects with very complicated geometry, has been studied in [18, 19, 20]. The asymptotic behaviour of eigenvalues and eigenfunctions of membranes and bodies with many concentrated masses near the boundary has been investigated in [21, 22, 23, 24, 25]. In [26, 27] the asymptotic analysis has been applied to the spectral problems for membranes and plates with the density perturbed in a thin neighbourhood of a closed smooth curve. The spectral problems on metric graphs that describe the eigenvibrations of elastic networks with heavy nodes have been studied in [28, 29].

A characteristic feature of such problems is the presence of perturbed density ρ_ε at the spectral parameter, which in turn leads to a self-adjoint operator realization of the problem in a Hilbert space (a weighted Lebesgue space) that also depends on the small parameter. The study of families of operators acting on varying spaces entails some mathematical difficulties. First of all, the question arises how to understand the convergence of such families. Next, if these operators do converge in some sense, does this convergence implies the convergence of their spectra (see [15, III.1], [31, 32, 33] for more details). Most of the above-mentioned publications deal with asymptotic approximations of eigenvalues and eigenfunctions; justifying such asymptotics, the researchers used the theory of quasimodes [34], and therefore the question of the operator convergence can be avoided in the studies.

In this paper we consider the Sturm-Liouville operators and investigate the eigenvalue problems with general boundary conditions and the weight function perturbed by the so-called δ'-like sequence $\varepsilon^{-2}h(x/\varepsilon)$. By abandoning the self-adjointness, we realize the perturbed problem as a family of non-self-adjoint matrix operators $\mathcal{A}_\varepsilon$ acting on

a fixed Hilbert space and prove the norm resolvent convergence of $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0$. The operators $\mathcal{A}_\varepsilon$ are certainly similar to self-adjoint ones for each ε and their spectra are real, discrete and simple. Surprisingly enough, the limit operator is essentially non-self-adjoint, because it possesses multiple eigenvalues with non-trivial Jordan cells. Actually the singularly perturbed problem gives us an example of some self-adjoint operators T_ε with compact resolvents acting on varying spaces H_ε that "converge" to a non-self-adjoint operator T_0 in the space H_0. More precisely, the spectra of T_ε converge to the spectrum of T_0 in the Hausdorff sense, taking account of the algebraic multiplicities of eigenvalues; moreover the limit position, as $\varepsilon \to 0$, of the eigensubspaces of T_ε can be described by means of the root subspaces of T_0.

Note that a partial case of the problem, namely the Sturm-Liouville operator without a potential subject to the Dirichlet type boundary condition, was previously studied in [13]. In Theorem 9, the Hausdorff convergence of the perturbed spectrum to some limit set was proved. This limit set was treated as a union of spectra of three self-adjoint operators (cf. Theorem 2 below), but the limit operator was not constructed and the question of eigenvalue multiplicity was not discussed.

We use the following notation. Let $L_2(r, I)$ be the weighted Lebesgue space with the norm

$$\|f\|_{L_2(r,I)} = \left(\int\limits_I r(x)|f(x)|^2 \, dx \right)^{1/2},$$

provided r is positive. Throughout the paper, $W_2^k(I)$ stands for the Sobolev space of the functions defined on $I \subset \mathbb{R}$ that belong to $L_2(I)$ together with their derivatives up to the order k. The norm in $W_2^k(I)$ is given by

$$\|f\|_{W_2^k(I)} := \left(\|f^{(k)}\|_{L_2(I)}^2 + \|f\|_{L_2(I)}^2 \right)^{1/2},$$

where $\|f\|_{L_2(I)}$ is the usual L_2-norm. The spectrum, point spectrum and resolvent set of a linear operator T are denoted by $\sigma(T)$, $\sigma_p(T)$ and $\rho(T)$, respectively, and the Hilbert space adjoint operator of T is T^*. For any complex number $z \in \rho(T)$, the resolvent operator $\mathrm{R}_z(T)$ is defined by $\mathrm{R}_z(T) = (T - z)^{-1}$. Also, we will sometimes abuse notation and write column vectors as row vectors.

2. Statement of Problem

Let $\mathcal{I} = (a, b)$ be a finite interval in $\mathbb{R}$ containing the origin and ε be a small positive parameter. Set $\mathcal{I}_a = (a, 0)$, $\mathcal{I}_b = (0, b)$, $\mathcal{I}_a^\varepsilon = (a, -\varepsilon)$, $\mathcal{I}_b^\varepsilon = (\varepsilon, b)$ and $\mathcal{J} = (-1, 1)$. We study the limiting behavior as $\varepsilon \to 0$ of eigenvalues λ^ε and eigenfunctions y_ε of the problem

$$-y_\varepsilon'' + q(x)y_\varepsilon = \lambda^\varepsilon r_\varepsilon(x)y_\varepsilon, \quad x \in \mathcal{I}, \tag{1}$$

$$y_\varepsilon(a)\cos\alpha + y_\varepsilon'(a)\sin\alpha = 0, \tag{2}$$

$$y_\varepsilon(b)\cos\beta + y_\varepsilon'(b)\sin\beta = 0 \tag{3}$$

with the singularly perturbed weight function

$$r_\varepsilon(x) = \begin{cases} r(x), & x \in \mathcal{I}_a^\varepsilon \cup \mathcal{I}_b^\varepsilon, \\ \varepsilon^{-2}h(\varepsilon^{-1}x), & x \in (-\varepsilon, \varepsilon). \end{cases}$$

Assume that $\alpha, \beta \in \mathbb{R}$, $q, r \in L^\infty(\mathcal{I})$ and $h \in L^\infty(\mathcal{J})$; r and h are uniformly positive.

For any fixed real α, β and positive ε small enough, problem (1)–(3) admits a self-adjoint realization in the weighted space $L_2(r_\varepsilon, \mathcal{I})$. Let us consider the Sturm-Liouville differential expression $\tau(\phi) = -\phi'' + q\phi$. We introduce the operator T_ε defined by $T_\varepsilon\phi = r_\varepsilon^{-1}\tau(\phi)$ on the functions $\phi \in W_2^2(\mathcal{I})$ obeying boundary conditions (2) and (3). Hence $\{T_\varepsilon\}_{\varepsilon>0}$ is a family of self-adjoint operators in the varying Hilbert spaces $L_2(r_\varepsilon, \mathcal{I})$. Of course the spectrum of T_ε is real, discrete and simple.

Problem (1)–(3) can be also associated with a non-self-adjoint matrix operator in the fixed Hilbert space $\mathcal{L} = L_2(r, \mathcal{I}_a) \times L_2(h, \mathcal{J}) \times L_2(r, \mathcal{I}_b)$ as follows. Subsequently, we will write boundary conditions (2) and (3) for a function ϕ as $\ell_a\phi = 0$ and $\ell_b\phi = 0$ respectively. Let us introduce the new variable $t = x/\varepsilon$ and set $w_\varepsilon(t) = y_\varepsilon(\varepsilon t)$. Then the eigenvalue problem can be written in the form

$$- y_\varepsilon'' + q(x)y_\varepsilon = \lambda^\varepsilon r(x)y_\varepsilon, \quad x \in \mathcal{I}_a^\varepsilon, \qquad \ell_a y_\varepsilon = 0, \tag{4}$$

$$- w_\varepsilon'' + \varepsilon^2 q(\varepsilon t)w_\varepsilon = \lambda^\varepsilon h(t)w_\varepsilon, \quad t \in \mathcal{J}, \tag{5}$$

$$- y_\varepsilon'' + q(x)y_\varepsilon = \lambda^\varepsilon r(x)y_\varepsilon, \quad x \in \mathcal{I}_b^\varepsilon, \qquad \ell_b y_\varepsilon = 0 \tag{6}$$

with the coupling conditions

$$y_\varepsilon(-\varepsilon) = w_\varepsilon(-1), \quad y_\varepsilon(\varepsilon) = w_\varepsilon(1), \tag{7}$$

$$\varepsilon y_\varepsilon'(-\varepsilon) = w_\varepsilon'(-1), \quad \varepsilon y_\varepsilon'(\varepsilon) = w_\varepsilon'(1). \tag{8}$$

Let $\mathring{A}_a$ be the operator in $L_2(r, \mathcal{I}_a)$ that is defined by $\mathring{A}_a\phi = r^{-1}\tau(\phi)$ on the functions ϕ belonging to the set $D(\mathring{A}_a) = \{\phi \in W_2^2(\mathcal{I}_a) \colon \ell_a\phi = 0\}$. Similarly, let $\mathring{A}_b$ be the operator in $L_2(r, \mathcal{I}_b)$ such that $\mathring{A}_b\phi = r^{-1}\tau(\phi)$ and $D(\mathring{A}_b) = \{\phi \in W_2^2(\mathcal{I}_b) \colon \ell_b\phi = 0\}$. We also introduce the operator $\mathring{B} = -h^{-1}\dfrac{d^2}{dt^2}$ in $L_2(h, \mathcal{J})$ with domain $D(\mathring{B}) = W_2^2(\mathcal{J})$ and its potential perturbation $\mathring{B}_\varepsilon = \mathring{B} + \varepsilon^2\dfrac{q(\varepsilon t)}{h(t)}$.

Let us consider the matrix operator

$$\mathcal{A}_\varepsilon = \begin{pmatrix} \mathring{A}_a & 0 & 0 \\ 0 & \mathring{B}_\varepsilon & 0 \\ 0 & 0 & \mathring{A}_b \end{pmatrix}$$

in $\mathcal{L}$, acting on the domain

$$D(\mathcal{A}_\varepsilon) = \{(\phi_a, \psi, \phi_b) \in D(\mathring{A}_a) \times D(\mathring{B}_\varepsilon) \times D(\mathring{A}_b) \colon$$
$$\phi_a(-\varepsilon) = \psi(-1), \ \phi_b(\varepsilon) = \psi(1), \ \varepsilon\phi_a'(-\varepsilon) = \psi'(-1), \ \varepsilon\phi_b'(\varepsilon) = \psi'(1)\}.$$

A straightforward calculation shows that $\mathcal{A}_\varepsilon$ is non-self-adjoint. Note that the spectral equation $(\mathcal{A}_\varepsilon - \lambda^\varepsilon)Y_\varepsilon = 0$ is slightly different from eigenvalue problem (4)–(8). In fact, if we display the components of the vector Y_ε by writing $Y_\varepsilon = (y_\varepsilon^a, w_\varepsilon, y_\varepsilon^b)$, then we see at once that y_ε^a is a solution of (4) on the whole interval $\mathcal{I}_a$ (not only in $\mathcal{I}_a^\varepsilon$), and y_ε^b

is a solution of (6) on the whole interval $\mathcal{I}_b$. However, this "extra information", namely the extensions of the solutions to the intervals $\mathcal{I}_a$ and $\mathcal{I}_b$, does not prevent the operator $\mathcal{A}_\varepsilon$ from adequately describing the spectrum and the eigenfunctions of (4)–(8) (or also (1)–(3)), because of the uniqueness of such extensions.

Proposition 1. $\sigma(\mathcal{A}_\varepsilon) = \sigma(T_\varepsilon)$.

Proof. Fix a positive ε. We will show that $\rho(\mathcal{A}_\varepsilon) = \rho(T_\varepsilon)$. Suppose first that $\zeta \in \rho(T_\varepsilon)$ and consider the equation $(\mathcal{A}_\varepsilon - \zeta)Y = F$, where F belongs to $\mathcal{L}$. Suppose that $F = (f_a, f_0, f_b)$. Then we can construct the function

$$f(x) = \begin{cases} f_a(x) & \text{for } x \in \mathcal{I}_a^\varepsilon, \\ f_0(x/\varepsilon) & \text{for } x \in (-\varepsilon, \varepsilon), \\ f_b(x) & \text{for } x \in \mathcal{I}_b^\varepsilon \end{cases}$$

belonging to $L_2(r_\varepsilon, \mathcal{I})$. Next, $y = (T_\varepsilon - \zeta)^{-1}f$ is a unique solution of the problem

$$- y'' + qy - \zeta ry = rf_a \quad \text{in } \mathcal{I}_a^\varepsilon, \qquad \ell_a y = 0, \tag{9}$$

$$- \varepsilon^2 y'' + \varepsilon^2 qy - \zeta hy = hf_0 \quad \text{in } (-\varepsilon, \varepsilon), \tag{10}$$

$$- y'' + qy - \zeta ry = rf_b \quad \text{in } \mathcal{I}_b^\varepsilon, \qquad \ell_b y = 0, \tag{11}$$

$$[y]_{-\varepsilon} = 0, \quad [y]_\varepsilon = 0, \quad [y']_{-\varepsilon} = 0, \quad [y']_\varepsilon = 0, \tag{12}$$

where $[y]_{x_0}$ is a jump of y at the point x_0. Denote by y_a the extension of y from $\mathcal{I}_a^\varepsilon$ to $\mathcal{I}_a$ as a solution of (9). Recall that the right hand side f_a is defined on the whole interval $\mathcal{I}_a$. This extension is uniquely defined. Similarly, we denote by y_b the solution of (11) in $\mathcal{I}_b$ such that $y_b(x) = y(x)$ for $x \in \mathcal{I}_b^\varepsilon$. Then the vector $Y(x) = (y_a(x), y(x/\varepsilon), y_b(x))$ belongs to $D(\mathcal{A}_\varepsilon)$ and solves $(\mathcal{A}_\varepsilon - \zeta)Y = F$. The last equation admits a unique solution Y; if we assume that there are more such solutions, then we immediately obtain a contradiction with the uniqueness of y. Therefore, $\rho(\mathcal{A}_\varepsilon) \subset \rho(T_\varepsilon)$.

Conversely, suppose $\zeta \in \rho(\mathcal{A}_\varepsilon)$. We prove that $(T_\varepsilon - \zeta)y = f$ is uniquely solvable for all $f \in L_2(r_\varepsilon, \mathcal{I})$. Given f, construct the vector $F = (f_a(x), f(\varepsilon t), f_b(x))$, where f_a and f_b are the restrictions of f to $\mathcal{I}_a$ and $\mathcal{I}_b$ respectively. Then the problem

$$- \phi_a'' + q\phi_a - \zeta r\phi_a = rf_a \quad \text{in } \mathcal{I}_a, \qquad \ell_a \phi_a = 0,$$
$$- \psi'' + \varepsilon^2 q(\varepsilon \cdot)\psi - \zeta h\psi = hf(\varepsilon \cdot) \quad \text{in } \mathcal{J},$$
$$- \phi_b'' + q\phi_b - \zeta r\phi_b = rf_b \quad \text{in } \mathcal{I}_b, \qquad \ell_b \phi_b = 0,$$
$$\phi_a(-\varepsilon) = \psi(-1), \ \ \phi_b(\varepsilon) = \psi(1), \ \ \varepsilon\phi_a'(-\varepsilon) = \psi'(-1), \ \ \varepsilon\phi_b'(\varepsilon) = \psi'(1).$$

admits a unique solution $Y = (\mathcal{A}_\varepsilon - \zeta)^{-1}F$. If $Y = (\phi_a, \psi, \phi_b)$, then function

$$y(x) = \begin{cases} \phi_a(x) & \text{for } x \in \mathcal{I}_a^\varepsilon, \\ \psi(x/\varepsilon) & \text{for } x \in (-\varepsilon, \varepsilon), \\ \phi_b(x) & \text{for } x \in \mathcal{I}_b^\varepsilon \end{cases}$$

is a solution of (9)–(12). Since the spectrum of T_ε is discrete, the solvability of $(T_\varepsilon - \zeta)y = f$ for all $f \in L_2(r_\varepsilon, \mathcal{I})$ ensures $\zeta \in \rho(T_\varepsilon)$, and hence $\rho(T_\varepsilon) \subset \rho(\mathcal{A}_\varepsilon)$. $\square$

3. Norm Resolvent Convergence of $\mathcal{A}_\varepsilon$

In this section we will prove that the family of operators $\mathcal{A}_\varepsilon$ converges in the norm resolvent sense as $\varepsilon \to 0$. Let B be the restriction of $\mathring{B}$ to the domain

$$D(B) = \left\{ \psi \in D(\mathring{B}) \colon \psi'(-1) = 0, \ \ \psi'(1) = 0 \right\}.$$

We introduce the matrix operator

$$\mathcal{A} = \begin{pmatrix} \mathring{A}_a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \mathring{A}_b \end{pmatrix}$$

in the space $\mathcal{L}$ acting on

$$D(\mathcal{A}) = \left\{ (\phi_a, \psi, \phi_b) \in D(\mathring{A}_a) \times D(B) \times D(\mathring{A}_b) \colon \phi_a(0) = \psi(-1), \ \ \phi_b(0) = \psi(1) \right\}.$$

This operator is associated with the eigenvalue problem

$$-u'' + qu = \lambda r u \quad \text{in } \in \mathcal{I}_a, \quad \ell_a u = 0, \tag{13}$$

$$-w'' = \lambda h w, \quad \text{in } \in \mathcal{J}, \quad w'(-1) = 0, \quad w'(1) = 0, \tag{14}$$

$$-v'' + qv = \lambda r v \quad \text{in } \in \mathcal{I}_b, \quad \ell_b v = 0, \tag{15}$$

$$u(0) = w(-1), \quad v(0) = w(1) \tag{16}$$

which can be regarded as *the limit problem.* The following assertion is one of the main results of this paper.

Theorem 1. *The family of operators $\mathcal{A}_\varepsilon$ converges to $\mathcal{A}$ as $\varepsilon \to 0$ in the norm resolvent sense. In addition,*

$$\|\mathrm{R}_\zeta(\mathcal{A}_\varepsilon) - \mathrm{R}_\zeta(\mathcal{A})\| \leqslant c\sqrt{\varepsilon}, \tag{17}$$

the constant c being independent of ε.

For the convenience of the reader we collect together the definitions of all operators which will be used in the proof.

- *Operators $T_a^\varepsilon(\zeta)$, $T_b^\varepsilon(\zeta)$, $T_a(\zeta)$ and $T_b(\zeta)$.* We endow $D(\mathring{B})$ with the graph norm, i.e., the norm of the Sobolev space $W_2^2(\mathcal{J})$. Let $T_a^\varepsilon(\zeta) \colon D(\mathring{B}) \to L_2(r, \mathcal{I}_a)$ be defined as follows. Given $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and $\psi \in D(\mathring{B})$, we compute $\psi(-1)$, find then a unique solution u_a of the problem

$$-u'' + qu - \zeta r u = 0 \quad \text{in } \mathcal{I}_a, \qquad \ell_a u = 0, \quad u(-\varepsilon) = \psi(-1) \tag{18}$$

  and finally set $T_a^\varepsilon(\zeta)\psi = u_a$. Similarly, we define $T_b^\varepsilon(\zeta) \colon D(\mathring{B}) \to L_2(r, \mathcal{I}_b)$ which solves the problem

$$-v'' + qv - \zeta r v = 0 \quad \text{in } \mathcal{I}_b, \qquad v(\varepsilon) = \psi(1), \quad \ell_b v = 0 \tag{19}$$

  for given $\psi \in D(\mathring{B})$. Next, the operators $T_a(\zeta)$ and $T_b(\zeta)$ stand for $T_a^\varepsilon(\zeta)$ and $T_b^\varepsilon(\zeta)$, provided $\varepsilon = 0$. So $T_a(\zeta)$ (resp. $T_b(\zeta)$) solves problem (18) (resp. (19)) for given $\psi \in D(\mathring{B})$ and $\varepsilon = 0$.

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- *Operators $S_a^\varepsilon(\zeta)$ and $S_b^\varepsilon(\zeta)$.* Suppose that $D(\mathring{A}_a)$ and $D(\mathring{A}_b)$ are equipped by the graph norms. These norms are equivalent to the norms of $W_2^2(\mathcal{I}_a)$ and $W_2^2(\mathcal{I}_b)$ respectively. The operator $S_a^\varepsilon(\zeta)\colon D(\mathring{A}_a) \to L_2(h, \mathcal{J})$ is defined by $S_a^\varepsilon(\zeta)\phi = w_a$, where w_a is a unique solution of

$$-w'' + \varepsilon^2 q(\varepsilon \cdot)w = \zeta h w \quad \text{in } \mathcal{J}, \quad w'(-1) = \phi'(-\varepsilon), \ w'(1) = 0 \qquad (20)$$

for given $\phi \in D(\mathring{A}_b)$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}$. Similarly, the operator $S_b^\varepsilon(\zeta)\colon D(\mathring{A}_b) \to L_2(h, \mathcal{J})$ solves

$$-w'' + \varepsilon^2 q(\varepsilon \cdot)w = \zeta h w \quad \text{in } \mathcal{J}, \qquad w'(-1) = 0, \ w'(1) = \phi'(\varepsilon) \qquad (21)$$

for some $\phi \in D(\mathring{A}_a)$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}$.

- *Operator B_ε.* This operator is the restriction of $\mathring{B}_\varepsilon$ to the domain

$$D(B_\varepsilon) = \left\{ \psi \in D(\mathring{B}_\varepsilon)\colon \psi'(-1) = 0, \ \psi'(1) = 0 \right\}.$$

- *Operators A_a^ε, A_b^ε, A_a and A_b.* Let A_a^ε and A_b^ε be the restrictions of $\mathring{A}_a$ and $\mathring{A}_b$ respectively to the domains

$$D(A_a^\varepsilon) = \{\phi \in D(\mathring{A}_a)\colon \phi(-\varepsilon) = 0\},$$
$$D(A_b^\varepsilon) = \{\phi \in D(\mathring{A}_b)\colon \phi(\varepsilon) = 0\}.$$

The operators A_a and A_b stand for A_a^ε and A_b^ε, provided $\varepsilon = 0$.

We now construct the resolvents of $\mathcal{A}_\varepsilon$ and $\mathcal{A}$ in the explicit form as follows. Fix $\zeta \in \mathbb{C} \setminus \mathbb{R}$. First of all, note that the operators $T_a^\varepsilon(\zeta)$, $T_b^\varepsilon(\zeta)$, $S_a^\varepsilon(\zeta)$ and $S_b^\varepsilon(\zeta)$ are well-defined for such values of ζ. Moreover these operators are compact. Given $F = (f_a, f_0, f_b) \in \mathcal{L}$, solve the equation $(\mathcal{A}_\varepsilon - \zeta)Y = F$. The first component of $Y = (\phi_a, \psi, \phi_b)$ is a solution of the Dirichlet type problem

$$-\phi'' + q\phi - \zeta r\phi = r f_a \quad \text{in } \mathcal{I}_a, \qquad \ell_a \phi = 0, \quad \phi(-\varepsilon) = \psi(-1).$$

This solution can be represented as the sum of a solution of the non-homogeneous equation subject to the homogeneous boundary conditions and a solution of (18):

$$\phi_a = \mathrm{R}_\zeta(A_a^\varepsilon)f_a + T_a^\varepsilon(\zeta)\psi. \qquad (22)$$

The same argument yields

$$\phi_b = \mathrm{R}_\zeta(A_b^\varepsilon)f_b + T_b^\varepsilon(\zeta)\psi. \qquad (23)$$

The middle element ψ of Y is a solution of the Neumann type problem

$$-\psi'' + \varepsilon^2 q(\varepsilon \cdot)\psi - \zeta h\psi = h f_0 \quad \text{in } \mathcal{J}, \quad \psi'(-1) = \varepsilon\phi_a'(-\varepsilon), \quad \psi'(1) = \varepsilon\phi_b'(\varepsilon),$$

and it can be written as

$$\psi = \mathrm{R}_\zeta(B_\varepsilon)f_0 + \varepsilon S_a^\varepsilon(\zeta)\phi_a + \varepsilon S_b^\varepsilon(\zeta)\phi_b. \qquad (24)$$

Then (22)–(24) taken together yield

$$\phi_a - T_a^\varepsilon(\zeta)\psi = \mathrm{R}_\zeta(A_a^\varepsilon)f_a,$$
$$-\varepsilon S_a^\varepsilon(\zeta)\phi_a + \psi - \varepsilon S_b^\varepsilon(\zeta)\phi_b = \mathrm{R}_\zeta(B_\varepsilon)f_0,$$
$$-T_b^\varepsilon(\zeta)\psi + \phi_b = \mathrm{R}_\zeta(A_b^\varepsilon)f_b.$$

It follows that the resolvent of $\mathcal{A}_\varepsilon$ has the form

$$\mathrm{R}_\zeta(\mathcal{A}_\varepsilon) = \mathcal{H}_\varepsilon(\zeta)^{-1}\mathcal{R}_\varepsilon(\zeta), \qquad (25)$$

where

$$\mathcal{R}_\varepsilon(\zeta) = \begin{pmatrix} \mathrm{R}_\zeta(A_a^\varepsilon) & 0 & 0 \\ 0 & \mathrm{R}_\zeta(B_\varepsilon) & 0 \\ 0 & 0 & \mathrm{R}_\zeta(A_b^\varepsilon) \end{pmatrix}, \qquad (26)$$

$$\mathcal{H}_\varepsilon(\zeta) = \begin{pmatrix} E & -T_a^\varepsilon(\zeta) & 0 \\ -\varepsilon S_a^\varepsilon(\zeta) & E & -\varepsilon S_b^\varepsilon(\zeta) \\ 0 & -T_b^\varepsilon(\zeta) & E \end{pmatrix}, \qquad (27)$$

and E denotes the identity operator in the corresponding spaces. We shall prove below that $\mathcal{H}_\varepsilon(\zeta)$ is invertible for ε small enough.

Now we consider the equation

$$(\mathcal{A} - \zeta)Y = F$$

for $F \in \mathcal{L}$. In the coordinate representation we have $(\mathring{A}_a - \zeta)\phi_a = f_a$, $(B - \zeta)\psi = f_0$ and $(\mathring{A}_b - \zeta)\phi_b = f_b$, where $Y = (\phi_a, \psi, \phi_b)$ and $F = (f_a, f_0, f_b)$. Obviously, $\psi = \mathrm{R}_\zeta(B)f_0$. The functions ϕ_a and ϕ_b are solutions of the problems

$$-\phi'' + q\phi - \zeta r\phi = rf_a \quad \text{in } \mathcal{I}_a, \qquad \ell_a\phi = 0, \quad \phi(0) = \psi(-1);$$
$$-\phi'' + q\phi - \zeta r\phi = rf_b \quad \text{in } \mathcal{I}_b, \qquad \phi(0) = \psi(-1), \quad \ell_b\phi = 0$$

respectively. By reasoning similar to that for (22) and (23), we find

$$\phi_a = \mathrm{R}_\zeta(A_a)f_a + T_a(\zeta)\mathrm{R}_\zeta(B)f_0, \qquad \phi_b = \mathrm{R}_\zeta(A_b)f_b + T_b(\zeta)\mathrm{R}_\zeta(B)f_0.$$

Hence the resolvent of $\mathcal{A}$ can be written in the form

$$\mathrm{R}_\zeta(\mathcal{A}) = \begin{pmatrix} \mathrm{R}_\zeta(A_a) & T_a(\zeta)\mathrm{R}_\zeta(B) & 0 \\ 0 & \mathrm{R}_\zeta(B) & 0 \\ 0 & T_b(\zeta)\mathrm{R}_\zeta(B) & \mathrm{R}_\zeta(A_b) \end{pmatrix}. \qquad (28)$$

To compare the resolvents of $\mathcal{A}_\varepsilon$ and $\mathcal{A}$, we need some auxiliary assertions.

Proposition 2. *The operators A_a^ε, A_b^ε and B_ε converge as $\varepsilon \to 0$ to A_a, A_b and B respectively in the norm resolvent sense. Moreover*

$$\|\mathrm{R}_\zeta(A_a^\varepsilon) - \mathrm{R}_\zeta(A_a)\| \leqslant C_1\sqrt{\varepsilon}, \qquad \|\mathrm{R}_\zeta(A_b^\varepsilon) - \mathrm{R}_\zeta(A_b)\| \leqslant C_2\sqrt{\varepsilon}, \qquad (29)$$

$$\|\mathrm{R}_\zeta(B_\varepsilon) - \mathrm{R}_\zeta(B)\| \leqslant C_3\varepsilon^2, \qquad (30)$$

where the constants C_k do not depend on ε.

Proof. Fix $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and let us compare the elements $u_\varepsilon = \mathrm{R}_\zeta(A_b^\varepsilon)f$ and $u = \mathrm{R}_\zeta(A_b)f$ for given $f \in L_2(r, \mathcal{I}_b)$. Since u_ε and u solve the problems

$$-u_\varepsilon'' + qu_\varepsilon - \zeta ru_\varepsilon = rf \quad \text{in } \mathcal{I}_b, \qquad u_\varepsilon(\varepsilon) = 0, \quad \ell_b u_\varepsilon = 0;$$
$$-u'' + qu - \zeta ru = rf \quad \text{in } \mathcal{I}_b, \qquad u(0) = 0, \quad \ell_b u = 0,$$

they are related by the equality

$$u_\varepsilon(x) = u(x) - \frac{u(\varepsilon)}{z(\varepsilon)}\,z(x), \qquad x \in \mathcal{I}_b,$$

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where z is a solution of the problem

$$-z'' + qz - \zeta r z = 0 \quad \text{in } \mathcal{I}_b, \qquad z(0) = 1, \quad \ell_b z = 0. \tag{31}$$

Obviously, $z(\varepsilon)$ is different from zero for ε small enough. Then we have

$$\|u_\varepsilon - u\|_{L_2(r,\mathcal{I}_b)} \leqslant \frac{|u(\varepsilon)|}{|z(\varepsilon)|}\, \|z\|_{L_2(r,\mathcal{I}_b)} \leqslant c_1|u(\varepsilon)| \leqslant c_2\sqrt{\varepsilon}\, \|u\|_{W_2^1(\mathcal{I}_b)},$$

because $z(\varepsilon) \to 1$ as $\varepsilon \to 0$ and

$$|u(\varepsilon)| = \left| \int\limits_0^\varepsilon u'(x)\, dx \right| \leqslant c_3\sqrt{\varepsilon}\, \|u\|_{W_2^1(\mathcal{I}_b)}.$$

Observe that $\mathrm{R}_\zeta(A_b)$ is a bounded operator from $L_2(r,\mathcal{I}_b)$ to the domain of A_b equipped with the graph norm. Since the domain is a subspace of $W_2^1(\mathcal{I}_b)$, there exists a constant c_4 independent of f such that

$$\|u\|_{W_2^1(\mathcal{I}_b)} \leqslant c_4\|f\|_{L_2(r,\mathcal{I}_b)}.$$

Therefore

$$\left\| \big(\mathrm{R}_\zeta(A_b^\varepsilon) - \mathrm{R}_\zeta(A_b)\big)f \right\|_{L_2(r,\mathcal{I}_b)} \leqslant C_2\sqrt{\varepsilon}\, \|f\|_{L_2(r,\mathcal{I}_b)},$$

which establishes the norm resolvent convergence $A_b^\varepsilon \to A_b$ as $\varepsilon \to 0$ and the corresponding estimate in (29). The proof for the operators A_a^ε is similar to that just given.

We now turn to the operators B_ε and first we establish that $\|\mathrm{R}_\zeta(B_\varepsilon)\| \leqslant c$ for all ε small enough. Given $g \in L_2(h,\mathcal{J})$, consider $w_\varepsilon = \mathrm{R}_\zeta(B_\varepsilon)g$ which solves

$$-w_\varepsilon'' + \varepsilon^2 q(\varepsilon\cdot)w_\varepsilon - \zeta h w_\varepsilon = hg \quad \text{in } \mathcal{J}, \quad w_\varepsilon'(-1) = 0, \quad w_\varepsilon'(1) = 0.$$

Recall that q and h are bounded in $\mathcal{I}$ and $\mathcal{J}$ respectively, and h is uniformly positive on $\mathcal{I}$. Then we have

$$\|\mathrm{R}_\zeta(B_\varepsilon)g\|_{L_2(h,\mathcal{J})} = \left\|\mathrm{R}_\zeta(B)\big(g - \varepsilon^2 q(\varepsilon\cdot)h^{-1}w_\varepsilon\big)\right\|_{L_2(h,\mathcal{J})} \leqslant$$
$$\leqslant \|\mathrm{R}_\zeta(B)g\|_{L_2(h,\mathcal{J})} + \varepsilon^2\|q\|_{L^\infty(\mathcal{I})}\,\|h^{-1}\|_{L^\infty(\mathcal{J})}\,\|w_\varepsilon\|_{L_2(h,\mathcal{J})} \leqslant$$
$$\leqslant c_0\|g\|_{L_2(h,\mathcal{J})} + c_1\varepsilon^2\|\mathrm{R}_\zeta(B_\varepsilon)g\|_{L_2(h,\mathcal{J})}$$

and therefore

$$\|\mathrm{R}_\zeta(B_\varepsilon)g\|_{L_2(h,\mathcal{J})} \leqslant \frac{c_0}{1 - c_1\varepsilon^2}\, \|g\|_{L_2(h,\mathcal{J})} \leqslant c\|g\|_{L_2(h,\mathcal{J})} \tag{32}$$

if ε is small enough.

Next, we set $w = \mathrm{R}_\zeta(B)g$. Then the difference $s_\varepsilon = w_\varepsilon - w$ solves the problem

$$-s_\varepsilon'' - \zeta h s_\varepsilon = -\varepsilon^2 q(\varepsilon\cdot)w_\varepsilon \quad \text{in } \mathcal{J}, \quad s_\varepsilon'(-1) = 0, \quad s_\varepsilon'(1) = 0.$$

Hence in view of (32) we deduce

$$\|(\mathrm{R}_\zeta(B_\varepsilon) - \mathrm{R}_\zeta(B))g\|_{L_2(h,\mathcal{J})} = \|s_\varepsilon\|_{L_2(h,\mathcal{J})} \leqslant$$
$$\leqslant c_2\varepsilon^2\|w_\varepsilon\|_{L_2(h,\mathcal{J})} =$$
$$= c_2\varepsilon^2\|\mathrm{R}_\zeta(B_\varepsilon)g\|_{L_2(h,\mathcal{J})} \leqslant$$
$$\leqslant c_3\varepsilon^2\|g\|_{L_2(h,\mathcal{J})},$$

which finishes the proof. $\qquad\qquad\square$

Proposition 3. (i) *For each* $\zeta \in \mathbb{C} \setminus \mathbb{R}$, *we have the bounds*

$$\|T_a^\varepsilon(\zeta) - T_a(\zeta)\| \leqslant c\varepsilon, \qquad \|T_b^\varepsilon(\zeta) - T_b(\zeta)\| \leqslant c\varepsilon,$$

the constant c *being independent of* ε.
(ii) *There exists a constant* C *such that*

$$\|S_a^\varepsilon(\zeta)\| + \|S_b^\varepsilon(\zeta)\| \leqslant C$$

for all ε *small enough.*

Proof. (i) Let us show that $T_b^\varepsilon(\zeta)$ converge to $T_b(\zeta)$ in the norm as $\varepsilon \to 0$. The same proof remains valid for $T_a^\varepsilon(\zeta)$. Suppose that $u_\varepsilon = T_b^\varepsilon(\zeta)\psi$ is a solution of (19) for given $\psi \in D(\mathring{B})$. It is easily seen that

$$u_\varepsilon(x) = \frac{\psi(1)}{z(\varepsilon)}\, z(x), \quad x \in \mathcal{I}_a,$$

where z is defined by (31). If $u = T_b(\zeta)\psi$, then we have $u = \psi(1)z$. Hence

$$\|(T_b^\varepsilon(\zeta) - T_b(\zeta))\psi\|_{L_2(r,\mathcal{I}_b)} = \left\|\frac{\psi(1)}{z(\varepsilon)}\, z - \psi(1)z\right\|_{L_2(r,\mathcal{I}_b)} \leqslant$$
$$\leqslant \left|\frac{z(\varepsilon) - 1}{z(\varepsilon)}\right| |\psi(1)|\, \|z\|_{L_2(r,\mathcal{I}_b)} \leqslant$$
$$\leqslant c_1\varepsilon\, \|\psi\|_{D(\mathring{B})},$$

because z belongs to $C^1(\mathcal{I}_b)$ and $z(0) = 1$. Recall also that $D(\mathring{B}) = W_2^2(\mathcal{J})$ and hence $\|\psi\|_{C(\mathcal{J})} \leqslant C\|\psi\|_{D(\mathring{B})}$ by the Sobolev embedding theorem.

(ii) For each $\phi \in D(\mathring{A}_b)$, the function $w_\varepsilon = S_b^\varepsilon(\zeta)\phi$ is a solution of (21) and satisfies the estimate $\|w_\varepsilon\|_{L_2(h,\mathcal{J})} \leqslant c_2\,|\phi'(\varepsilon)|$ with a constant c_2 independent of ε, since the resolvents $\mathrm{R}_\zeta(B_\varepsilon)$ are uniformly bounded on ε by Proposition 2. The trace operator $j_\varepsilon\colon D(\mathring{A}_b) \to \mathbb{C}$, $j_\varepsilon\phi = \phi'(\varepsilon)$, is also uniformly bounded on ε. Therefore

$$\|S_b^\varepsilon(\zeta)\phi\|_{L_2(h,\mathcal{J})} = \|w_\varepsilon\|_{L_2(h,\mathcal{J})} \leqslant C\|\phi\|_{W_2^2(\mathcal{I}_b)}.$$

The same proof works for $S_a^\varepsilon(\zeta)$. $\qquad\square$

We are now in a position to prove Theorem 1. In view of Proposition 3, we conclude that the family of matrix operators $\mathcal{H}_\varepsilon(\zeta)$, given by (27), converges as $\varepsilon \to 0$ towards

$$\mathcal{H}(\zeta) = \begin{pmatrix} E & -T_a(\zeta) & 0 \\ 0 & E & 0 \\ 0 & -T_b(\zeta) & E \end{pmatrix}$$

in the norm. Moreover $\|\mathcal{H}_\varepsilon(\zeta) - \mathcal{H}(\zeta)\| \leqslant c_1\varepsilon$. Observe that $\mathcal{H}(\zeta)$ is invertible and

$$\mathcal{H}(\zeta)^{-1} = \begin{pmatrix} E & T_a(\zeta) & 0 \\ 0 & E & 0 \\ 0 & T_b(\zeta) & E \end{pmatrix}.$$

Therefore $\mathcal{H}_\varepsilon(\zeta)$ is also invertible for ε small enough, and

$$\|\mathcal{H}_\varepsilon(\zeta)^{-1} - \mathcal{H}(\zeta)^{-1}\| \leqslant c_2\varepsilon. \tag{33}$$

Recalling (25) and applying Proposition 2, we deduce

$$\mathrm{R}_\zeta(\mathcal{A}_\varepsilon) = \mathcal{H}_\varepsilon(\zeta)^{-1}\mathcal{R}_\varepsilon(\zeta)$$

$$= \begin{pmatrix} E & -T_a^\varepsilon(\zeta) & 0 \\ -\varepsilon S_a^\varepsilon(\zeta) & E & -\varepsilon S_b^\varepsilon(\zeta) \\ 0 & -T_b^\varepsilon(\zeta) & E \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{R}_\zeta(A_a^\varepsilon) & 0 & 0 \\ 0 & \mathrm{R}_\zeta(B_\varepsilon) & 0 \\ 0 & 0 & \mathrm{R}_\zeta(A_b^\varepsilon) \end{pmatrix}$$

$$\to \begin{pmatrix} E & T_a(\zeta) & 0 \\ 0 & E & 0 \\ 0 & T_b(\zeta) & E \end{pmatrix} \begin{pmatrix} \mathrm{R}_\zeta(A_a) & 0 & 0 \\ 0 & \mathrm{R}_\zeta(B) & 0 \\ 0 & 0 & \mathrm{R}_\zeta(A_b) \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{R}_\zeta(A_a) & T_a(\zeta)\mathrm{R}_\zeta(B) & 0 \\ 0 & \mathrm{R}_\zeta(B) & 0 \\ 0 & T_b(\zeta)\mathrm{R}_\zeta(B) & \mathrm{R}_\zeta(A_b) \end{pmatrix} = \mathrm{R}_\zeta(\mathcal{A}) \quad \text{as } \varepsilon \to 0,$$

by (28). Estimate (17) follows from the equality

$$\mathrm{R}_\zeta(\mathcal{A}_\varepsilon) - \mathrm{R}_\zeta(\mathcal{A}) = \mathcal{H}_\varepsilon(\zeta)^{-1}(\mathcal{R}_\varepsilon(\zeta) - \mathcal{R}(\zeta)) - (\mathcal{H}_\varepsilon(\zeta)^{-1} - \mathcal{H}(\zeta)^{-1})\mathcal{R}(\zeta)$$

and bounds (29), (30) and (33). Here $\mathcal{R}(\zeta) = \mathrm{diag}\left\{\mathrm{R}_\zeta(A_a), \mathrm{R}_\zeta(B), \mathrm{R}_\zeta(A_b)\right\}$.

4. Spectrum of $\mathcal{A}$

The limit operator

$$\mathcal{A} = \begin{pmatrix} \mathring{A}_a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \mathring{A}_b \end{pmatrix}, \qquad D(\mathcal{A}) = \Big\{(\phi_a, \psi, \phi_b) \in D(\mathring{A}_a) \times D(B) \times D(\mathring{A}_b)\colon$$
$$\phi_a(0) = \psi(-1), \ \ \phi_b(0) = \psi(1)\Big\}.$$

constructed above is non-self-adjoint. Direct computations show that the adjoint operator $\mathcal{A}^*$ in $\mathcal{L}$ has the form

$$\mathcal{A}^* = \begin{pmatrix} A_a & 0 & 0 \\ 0 & \mathring{B} & 0 \\ 0 & 0 & A_b \end{pmatrix}, \qquad D(\mathcal{A}^*) = \Big\{(\phi_a, \psi, \phi_b) \in D(A_a) \times D(\mathring{B}) \times D(A_b)\colon$$
$$\phi_a'(0) = \psi'(-1), \ \ \phi_b'(0) = \psi'(1)\Big\}.$$

In what follows we will denote by u_λ, v_λ and w_λ the eigenfunctions of A_a, A_b and B respectively which correspond to an eigenvalue λ. So u_λ, v_λ and w_λ are non-trivial solutions of the problems

$$- u'' + qu = \lambda ru \quad \text{in } \mathcal{I}_a, \qquad \ell_a u = 0, \quad u(0) = 0; \tag{34}$$

$$- v'' + qv = \lambda rv \quad \text{in } \mathcal{I}_b, \qquad v(0) = 0, \quad \ell_b v = 0; \tag{35}$$

$$- w'' = \lambda h w \quad \text{in } \mathcal{J}, \qquad w'(-1) = 0, \quad w'(1) = 0 \tag{36}$$

respectively. Let us normalize these eigenfunctions by setting

$$\|u_\lambda\|_{L_2(r,\mathcal{I}_a)} = \|v_\lambda\|_{L_2(r,\mathcal{I}_b)} = \|w_\lambda\|_{L_2(h,\mathcal{J})} = 1. \tag{37}$$

Denote also by X_λ the root subspace of $\mathcal{A}$ for λ, that is

$$X_\lambda = \mathrm{span}\left\{\ker(\mathcal{A} - \lambda)^k \colon k \in \mathbb{N}\right\}.$$

The eigenvectors and root vectors of a non-self-adjoint operator are also called generalized eigenvectors. So X_λ is a subspace of the generalized eigenfunctions corresponding to the eigenvalue λ.

Theorem 2.   *(i) The spectrum of $\mathcal{A}$ is real and discrete, and*

$$\sigma(\mathcal{A}) = \sigma(A_a) \cup \sigma(B) \cup \sigma(A_b). \tag{38}$$

  (ii) If λ belongs to only one of the sets $\sigma(A_a)$, $\sigma(B)$ or $\sigma(A_b)$, then λ is a simple eigenvalue of $\mathcal{A}$.

 (iii) If $\lambda \in \sigma(A_a) \cap \sigma(A_b)$, but λ is not an eigenvalue of B, then λ is a double eigenvalue and $X_\lambda = \ker(\mathcal{A} - \lambda E)$.

 (iv) Suppose that λ belongs to $\sigma(A_a) \cap \sigma(B)$ (resp. $\sigma(A_b) \cap \sigma(B)$), but λ is not an eigenvalue of A_b (resp. A_a), then λ is a double eigenvalue of $\mathcal{A}$. Finally, if $\lambda \in \sigma(A_a) \cap \sigma(A_b) \cap \sigma(B)$, then λ is an eigenvalue of $\mathcal{A}$ with multiplicity 3. In both the cases we have $X_\lambda = \ker(\mathcal{A} - \lambda)^2$, but $X_\lambda \neq \ker(\mathcal{A} - \lambda)$.

Proof. (*i*) Equality (38) follows directly from the explicit representation (28) of $\mathrm{R}_\zeta(\mathcal{A})$. Indeed, each of spectra $\sigma(A_a)$, $\sigma(A_b)$ and $\sigma(B)$ is contained in the spectrum of $\mathcal{A}$. If ζ does not belongs to set $\sigma(A_a) \cup \sigma(A_b) \cup \sigma(B)$, then not only $\mathrm{R}_\zeta(A_a)$, $\mathrm{R}_\zeta(A_b)$, $\mathrm{R}_\zeta(B)$, but also $T_a(\zeta)$ and $T_b(\zeta)$ are bounded, because in this case problems (18) and (19) for $\varepsilon = 0$ are uniquely solvable for all $\psi \in W_2^2(\mathcal{J})$. Therefore operator $\mathrm{R}_\zeta(\mathcal{A})$ is also bounded. The operators A_a, A_b and B associated with eigenvalue problems (34), (35) and (36) are self-adjoint and have compact resolvents. Consequently $\sigma(A)$ is real and discrete.

    (*ii*) Observe that the spectra of A_a, A_b and B are simple. A trivial verification shows that if λ belongs to only one of the sets $\sigma(A_a)$, $\sigma(A_b)$ or $\sigma(B)$, then λ is a simple eigenvalue of $\mathcal{A}$ with eigenvector $(u_\lambda, 0, 0)$ if $\lambda \in \sigma(A_a)$, and $(0, 0, v_\lambda)$ if $\lambda \in \sigma(A_b)$, and $(T_a(\lambda)w_\lambda, w_\lambda, T_b(\lambda)w_\lambda)$ if $\lambda \in \sigma(B)$.

    (*iii*) In the case $\lambda \in \sigma(A_a) \cap \sigma(A_b)$ and $\lambda \notin \sigma(B)$, there are two linearly independent eigenvectors $U = (u_\lambda, 0, 0)$ and $V = (0, 0, v_\lambda)$. Moreover, equation

$$(\mathcal{A} - \lambda)Y = c_1 U + c_2 V$$

is unsolvable for any c_1 and c_2 such that $c_1^2 + c_2^2 \neq 0$. If for instance c_1 is different from zero, then the problem

$$-u'' + qu - \lambda ru = c_1 r u_\lambda \quad \text{in } \mathcal{I}_a, \qquad \ell_a u = 0, \quad u(0) = 0 \tag{39}$$

has no solutions. Suppose, contrary to our claim, that such solution exists. Then multiplying equation (39) by u_λ and integrating by parts yield $c_1 \|u_\lambda\|_{L_2(r, \mathcal{I}_a)}^2 = 0$. Therefore $X_\lambda = \ker(\mathcal{A} - \lambda)$ and $\dim X_\lambda = 2$.

    (*iv*) Suppose that $\lambda \in \sigma(A_a) \cap \sigma(B)$ and $\lambda \notin \sigma(A_b)$. In this case there exists the eigenvector $U = (u_\lambda, 0, 0)$. Furthermore, we will show that the equation

$$(\mathcal{A} - \lambda)U_* = U$$

is solvable. We are thus looking for a solution $U_* = (u, w, v)$ of

$$-u'' + qu - \lambda ru = ru_\lambda \quad \text{in } \mathcal{I}_a, \qquad \ell_a u = 0, \quad u(0) = w(-1); \tag{40}$$

$$-w'' - \lambda hw = 0 \quad \text{in } \mathcal{J}, \qquad w'(-1) = 0, \quad w'(1) = 0; \tag{41}$$

$$-v'' + qv - \lambda rv = 0 \quad \text{in } \mathcal{I}_b, \qquad v(0) = w(1), \quad \ell_b v = 0. \tag{42}$$

Obviously, $w = c_0 w_\lambda$ for some constant c_0, where w_λ is a normalized eigenfunction of B. Then (42) admits a unique solution $v_* = c_0 T_b(\lambda)w_\lambda$ for each c_0, since $\lambda \in \varrho(A_b)$. Next, (40) is in general unsolvable, since λ is a point of $\sigma(A_a)$. But we have the free parameter

c_0 in the boundary condition; (40) with the condition $u(0) = c_0 w_\lambda(-1)$ is solvable if and only if

$$c_0 = \frac{1}{w_\lambda(-1)u'_\lambda(0)}. \tag{43}$$

This equality can be easily obtained by multiplying the equation in (40) by u_λ and integrating by parts. Remark that both of the values $w_\lambda(-1)$ and $u'_\lambda(0)$ are different from zero. If u_0 is a solution of (40), then the operator $\mathcal{A}$ has a root vector

$$U_* = (u_0, c_0\, w_\lambda, c_0\, T_b(\lambda)w_\lambda),$$

where c_0 is given by (43). Hence, the subspace X_λ is a linear span of the eigenvector U and the root vector U_*. In addition, there are no other root vectors, because the equation

$$(\mathcal{A} - \lambda)Y = U_*$$

leads to the problem

$$-w'' - \lambda h w = c h w_\lambda \quad \text{in } \mathcal{J}, \qquad w'(-1) = 0, \quad w'(1) = 0, \tag{44}$$

which is unsolvable for $c \neq 0$. The case $\lambda \in \sigma(A_b) \cap \sigma(B)$ and $\lambda \notin \sigma(A_a)$ is treated similarly.

Now we suppose that

$$\lambda \in \sigma(A_a) \cap \sigma(A_b) \cap \sigma(B).$$

Then the operator $\mathcal{A}$ has two linearly independent eigenvectors $U = (u_\lambda, 0, 0)$ and $V = (0, 0, v_\lambda)$. Note also that $\mathcal{A}$ has no eigenvectors $Y = (u, w, v)$, where w is different from zero. In this case, the values $w(-1)$ and $w(1)$ are always different from zero and hence the problems for u and v are unsolvable. We will prove that $X_\lambda = \ker(\mathcal{A} - \lambda)^2$ and $\dim X_\lambda = 3$. Let us consider the equation

$$(\mathcal{A} - \lambda)Y = c_1 U + c_2 V$$

with arbitrary constants c_1 and c_2, that is to say,

$$-u'' + qu - \lambda r u = c_1 r u_\lambda \quad \text{in } \mathcal{I}_a, \qquad \ell_a u = 0, \quad u(0) = w(-1); \tag{45}$$

$$-w'' - \lambda h w = 0 \quad \text{in } \mathcal{J}, \qquad w'(-1) = 0, \quad w'(1) = 0; \tag{46}$$

$$-v'' + qv - \lambda r v = c_2 r v_\lambda \quad \text{in } \mathcal{I}_b, \qquad v(0) = w(1), \quad \ell_b v = 0. \tag{47}$$

Reasoning as above, we establish that $w = c_0 w_\lambda$ and problems (45) and (47) admit solutions simultaneously if and only if the following equalities

$$c_1 = c_0 w_\lambda(-1) u'_\lambda(0), \qquad c_2 = -c_0 w_\lambda(1) v'_\lambda(0)$$

hold. Then the conditions $c_0 \neq 0$ and

$$c_1 = -\frac{w_\lambda(-1)u'_\lambda(0)}{w_\lambda(1)v'_\lambda(0)}\, c_2$$

ensure the existence of a root vector Y_* of $\mathcal{A}$. Furthermore there are no other root vectors, by reasoning similar to that in the previous case. Hence the subspace X_λ for a triple eigenvalue λ is generated by the eigenvectors U, V and the root vector Y_*. $\quad\square$

5. Convergence of Spectra

Let us denote by

$$\lambda_1^\varepsilon < \lambda_2^\varepsilon < \cdots < \lambda_n^\varepsilon < \cdots$$

the eigenvalues of problem (1)–(3), i.e., the eigenvalues of $\mathcal{A}_\varepsilon$. Note that each eigenvalue λ_n^ε is simple. Let

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$$

be the eigenvalues of limit problem (13)–(16) (or also the operator $\mathcal{A}$), counted with algebraic multiplicities.

Theorem 3. *For each $n \in \mathbb{N}$, the eigenvalue λ_n^ε of problem (13)–(16) converges as $\varepsilon \to 0$ to the eigenvalue λ_n of (13)–(16) with the same number. That is, if λ is an eigenvalue of (13)–(16) with algebraic multiplicity m, then there exists a neighbourhood of λ which contains exactly m eigenvalues of (1)–(3) for ε small enough.*

Proof. The theorem follows from the norm resolvent convergence of $\mathcal{A}_\varepsilon$ proved in Theorem 1 and some general results on the approximation of eigenvalues of compact operators. Let K be a compact operator in a separable Hilbert space $\mathcal{H}$. Suppose that $\{K_\varepsilon\}_{\varepsilon>0}$ is a sequence of compact operators in $\mathcal{H}$ such that $K_\varepsilon \to K$ as $\varepsilon \to 0$ in the uniform norm. Let $\mu_1, \mu_2, \ldots$ be the nonzero eigenvalues of K ordered by decreasing magnitude taking account of algebraic multiplicities. Then for each $\varepsilon > 0$ there is an ordering of the eigenvalues $\mu_1(\varepsilon), \mu_2(\varepsilon), \ldots$ of K_ε such that $\lim_{\varepsilon \to 0} \mu_n(\varepsilon) = \mu_n$, for each natural number n. Suppose that μ is a nonzero eigenvalue of K with algebraic multiplicity m and Γ_μ is a circle centered at μ which lies in $\rho(K)$ and contains no other points of $\sigma(K)$. Then, there is an ε_0 such that, for $0 < \varepsilon \leqslant \varepsilon_0$, there are exactly m eigenvalues (counting algebraic multiplicities) of K_ε lying inside Γ_μ and all points of $\sigma(K_\varepsilon)$ are bounded away from Γ_μ [35, Ch.1], [36, Ch.XI-9], [37].

We apply these results to $K = \mathrm{R}_\zeta(\mathcal{A})$ and $K_\varepsilon = \mathrm{R}_\zeta(\mathcal{A}_\varepsilon)$. Then we have

$$\sigma_p(\mathrm{R}_\zeta(\mathcal{A})) = \left\{ \frac{1}{\lambda_n - \zeta},\ n \in \mathbb{N} \right\}, \qquad \sigma_p(\mathrm{R}_\zeta(\mathcal{A}_\varepsilon)) = \left\{ \frac{1}{\lambda_n^\varepsilon - \zeta},\ n \in \mathbb{N} \right\};$$

both eigenvalue sequences are ordered by decreasing magnitude. Since $\mathcal{A}_\varepsilon \to \mathcal{A}$ in the norm resolvent sense as $\varepsilon \to 0$, that is, $\|\mathrm{R}_\zeta(\mathcal{A}_\varepsilon) - \mathrm{R}_\zeta(\mathcal{A})\| \to 0$ as $\varepsilon \to 0$, we have the "number-by-number" convergence of the eigenvalues

$$\frac{1}{\lambda_n^\varepsilon - \zeta} \to \frac{1}{\lambda_n - \zeta}, \qquad \text{as } \varepsilon \to 0,$$

from which the desired conclusion follows. $\square$

Remark 1. We expect that the estimate

$$|\lambda_n^\varepsilon - \lambda_n| \leqslant C_n \sqrt{\varepsilon}$$

to be correct for each $n \in \mathbb{N}$ and some constants C_n. However, it does not follow directly from bound (17), because the resolvents $\mathrm{R}_\zeta(\mathcal{A})$ and $\mathrm{R}_\zeta(\mathcal{A}_\varepsilon)$ are not in general normal operators.

6. Some Remarks On Eigenfunction Convergence

Since the multiplicity of eigenvalues of the limit operator is up to 3, the bifurcation pictures for multiple eigenvalues of (13)–(16) are quite complicated. The bifurcations of eigenvalues as well the eigensubspaces can be described by a more accurate asymptotic analysis. We omit the details here, because we will consider these questions in a forthcoming publication. However we can obtain some results on the limit behaviour of eigenfunctions that follow directly from the norm resolvent convergence $\mathcal{A}_\varepsilon \to \mathcal{A}$.

Let us return to the compact operators K and K_ε which appeared in the previous section. We consider the Riesz spectral projections

$$E(\mu) = \frac{1}{2\pi i} \int\limits_{\Gamma_\mu} \mathrm{R}_z(\mathcal{A})\, dz, \qquad E_\varepsilon(\mu) = \frac{1}{2\pi i} \int\limits_{\Gamma_\mu} \mathrm{R}_z(\mathcal{A}_\varepsilon)\, dz.$$

The range $R(E(\mu))$ of $E(\mu)$ is the space of generalized eigenfunctions of K corresponding to μ and $R(E_\varepsilon(\mu))$ is the direct sum of the subspaces of generalized eigenfunctions of K_ε associated with the eigenvalues of K_ε inside Γ_μ. If $K_\varepsilon \to K$ as $\varepsilon \to 0$ in the norm, then $E_\varepsilon(\mu) \to E(\mu)$ in the norm, and therefore

$$\dim R(E_\varepsilon(\mu)) = \dim R(E(\mu)) = m,$$

where m is the algebraic multiplicity of μ.

Theorem 4. *Let $y_{\varepsilon,n}$ be the eigenfunction of (1)–(3) which corresponds to the eigenvalue λ_n^ε and $\|y_{\varepsilon,n}\|_{L_2(r,\mathcal{I})} = 1$.*

Suppose that $\lambda_n^\varepsilon \to \lambda_n$, where λ_n is a simple eigenvalue of $\mathcal{A}$ belonging to $\sigma(A_a)$. Then the eigenfunction $y_{\varepsilon,n}$ converges in $L_2(\mathcal{I})$ as $\varepsilon \to 0$ to the function

$$y(x) = \begin{cases} u_n(x), & \text{if } x \in \mathcal{I}_a, \\ 0, & \text{if } x \in \mathcal{I}_b \end{cases},$$

where u_n is an normalized eigenfunction of A_a associated with λ_n, that is,

$$-u_n'' + qu_n = \lambda_n r u_n \quad \text{in } \mathcal{I}_a, \quad \ell_a u_n = 0, \quad u_n(0) = 0, \qquad \|u_n\|_{L_2(r,\mathcal{I}_a)} = 1.$$

Similarly if λ_n belongs to $\sigma(A_b)$ and λ_n is simple, then $y_{\varepsilon,n} \to y$ in $L_2(\mathcal{I})$ as $\varepsilon \to 0$, where

$$y(x) = \begin{cases} 0, & \text{if } x \in \mathcal{I}_a, \\ v_n(x), & \text{if } x \in \mathcal{I}_b \end{cases}$$

and v_n is an normalized eigenfunction of A_b with eigenvalue λ_n, i.e.,

$$-v_n'' + qv_n = \lambda_n r v_n \quad \text{in } \mathcal{I}_b, \quad v_n(0) = 0, \quad \ell_b v_n = 0, \qquad \|v_n\|_{L_2(r,\mathcal{I}_b)} = 1.$$

Assume $\lambda_n^\varepsilon \to \lambda_n$, where λ_n is a simple eigenvalue of $\mathcal{A}$ belonging to $\sigma(B)$. Then the eigenfunction $y_{\varepsilon,n}$ converges in $L_2(\mathcal{I})$ to a solution y of the problem

$$-y'' + qy = \lambda_n r y \quad \text{in } \mathcal{I} \setminus \{0\}, \quad \ell_a y = 0, \quad \ell_b y = 0,$$
$$y(-0) = \theta w_n(-1), \quad y(+0) = \theta w_n(1),$$

where w_n is the corresponding eigenfunction of B such that $\|w_n\|_{L_2(h,\mathcal{J})} = 1$. Normalizing factor θ is given by

$$\theta = \left(\|T_a(\lambda_n)w_n\|^2_{L_2(r,\mathcal{I}_a)} + \|T_b(\lambda_n)w_n\|^2_{L_2(r,\mathcal{I}_b)} \right)^{-1}.$$

Proof. In the case when $K = R_\zeta(\mathcal{A})$, $K_\varepsilon = R_\zeta(\mathcal{A}_\varepsilon)$, λ is a unique point of $\sigma(\mathcal{A})$ lying inside Γ_λ, and ε is small enough, we see that $X_\lambda = R(E(\frac{1}{\lambda-\zeta}))$ is a subspace of generalized eigenfunctions of $\mathcal{A}$ corresponding to the eigenvalue λ, and the subspace $X_\lambda^\varepsilon = R(E_\varepsilon(\frac{1}{\lambda-\zeta}))$ is generated by all eigenfunctions of $\mathcal{A}_\varepsilon$ for which $\lambda_n^\varepsilon \to \lambda$ as $\varepsilon \to 0$. Then the norm resolvent convergence $\mathcal{A}_\varepsilon \to \mathcal{A}$ implies that the gap between X_λ^ε and X_λ tends to zero as $\varepsilon \to 0$ for each $\lambda \in \sigma_p(\mathcal{A})$. In particular, if λ_n is a simple eigenvalue of $\mathcal{A}$ with eigenvector Y_n and $Y_{\varepsilon,n}$ is an eigenvector of $\mathcal{A}_\varepsilon$ that corresponds to λ_n^ε, then $Y_{\varepsilon,n} \to Y_n$ in $\mathcal{L}$ as $\varepsilon \to 0$, provided $\|Y_{\varepsilon,n}\|_{\mathcal{L}} = \|Y_n\|_{\mathcal{L}} = 1$.

Assume λ_n is a simple eigenvalue of $\mathcal{A}$ and $\lambda_n \in \sigma(A_a)$. In view of Theorem 2, the subspace X_λ is generated by vector $Y_n = (u_n, 0, 0)$. Then $Y_{\varepsilon,n} \to Y_n$ as $\varepsilon \to 0$ in the norm of $\mathcal{L}$. If we set $Y_{\varepsilon,n} = (y_\varepsilon^a, w_\varepsilon, y_\varepsilon^b)$, then the eigenfunction $y_{\varepsilon,n}$ of (1)–(3) can be written as

$$y_{\varepsilon,n}(x) = \begin{cases} y_\varepsilon^a(x), & \text{if } x \in \mathcal{I}_a^\varepsilon, \\ w_\varepsilon(x/\varepsilon), & \text{if } x \in (-\varepsilon, \varepsilon), \\ y_\varepsilon^b(x), & \text{if } x \in \mathcal{I}_b^\varepsilon. \end{cases}$$

So we have

$$\|y_{\varepsilon,n} - y_n\|^2_{L_2(\mathcal{I})} = \int_a^{-\varepsilon} |y_\varepsilon^a - u_n|^2 \, dx + \int_\varepsilon^b |y_\varepsilon^a|^2 \, dx$$

$$+ \int_{-\varepsilon}^0 |w_\varepsilon(\tfrac{x}{\varepsilon}) - u_n(x)|^2 \, dx + \int_0^\varepsilon |w_\varepsilon(\tfrac{x}{\varepsilon})|^2 \, dx \leqslant c_1 \|y_\varepsilon^a - u_n\|^2_{L_2(r,\mathcal{I}_a)}$$

$$+ c_2 \|y_\varepsilon^b\|^2_{L_2(r,\mathcal{I}_b)} + c_3 \varepsilon \|w_\varepsilon\|^2_{L_2(h,\mathcal{J})} + \int_{-\varepsilon}^0 |u_n|^2 \, dx \leqslant c_4 \|Y_{\varepsilon,n} - Y_n\|^2_{\mathcal{L}} + c_5 \varepsilon.$$

The right-hand side tends to zero as $\varepsilon \to 0$, since $Y_{\varepsilon,n} \to Y_n$ in $\mathcal{L}$ and u_n is bounded on $\mathcal{I}_a$ as an element of $W_2^2(\mathcal{I}_a)$. The same proof works for the cases $\lambda_n \in \sigma(A_b)$ and $\lambda_n \in \sigma(B)$. $\qquad \square$

Remark 2. Of course, in the case of multiple eigenvalues, we also have some information about the convergence of eigenfunctions. For instance, if we suppose that $\lambda \in \sigma(A_a) \cap \sigma(A_b)$, but λ is not an eigenvalue of B, and two eigenvalues λ_n^ε and $\lambda_{n+1}^\varepsilon$ tend to λ as $\varepsilon \to 0$, then the gap between the eigensubspace X_λ of $\mathcal{A}$ and the subspace $X_\lambda^\varepsilon = \text{span}\{y_{\varepsilon,n}, y_{\varepsilon,n+1}\}$ vanishes as $\varepsilon \to 0$. Therefore, the eigenfunctions $y_{\varepsilon,n}$ and $y_{\varepsilon,n+1}$ converge in $L_2(\mathcal{I})$ to some linear combinations $c_1 u_\lambda + c_2 v_\lambda$, where u_λ and v_λ are eigenfunctions of A_a and A_b respectively that correspond to λ. However, without a deeper analysis of the problem, we will not know what the linear combinations are limit positions of vectors $y_{\varepsilon,n}$ and $y_{\varepsilon,n+1}$ in the plane X_λ.

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ПРО СПЕКТР СТРУН З δ'-ПОДІБНИМИ ЗБУРЕННЯМИ ГУСТИНИ МАСИ

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Коливні системи з приєднаними масами почали досліджувати ще Пуассон та Бессель, і сьогодні маємо тисячі наукових праць, в яких вивчають коливні процеси у середовищах з неоднорідно розподіленими масами. Теорія сильно неоднорідних середовищ почала особливо активно розвиватися з появою композитних матеріалів, які володіють різноманітними корисними властивостями і широко застосовуються в сучасних технологіях. Важливою частиною цієї теорії є дослідження динамічних та спектральних задач для диференціальних операторів із коефіцієнтами, що містять сильні локальні збурення. Такі задачі вимагають нових підходів в теорії крайових задач для диференціальних рівнянь, створення нових асимптотичних методів і нових обчислювальних схем.

Ми вивчаємо спектральні властивості операторів Штурма-Ліувілля з сингулярним збуренням вагової функції у випадку загальних крайових умов.

Збуренням є так звана δ'-подібна послідовність вигляду $\varepsilon^{-2}h(x/\varepsilon)$. Характерною особливістю таких задач є присутність залежної від малого параметра ε густини ρ_ε при спектральному параметрі, внаслідок чого самоспряжена операторна реалізація задачі можлива лише у вагових просторах Лебега, які теж залежать від параметра ε. Дослідження сімей операторів, які діють у різних просторах, пов'язане з багатьма труднощами математичної природи. По-перше, як трактувати збіжність таких сімей. По-друге, якщо оператори і збігаються в певному сенсі, то чи така збіжність гарантує збіжність спектрів та власних підпросторів, позаяк в цьому випадку не вдається застосувати класичні теореми теорії операторів. Багато дослідників уникали питання операторної збіжності у таких задач, будуючи формальні асимптотики власних значень і власних функцій та застосовуючи теорію квазімод. Такий підхід не завжди дає цілковитий опис граничної поведінки спектру, його граничної кратності та структури граничних власних підпросторів.

У цій статті застосовано інший підхід до задач з концентрованими масами. Відмовившись від переваг самоспряжених операторів, ми реалізували сингулярно збурену задачу як сім'ю деяких несамоспряжених матричних операторів $\mathcal{A}_\varepsilon$, що діють в тому самому гільбертовому просторі. Ми довели рівномірну резольвентну збіжність $\mathcal{A}_\varepsilon$ при as $\varepsilon \to 0$ і, як наслідок, довели, що їхні спектри збігаються в сенсі Гаусдорфа до спектра деякого оператора $\mathcal{A}$. Цікаво, що хоча оператори $\mathcal{A}_\varepsilon$ були подібними до самоспряжених і володіли дійсним дискретним і простим спектром, граничний оператор $\mathcal{A}$ виявився суттєво несамоспряженим з кратним спектром і кореневими підпросторами, що містили приєднані вектори. На мові самоспряженої операторної реалізації, ми отримали приклад сім'ї самоспряжених операторів T_ε з компактною резольвентою, що діють у деяких просторах H_ε і які "збігаються" до несамоспряженого оператора T_0 в просторі H_0. Тобто, спектри операторів T_ε збігаються в сенсі Гаусдорфа до спектру оператора T_0 із врахуванням алгебричної кратності, а граничне розташування власних підпросторів операторів T_ε можна описати лише за допомогою кореневих підпросторів оператора T_0.

Ключові слова: оператор Штурма-Ліувілля, приєднана маса, сингулярні збурення, спектральна задача, рівномірна резольвентна збіжність, збіжність за Гаусдорфом, несамоспряжений оператор.