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HOMEOMORPHISMS OF THE SPACE OF NONZERO INTEGERS WITH THE KIRCH TOPOLOGY

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The *Golomb* (resp. *Kirch*) topology on the set $\mathbb{Z}^\bullet$ of nonzero integers is generated by the base consisting of arithmetic progressions $a + b\mathbb{Z} = \{a + bn : n \in \mathbb{Z}\}$ where $a \in \mathbb{Z}^\bullet$ and b is a (square-free) number, coprime with a. In 2019 Dario Spirito proved that the space of nonzero integers endowed with the Golomb topology admits only two self-homeomorphisms. In this paper we prove an analogous fact for the space of nonzero integers endowed with the Kirch topology: it also admits exactly two self-homeomorphisms.

Key words: Kirch topology, superconnected space, superconnecting poset.

In this paper we describe the homeomorphism group of the space $\mathbb{Z}^\bullet$ of nonzero integers endowed with the *Kirch topology* τ_K, which is generated by the subbase consisting of the cosets $a + p\mathbb{Z}$ where $a \in \mathbb{Z}^\bullet$ and p is a prime number that does not divide a. On the subspace $\mathbb{N}$ of $\mathbb{Z}^\bullet$ this topology was introduced by Kirch in [6].

Banakh, Stelmakh and Turek [3] proved that the subspace $\mathbb{N}$ of $(\mathbb{Z}^\bullet, \tau_K)$ is topologically rigid in the sense that each self-homeomorphism of $\mathbb{N}$ endowed with the subspace topology $\tau_K{\restriction}\mathbb{N} = \{U \cap \mathbb{N} : U \in \tau\}$ is the identity map of $\mathbb{N}$.

On the other hand, the space $(\mathbb{Z}^\bullet, \tau_K)$ does admit a non-trivial self-homeomorpfism, namely the map

$$j : \mathbb{Z}^\bullet \to \mathbb{Z}^\bullet, \quad j : x \mapsto -x.$$

In this paper we prove that this is the unique non-trivial self-homeomorphism of the topological space $(\mathbb{Z}^\bullet, \tau_K)$. A similar result for the Golomb topology on $\mathbb{Z}^\bullet$ was proved by Dario Spirito [11]. The topological rigidity of the Golomb topology on $\mathbb{N}$ was proved by Banakh, Spirito and Turek in [2].

Theorem 1. *The space $\mathbb{Z}^\bullet = \mathbb{Z}\setminus\{0\}$ of nonzero integers endowed with the Kirch topology admits only two self-homeomorphisms.*

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The proof of this theorem follows the lines of the proof of the topological rigidity of the space $(\mathbb{N}, \tau_K{\restriction}\mathbb{N})$ from [3]. The proof is divided into 23 lemmas. A crucial role in the proof belongs to the superconnectedness of the Kirch space and the superconnecting poset of the Kirch space, which is defined in Section 2.

1. Four classical number-theoretic results

By Π we denote the set of prime numbers. For a number $x \in \mathbb{Z}$ by Π_x we denote the set of all prime divisors of x. Two numbers $x, y \in \mathbb{Z}$ are *coprime* iff $\Pi_x \cap \Pi_y = \varnothing$.

In the proof of Theorem 1 we shall exploit the following four known results of Number Theory. The first one is the famous Chinese Remainder Theorem (see. e.g. [5, 3.12]).

Theorem 2 (**Chinese Remainder Theorem**). *If $b_1, \dots, b_n \in \mathbb{Z}$ are pairwise coprime numbers, then for any numbers $a_1, \dots, a_n \in \mathbb{Z}$, the intersection $\bigcap_{i=1}^{n}(a_i + b_i\mathbb{N})$ is infinite.*

The second classical result is not elementary and is due to Dirichlet [4, S.VI], see also [1, Ch.7].

Theorem 3 (**Dirichlet**). *For any coprime numbers $a, b \in \mathbb{N}$ the arithmetic progression $a + b\mathbb{N}$ contains a prime number.*

The third classical result is a recent theorem of Mihaĭlescu [8] who solved old Catalan's Conjecture [7].

Theorem 4 (**Mihăilescu**). *If $a, b \in \{m^{n+1} : n, m \in \mathbb{N}\}$, then $|a - b| = 1$ if and only if $\{a, b\} = \{2^3, 3^2\}$.*

The fourth classical result we use is due to Karl Zsigmondy [12], see also [10, Theorem 3].

Theorem 5 (**Zsigmondy**). *For integer numbers $a, n \in \mathbb{N} \setminus \{1\}$ the inclusion*

$$\Pi_{a^n-1} \subseteq \bigcup_{0<k<n} \Pi_{a^k-1}$$

holds if and only if one of the following conditions is satisfied:

(1) *$n = 2$ and $a = 2^k - 1$ for some $k \in \mathbb{N}$; then*
$$a^2 - 1 = (a+1)(a-1) = 2^k(a-1);$$
(2) *$n = 6$ and $a = 2$; then*
$$a^n - 1 = 2^6 - 1 = 63 = 3^2 \times 7 = (a^2-1)^2 \times (a^3-1).$$

2. Superconnected spaces and their superconnecting posets

In this section we discuss superconnected topological spaces and some order structures related to such spaces.

First let us introduce some notation and recall some notions.

For a set A and $n \in \omega$ let $[A]^n = \{E \subseteq A : |A| = n\}$ be the family of n-element subsets of A, and $[A]^{<\omega} = \bigcup_{n\in\omega} [A]^n$ be the family of all finite subsets of A. For a function

$f : X \to Y$ and a subset $A \subseteq X$ by $f[A]$ we denote the image $\{f(a) : a \in A\}$ of the set A under the function f.

For a subset A of a topological space (X, τ) by $\overline{A}$ we denote the closure of A in X. For a point $x \in X$ we denote by $\tau_x := \{U \in \tau : x \in U\}$ the family of all open neighborhoods of x in (X, τ). A *poset* is an abbreviation for a partially ordered set.

A family $\mathcal{F}$ of subsets of a set X is called a *filter* if

- $\varnothing \notin \mathcal{F}$;
- for any $A, B \in \mathcal{F}$ their intersection $A \cap B \in \mathcal{F}$;
- for any sets $F \subseteq E \subseteq X$ the inclusion $F \in \mathcal{F}$ implies $E \in \mathcal{F}$.

A topological space (X, τ) is called *superconnected* if for any $n \in \mathbb{N}$ and non-empty open sets $U_1, \dots, U_n$ the intersection $\overline{U_1} \cap \dots \cap \overline{U_n}$ is non-empty. This allows us to define the filter

$$\mathcal{F}_\infty = \big\{B \subseteq X \colon \exists U_1, \dots, U_n \in \tau \setminus \{\varnothing\} \;\; (\overline{U_1} \cap \dots \cap \overline{U_n} \subseteq B)\big\}$$

called the *superconnecting filter* of X.

For every finite subset E of X consider the subfilter

$$\mathcal{F}_E := \Big\{B \subseteq X : \exists (U_x)_{x \in E} \in \prod_{x \in E} \tau_x \Big(\bigcap_{x \in E} \overline{U_x} \subseteq B\Big)\Big\}$$

of $\mathcal{F}_\infty$. Here we assume that $\mathcal{F}_\varnothing = \{X\}$. It is clear that for any finite sets $E \subseteq F$ in X we have $\mathcal{F}_E \subseteq \mathcal{F}_F$.

The family

$$\mathfrak{F} = \big\{\mathcal{F}_E : E \in [X]^{<\omega}\big\} \cup \{\mathcal{F}_\infty\}$$

is endowed with the inclusion partial order and is called the *superconnecting poset* of the superconnected space X. The filters $\mathcal{F}_\varnothing$ and $\mathcal{F}_\infty$ are the smallest and largest elements of the poset $\mathfrak{F}$, respectively.

The following obvious lemma shows that the superconnecting poset $\mathfrak{F}$ is a topological invariant of the superconnected space.

Proposition 1. *For any homeomorphism h of any superconnected topological space X, the map*

$$\tilde{h} : \mathfrak{F} \to \mathfrak{F}, \quad \tilde{h} \colon \mathcal{F} \mapsto \{h[A] \colon A \in \mathcal{F}\},$$

is an order isomorphism of the superconnecting poset $\mathfrak{F}$.

In the following sections we shall study the order properties of the poset $\mathfrak{F}$ for the Kirch space $(\mathbb{Z}^\bullet, \tau_K)$ and shall exploit the obtained information in the proof of the topological rigidity of the Kirch space.

3. Proof of Theorem 1

We divide the proof of Theorem 1 into 23 lemmas.

Lemma 1. *For any $a, b \in \mathbb{Z}^\bullet$ the closure $\overline{a + b\mathbb{Z}}$ of the arithmetic progression $a + b\mathbb{Z}$ in the Kirch space $(\mathbb{Z}^\bullet, \tau_K)$ is equal to*

$$\mathbb{Z}^\bullet \cap \bigcap_{p \in \Pi_b} \big(\{0, a\} + p\mathbb{Z}\big).$$

Proof. First we prove that $\overline{a+b\mathbb{Z}} \subseteq \{0,a\}+p\mathbb{Z}$ for every $p \in \Pi_b$. Take any point $x \in \overline{a+b\mathbb{Z}}$ and assume that $x \notin p\mathbb{Z}$. Then $x+p\mathbb{Z}$ is a neighborhood of x and hence the intersection $(x+p\mathbb{Z}) \cap (a+b\mathbb{Z})$ is not empty. Then there exist $u, v \in \mathbb{Z}$ such that $x+pu = a+bv$. Consequently, $x-a = bv-pu \in p\mathbb{Z}$ and $x \in a+p\mathbb{Z}$.

Next, take any point $x \in \mathbb{Z}^\bullet \cap \bigcap_{p\in\Pi_b}(\{0,a\}+p\mathbb{Z})$. Given any neighborhood O_x of x in $(\mathbb{Z}^\bullet, \tau_K)$, we should prove that $O_x \cap (a+b\mathbb{Z}) \neq \varnothing$. By the definition of the Kirch topology there exists a square-free number $d \in \mathbb{Z}^\bullet$ such that d, x are coprime and $x+d\mathbb{Z} \subseteq O_x$.

If $\Pi_b \subseteq \Pi_x$, then b, d are coprime and by the Chinese Remainder Theorem

$$\varnothing \neq (x+d\mathbb{Z}) \cap (a+b\mathbb{Z}) \subseteq O_x \cap (a+b\mathbb{Z}).$$

So, we can assume $\Pi_b \setminus \Pi_x \neq \varnothing$. The choice of $x \in \bigcap\limits_{p\in\Pi_b}(\{0,a\}+p\mathbb{Z})$ guarantees that

$$x \in \bigcap_{p\in\Pi_b\setminus\Pi_x}(a+p\mathbb{Z}) = a+q\mathbb{Z}$$

where $q = \prod\limits_{p\in\Pi_b\setminus\Pi_x} p$. Since the numbers x and d are coprime and d is square-free, the greatest common divisor of b and d divides the number q. Since $x-a \in q\mathbb{Z}$, the Euclides algorithm yields two numbers $u, v \in \mathbb{Z}$ such that $x-a = bu-dv$, which implies that $O_x \cap (a+b\mathbb{Z}) \supseteq (x+d\mathbb{Z}) \cap (a+b\mathbb{Z}) \neq \varnothing$. □

Lemma 1 implies that the Kirch space $(\mathbb{Z}^\bullet, \tau_K)$ is superconnected and hence possesses the superconnecting filter

$$\mathcal{F}_\infty = \Big\{F \subseteq \mathbb{Z}^\bullet : \exists U_1, \dots, U_n \in \tau_K \setminus \{\varnothing\} \quad \Big(\bigcap_{i=1}^n \overline{U_i} \subseteq F\Big)\Big\}$$

and the superconnecting poset

$$\mathfrak{F} = \big\{\mathcal{F}_E : E \in [\mathbb{Z}^\bullet]^{<\omega}\big\} \cup \{\mathcal{F}_\infty\}$$

consisting of the filters

$$\mathcal{F}_E = \Big\{F \subseteq \mathbb{Z}^\bullet : \exists (U_x)_{x\in E} \in \prod_{x\in E}\tau_x \Big(\bigcap_{x\in E}\overline{U_x} \subseteq F\Big)\Big\}.$$

Here for a point $x \in \mathbb{Z}^\bullet$ by $\tau_x := \{U \subseteq \mathbb{Z}^\bullet : x \in U\}$ we denote the family of open neighborhoods of x in the Kirch topology τ_K.

For a nonempty finite subset $E \subseteq \mathbb{Z}^\bullet$, let $\Pi_E = \bigcap\limits_{x\in E}\Pi_x$ be the set of common prime divisors of numbers in the set E. Also let

$$A_E = \{p \in \Pi : \exists k \in \mathbb{N} \ \ (E \subset \{0,k\}+p\mathbb{Z})\}.$$

Observe that $\Pi_E \subseteq A_E$ and $A_E \neq \varnothing$ because $2 \in A_E$. If E is a singleton, then $A_E = \Pi$; if $|E| \geq 2$, then

$$A_E \subseteq \Pi_x \cup \Pi_y \cup \Pi_{x-y} \subseteq \{2, \dots, \max E\}$$

for any distinct numbers $x, y \in E$. This inclusion follows from

Lemma 2. *For any two-element set $E = \{x,y\} \subset \mathbb{Z}^\bullet$ we have $A_E = \Pi_x \cup \Pi_y \cup \Pi_{x-y}$.*

Proof. Each number $p \in \Pi_x$ (resp. $p \in \Pi_y$) belongs to A_E because $\{x, y\} \subset \{0, y\} + p\mathbb{Z}$ (resp. $\{x, y\} \subset \{0, x\} + p\mathbb{Z}\}$). Each number $p \in \Pi_{x-y}$ belongs to A_E because $\{x, y\} \subset x + p\mathbb{Z} \subseteq \{0, x\} + p\mathbb{Z}$. This proves that $\Pi_x \cup \Pi_y \cup \Pi_{x-y} \subseteq A_E$.

Now take any prime number $p \in A_E$ and assume that $p \notin \Pi_x \cup \Pi_y$. It follows from $\{x, y\} = E \subset \{0, \alpha_E(p)\} + p\mathbb{Z}$ that $\{x, y\} \subseteq \alpha_E(p) + p\mathbb{Z}$ and hence $x - y \in p\mathbb{Z}$ and $p \in \Pi_{x-y}$. □

Let $\alpha_E \colon A_E \to \omega$ be the unique function satisfying the following conditions:

(i) $\alpha_E(p) < p$ for all $p \in A_E$;
(ii) $E \subseteq \{0, \alpha_E(p)\} + p\mathbb{Z}$ for all $p \in A_E$;
(iii) $\alpha_E(2) = 1$ and $\alpha_E(p) = 0$ for all $p \in \Pi_E \setminus \{2\}$.

Lemma 3. *Let $A \subset \Pi$ be a finite set containing 2 and $\alpha : A \to \mathbb{N}_0$ be a function such that $\alpha(2) = 1$ and $\alpha(p) \in \{0, \dots, p-1\}$ for all $p \in A \setminus \{2\}$. Let x be the product of odd prime numbers in the set A and y be any number in the set $\mathbb{Z}^\bullet \cap \bigcap_{p\in A}(\alpha(p) + p\mathbb{Z})$. Then the set $E = \{y, x, 2x\}$ has $A_E = A$ and $\alpha_E = \alpha$.*

Proof. For every prime number $p \in A$ we have $\{x, y\} \subset \{0, y\} + p\mathbb{Z}$, which implies that $p \in A_E$. Assuming that $A_E \setminus A$ contains some prime number p, we conclude that $x \notin p\mathbb{Z}$ and hence the inclusion $\{y, x, 2x\} = E \subset \{0, \alpha_E(p)\} + p\mathbb{Z}$ implies $\{x, 2x\} \subset \alpha_E(p) + p\mathbb{Z}$ and $x = 2x - x \in p\mathbb{Z}$. This contradiction shows that $A_E = A$. To show that $\alpha_E = \alpha$, take any prime number $p \in A = A_E$. If $p = 2$, then $\alpha(p) = 1 = \alpha_E(p)$. So, we assume that $p \neq 2$. If $\alpha(p) = 0$, then $y \in \alpha(p) + p\mathbb{Z} = p\mathbb{Z}$ and hence $p \in \Pi_E$. In this case $\alpha_E(p) = 0 = \alpha(p)$. If $\alpha(p) \neq 0$, then the number $y \in \alpha(p) + p\mathbb{Z}$ is not divisible by p and then the inclusions $\{y, x, 2x\} \subseteq \{0, \alpha(p)\} + p\mathbb{Z}$ and $\{y, x, 2x\} = E \subset \{0, \alpha_E(p)\} + p\mathbb{Z}$ imply that $\alpha(p) = \alpha_E(p)$. □

The following lemma yields an arithmetic description of the filters $\mathcal{F}_E$.

Lemma 4. *For any finite subset $E \subseteq \mathbb{Z}^\bullet$ with $|E| \geq 2$ we have*

$$\mathcal{F}_E = \Big\{B \subseteq \mathbb{Z}^\bullet \colon \exists L \in [\Pi \setminus A_E]^{<\omega} \quad \bigcap_{p\in L} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_E\setminus\Pi_E} (\{0, \alpha_E(p)\} + p\mathbb{Z}) \subseteq B\Big\}.$$

Here we assume that $\bigcap_{p\in\varnothing} p\mathbb{Z}^\bullet = \mathbb{Z}^\bullet$.

Proof. It suffices to verify two properties:

(1) for any $(U_x)_{x\in E} \in \prod_{x\in E} \tau_x$ there exists a finite set $L \subseteq \Pi \setminus A_E$ such that

$$\bigcap_{p\in L} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_E\setminus\Pi_E} (\{0, \alpha_E(p)\} + p\mathbb{Z}) \subseteq \bigcap_{x\in E} \overline{U_x};$$

(2) for any finite set $L \subseteq \Pi \setminus A_E$ there exists a sequence of neighborhoods $(U_x)_{x\in E} \in \prod_{x\in E} \tau_x$ such that

$$\bigcap_{x\in E} \overline{U_x} \subseteq \bigcap_{p\in L} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_E\setminus\Pi_E} (\{0, \alpha_E(p)\} + p\mathbb{Z}).$$

1. Given a sequence of neighborhoods $(U_x)_{x\in E} \in \prod_{x\in E}\tau_x$, for every $x \in E$ find a square-free number $q_x > x$ such that $\Pi_{q_x} \cap \Pi_x = \varnothing$ and $x + q_x\mathbb{Z} \subseteq U_x$. We claim that the finite set $L = \bigcup_{x\in E}\Pi_{q_x} \setminus A_E$ has the required property. Given any number

$$z \in \bigcap_{p\in L} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_E\setminus\Pi_E} (\{0, \alpha_E(p)\} + p\mathbb{Z}),$$

we should prove that $z \in \overline{U_x}$ for every $x \in E$. By Lemma 1,

$$\mathbb{Z}^\bullet \cap \bigcap_{p\in\Pi_{q_x}} (\{0, x\} + p\mathbb{Z}) = \overline{(x + q_x\mathbb{Z})} \subseteq \overline{U_x}.$$

So, it suffices to show that $z \in \{0, x\} + p\mathbb{Z}$ for any $p \in \Pi_{q_x}$. Since the numbers x and q_x are coprime, $p \notin \Pi_x$ and hence $p \notin \Pi_E$. If $p \notin A_E$, then $p \in \Pi_{q_x} \setminus A_E \subseteq L$ and hence $z \in p\mathbb{N} \subseteq \{0, x\} + p\mathbb{Z}$. If $p \in A_E$, then $x \in E \subseteq \{0, \alpha_E(p)\} + p\mathbb{Z}$ and $x \in \alpha_E(p) + p\mathbb{Z}$ (as $p \notin \Pi_x$). Then $x + p\mathbb{Z} = \alpha_E(p) + p\mathbb{Z}$ and $z \in \{0, \alpha_E(p)\} + p\mathbb{Z} = \{0, x\} + p\mathbb{Z}$.

2. Fix any finite set $L \subseteq \Pi \setminus A_E$. For every $x \in E$ consider the neighborhood $U_x = \bigcap_{p\in L\cup A_E\setminus\Pi_x} (x + p\mathbb{Z})$ of x in the Kirch topology. By Lemma 1,

$$\overline{U_x} = \mathbb{Z}^\bullet \cap \bigcap_{p\in L\cup A_E\setminus\Pi_x} (\{0, x\} + p\mathbb{Z}).$$

Given any number $z \in \bigcap_{x\in E}\overline{U_x}$, we should show that

$$z \in \bigcap_{p\in L} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_E\setminus\Pi_E} (\{0, \alpha_E(p)\} + p\mathbb{Z}).$$

This will follow as soon as we check that $z \in p\mathbb{Z}^\bullet$ for all $p \in L$ and $z \in \{0, \alpha_E(p)\} + p\mathbb{Z}$ for all $p \in A_E \setminus \Pi_E$.

Given any $p \in A_E \setminus \Pi_E$, we can find a point $x \in E \setminus p\mathbb{Z}$ and observe that $x \in E \subseteq \{0, \alpha_E(p)\} + p\mathbb{Z}$. Then

$$z \in \overline{U_x} \subseteq \overline{x + p\mathbb{Z}} \subseteq \{0, x\} + p\mathbb{Z} = \{0, \alpha_E(p)\} + p\mathbb{Z}.$$

Now take any prime number $p \in L$. Since $L \cap A_E = \varnothing$, we conclude that $E \not\subseteq p\mathbb{Z}$. So, we can fix a number $x \in E \setminus p\mathbb{Z}$. Taking into account that $p \notin A_E$, we conclude that $E \not\subseteq \{0, x\} + p\mathbb{Z}$ and hence there exists a number $y \in E$ such that $p\mathbb{Z} \neq y + p\mathbb{Z} \neq x + p\mathbb{Z}$. Then

$$z \in \overline{U_x} \cap \overline{U_y} \subseteq (\{0, x\} + p\mathbb{Z}) \cap (\{0, y\} + p\mathbb{Z}) = p\mathbb{Z}.$$

□

We shall use Lemma 4 for an arithmetic characterization of the partial order of the superconnecting poset $\mathfrak{F}$ of the Kirch space.

Lemma 5. *For two finite subsets $E, F \subseteq \Pi$ with $\min\{|E|, |F|\} \geq 2$ we have $\mathcal{F}_E \subseteq \mathcal{F}_F$ if and only if*

$$A_F \subseteq A_E,\ \ \Pi_F \setminus \{2\} \subseteq \Pi_E \quad \textit{and} \quad \alpha_E{\restriction}A_F \setminus \Pi_E = \alpha_F{\restriction}A_F \setminus \Pi_E.$$

Proof. To prove the "only if" part, assume that $\mathcal{F}_E \subseteq \mathcal{F}_F$. By Lemma 4, the set

$$\bigcap_{p\in A_F\setminus A_E} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_E\setminus\Pi_E} (\{0,\alpha_E(p)\}+p\mathbb{Z})$$

belongs to the filter $\mathcal{F}_E \subseteq \mathcal{F}_F$. By Lemma 4, there exists a finite set $L \subset \Pi\setminus A_F$ such that

$$\bigcap_{p\in L} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_F\setminus\Pi_F} (\{0,\alpha_F(p)\}+p\mathbb{Z}) \subseteq \bigcap_{p\in A_F\setminus A_E} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_E\setminus\Pi_E} (\{0,\alpha_E(p)\}+p\mathbb{Z}). \quad (1)$$

This inclusion combined with the Chinese Remainder Theorem 2 implies

$$A_F\setminus A_E \subseteq L \subset \Pi\setminus A_F,\ \ A_E\setminus(\Pi_E\cup\{2\}) \subseteq L\cup(A_F\setminus\Pi_F)$$

and

$$\alpha_E(p)=\alpha_F(p) \qquad \text{for any} \qquad p\in(A_F\setminus\Pi_F)\cap(A_E\setminus\Pi_E),$$

and

$$A_F\subseteq A_E, \qquad \Pi_F\setminus\{2\}\subseteq\Pi_E \qquad \text{and} \qquad \alpha_E{\restriction}A_F\setminus\Pi_E=\alpha_F{\restriction}A_F\setminus\Pi_E. \quad (2)$$

To prove the "if" part, assume that condition (2) holds. To prove that $\mathcal{F}_E\subseteq\mathcal{F}_F$, fix any set $\Omega\in\mathcal{F}_E$ and using Lemma 4, find a finite set $L\subseteq\Pi\setminus A_E$ such that

$$\bigcap_{p\in L} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_E\setminus\Pi_E} (\{0,\alpha_E(p)\}+p\mathbb{Z}) \subseteq \Omega.$$

Consider the finite set

$$\Lambda=(L\cup A_E)\setminus A_F = L\cup(A_E\setminus A_F)\supseteq L$$

and observe that condition (2) implies the inclusion

$$\mathcal{F}_F \ni \bigcap_{p\in\Lambda} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_F\setminus\Pi_F} (\{0,\alpha_F(p)\}+p\mathbb{Z}) \subseteq \bigcap_{p\in L} p\mathbb{Z}^\bullet \cap \bigcap_{p\in A_E\setminus\Pi_E} (\{0,\alpha_E(p)\}+p\mathbb{Z}) \subseteq \Omega, \quad (3)$$

yielding $\Omega\in\mathcal{F}_F$. □

Lemma 6. *For two nonempty subsets $E,F\subseteq\mathbb{N}$ with $\min\{|E|,|F|\}=1$ the relation $\mathcal{F}_E\subseteq\mathcal{F}_F$ holds if and only if $|E|=1$ and $E\subseteq F$.*

Proof. The "if" part is trivial. To prove the "only if" part, assume that $\mathcal{F}_E\subseteq\mathcal{F}_F$. First we prove that $|E|=1$. Assuming that $|E|>1$ and taking into account that $\min\{|E|,|F|\}=1$, we conclude that $|F|=1$. Choose a prime number $p>\max(E\cup F)$. Since $\bigcap_{y\in E}\overline{y+p\mathbb{Z}}\in\mathcal{F}_E\subseteq\mathcal{F}_F$, for the unique number x in the set F, there exists a square-free number d such that $\Pi_d\cap\Pi_x=\varnothing$ and $\overline{x+dp\mathbb{Z}}\subseteq\bigcap_{y\in E}\overline{y+p\mathbb{Z}}$. By Lemma 1,

$$x+qp\mathbb{Z}^\bullet\subseteq\overline{x+dp\mathbb{Z}}\subseteq\bigcap_{y\in E}\overline{y+p\mathbb{Z}}=\bigcap_{y\in E}(\{0,y\}+p\mathbb{Z})=p\mathbb{Z}.$$

The latter equality follows from $p>\max E$ and $|E|>1$. Then $x+dp\mathbb{Z}^\bullet\subseteq p\mathbb{Z}$ implies $x\in p\mathbb{Z}$, which contradicts the choice of $p>\max(E\cup F)\geq x$. This contradiction shows that $|E|=1$. Let z be the unique element of the set E.

It remains to prove that $z \in F$. To derive a contradiction, assume that $z \notin F$. Take any odd prime number $p > \max(E \cup F)$ and consider the set

$$\{0, z\} + p\mathbb{Z} = \overline{z + p\mathbb{Z}} \in \mathcal{F}_E \subseteq \mathcal{F}_F.$$

By the definition of the filter $\mathcal{F}_F$, for every $x \in F$ there exists a square-free number d_x such that $\Pi_{d_x} \cap \Pi_x = \varnothing$ and

$$\bigcap_{x \in F} \overline{x + d_x\mathbb{Z}} \subseteq \overline{z + p\mathbb{Z}} = \{0, z\} + p\mathbb{Z}.$$

Consider the set $P = \bigcup_{x \in F} \Pi_{d_x}$. If $p \in \Pi_{d_x}$ for some $x \in F$, we can use the Chinese Remainder Theorem 2 and find a number

$$c \in (x + p\mathbb{Z}) \cap \bigcap_{q \in P \setminus \{p\}} q\mathbb{Z} \subseteq \bigcap_{y \in F} \overline{y + d_y\mathbb{Z}} \subseteq \{0, z\} + p\mathbb{Z}.$$

Taking into account that x is not divisible by p, we conclude that $c \in (x + p\mathbb{Z}) \cap (z + p\mathbb{Z})$ and hence $x - z \in p\mathbb{Z}$, which contradicts the choice of $p > \max(E \cup F)$. This contradiction shows that $p \notin P$. Since $p \geq 3$, we can find a number $z' \notin \{0, z\} + p\mathbb{Z}$ and using the Chinese Remainder Theorem 2, find a number

$$u \in (z' + p\mathbb{Z}) \cap \bigcap_{q \in P} q\mathbb{Z}^\bullet \subseteq \bigcap_{y \in F} \overline{y + d_y\mathbb{Z}} \subseteq \{0, z\} + p\mathbb{Z},$$

which is a desired contradiction showing that $E \subseteq F$. $\square$

As we know, the largest element of the superconnecting poset $\mathfrak{F}$ is the superconnecting filter $\mathcal{F}_\infty$. This filter can be characterized as follows.

Lemma 7. *The superconnecting filter $\mathcal{F}_\infty$ of the Kirch space is generated by the base consisting of the sets $q\mathbb{N}$ for an odd square-free number $q \in \mathbb{N}$, i.e.*

$$\mathcal{F}_\infty = \Big\{ B \subseteq \mathbb{Z}^\bullet : \exists q \in (2\mathbb{N} - 1) \setminus \bigcup_{p \in \Pi} p^2\mathbb{N} \ \ (q\mathbb{Z}^\bullet \subseteq B) \Big\}.$$

Proof. Lemma 1 implies that each element $F \in \mathcal{F}_\infty$ contains the set $q\mathbb{Z}^\bullet$ for some odd square-free number q. Conversely, let q be an odd square-free number. Then the sets $U_1 = 1 + q\mathbb{Z}$ and $U_2 = 2 + q\mathbb{Z}$ are open in the Kirch topology on $\mathbb{Z}^\bullet$. By Lemma 1 we have

$$\overline{U_1} \cap \overline{U_2} = \mathbb{Z}^\bullet \cap \bigcap_{p \in \Pi_q} (\{0, 1\} + p\mathbb{Z}) \cap (\{0, 2\} + p\mathbb{Z}) = \mathbb{Z}^\bullet \cap \bigcap_{p \in \Pi_q} p\mathbb{Z} = q\mathbb{Z}^\bullet.$$

Hence $q\mathbb{Z}^\bullet \in \mathcal{F}_\infty$. $\square$

Lemma 8. *For a nonempty subset $E \subseteq \mathbb{Z}^\bullet$ the following conditions are equivalent:*

(1) $\mathcal{F}_E = \mathcal{F}_\infty$;
(2) $A_E = \{2\}$.

If $|E| = 2$, then conditions (1), (2) *are equivalent to*

(3) $E \in \big\{ \{2^n, 2^{n+1}\}, \{-2^n, -2^{n+1}\}, \{-2^n, 2^n\} : n \in \omega \big\}$.

Proof. (1) $\Rightarrow$ (2): Assume $\mathcal{F}_E = \mathcal{F}_\infty$. Consider the set $F = \{1, 2\}$ and observe that $A_F = \Pi_1 \cup \Pi_2 \cup \Pi_{2-1} = \{2\}$ and $\Pi_F = \varnothing$. Thus $A_F \subseteq A_E$, $\Pi_F \backslash \{2\} \subseteq \Pi_E$ and $\alpha_F {\restriction} A_F \backslash \Pi_E = \alpha_E {\restriction} A_F \backslash \Pi_E$. Lemma 5 implies $\mathcal{F}_E \subseteq \mathcal{F}_F$. Since $\mathcal{F}_E = \mathcal{F}_\infty$ is the largest element of $\mathfrak{F}$ we get $\mathcal{F}_E = \mathcal{F}_F$. By using again Lemma 5 we get $A_E \subseteq A_F$ which implies that $A_E = \{2\}$.

(2) $\Rightarrow$ (1): If $A_E = \{2\}$, then by the Lemma 4, the filter $\mathcal{F}_E$ is generated by the base consisting of the sets $q\mathbb{Z}^\bullet$ for an odd square-free number $q \in \mathbb{Z}^\bullet$. Therefore $\mathcal{F}_E = \mathcal{F}_\infty$ by the Lemma 7.

(2) $\Rightarrow$ (3): Assume that $|E| = 2$ and $A_E = \{2\}$. By Lemma 2, $E = \{\varepsilon 2^a, \delta 2^b\}$, where $a, b \in \omega$ and $\varepsilon, \delta \in \{-1, 1\}$. Without loss of generality we can assume that $b \leqslant a$. By Lemma 2, $\Pi_{\varepsilon 2^a - \delta 2^b} = \Pi_{\varepsilon 2^b(2^{a-b} - \delta/\varepsilon)} \subseteq \{2\}$. The last inclusion implies that $a - b = 1$ and $\delta/\varepsilon = 1$ or $a - b = 0$ and $\delta/\varepsilon = -1$. In the first case the set E equals $\{2^b, 2^{b+1}\}$ or $\{-2^b, -2^{b+1}\}$, in the second case $E = \{2^b, -2^b\}$.

(3) $\Rightarrow$ (2): The implication (3) $\Rightarrow$ (2) follows from Lemma 2. $\square$

In the following lemmas by $\mathfrak{F}'$ we denote the set of maximal elements of the poset $\mathfrak{F} \setminus \{\mathcal{F}_\infty\}$.

Lemma 9. *For a nonempty finite subset $E \subseteq \mathbb{Z}^\bullet$ the filter $\mathcal{F}_E$ belongs to the family $\mathfrak{F}'$ if and only if there exists an odd prime number $p \notin \Pi_E$ such that $A_E = \{2, p\}$.*

Proof. To prove the "if" part, assume that $A_E = \{2, p\}$ and $p \notin \Pi_E$ for some odd prime number p. By Lemma 8, $\mathcal{F}_E \neq \mathcal{F}_\infty$. To show that the filter $\mathcal{F}_E$ is maximal in $\mathfrak{F} \setminus \{\mathcal{F}_\infty\}$, take any finite set $F \subset \mathbb{Z}^\bullet$ such that $\mathcal{F}_E \subseteq \mathcal{F}_F \neq \mathcal{F}_\infty$. By Lemmas 5 and 8, $\{2\} \neq A_F \subseteq \{2, p\}$, $\Pi_F \subseteq \Pi_E \cup \{2\} = \{2\}$, and $\alpha_F {\restriction} A_F \setminus \Pi_E = \alpha_E {\restriction} A_F \setminus \Pi_E$. It follows that $A_F = \{2, p\} = A_E$, $\Pi_F \cup \{2\} = \Pi_E \cup \{2\}$ and $\alpha_F = \alpha_E$. Applying Lemma 5, we conclude that $\mathcal{F}_E = \mathcal{F}_F$, which means that the filter $\mathcal{F}_E$ is a maximal element of the poset $\mathcal{F} \setminus \{\mathcal{F}_\infty\}$.

To prove the "only if" part, assume that $\mathcal{F}_E \in \mathfrak{F}'$. By Lemma 8, $A_E \neq \{2\}$ and hence there exists an odd prime number $p \in A_E$. We claim that $p \notin \Pi_E$. To derive a contradiction, assume that $p \in \Pi_E$ and consider the sets $F = \{p, 2p\}$ and $G = \{1, p, 2p\}$. By Lemma 2, $A_F = A_G = \{2, p\}$, $\Pi_F = \{p\}$, and $\Pi_G = \varnothing$. Taking into account that $F \subset G$, $A_F = \{2, p\} \subseteq A_E$, $\Pi_F \setminus \{2\} = \{p\} \subseteq \Pi_E$ and $A_F \setminus \Pi_E \subseteq \{2\}$, we can apply Lemmas 5, 8 and conclude that $\mathcal{F}_E \subseteq \mathcal{F}_F \subseteq \mathcal{F}_G \neq \mathcal{F}_\infty$. The maximality of $\mathcal{F}_E$ implies $\mathcal{F}_E = \mathcal{F}_F = \mathcal{F}_G$. By Lemma 5, the equality $\mathcal{F}_G = \mathcal{F}_F$ implies $p \in \Pi_F \setminus \{2\} \subseteq \Pi_G = \varnothing$, which is a contradiction showing that $p \notin \Pi_E$.

Now consider the set $H = \{\alpha_E(p), p, 2p\}$ and observe that $A_H = \{2, p\}$, $\Pi_H = \varnothing$ and $\alpha_H(p) = \alpha_E(p)$. Lemmas 5 and 8 guarantee that $\mathcal{F}_E \subseteq \mathcal{F}_H \neq \mathcal{F}_\infty$. By the maximality of $\mathcal{F}_E$, we have $\mathcal{F}_E = \mathcal{F}_H$. Applying Lemma 5 once more, we conclude that $A_E = A_H = \{2, p\}$. $\square$

Lemma 9 implies the following description of the set $\mathfrak{F}'$.

Lemma 10. $\mathfrak{F}' = \big\{\mathcal{F}_{\{a,p,2p\}} : p \in \Pi \setminus \{2\},\ \ a \in \{1, \dots, p-1\}\big\}$.

Let $\mathfrak{F}''$ be the set of maximal elements of the poset $\mathfrak{F} \setminus (\mathfrak{F}' \cup \{\mathcal{F}_\infty\})$

Lemma 11. *For a finite set $E \subset \mathbb{Z}^\bullet$, the filter $\mathcal{F}_E$ belongs to the family $\mathfrak{F}''$ if and only if one of the following conditions holds:*

(1) *there exists an odd prime number* p *such that* $p \in \Pi_E$ *and* $A_E = \{2, p\}$*;*
(2) *there are two distinct odd prime numbers* p, q *such that* $A_E = \{2, p, q\}$ *and* $\Pi_E \subseteq \{2\}$*.*

Proof. To prove the "only if" part, assume that $\mathcal{F}_E \in \mathfrak{F}''$. By Lemma 8, there is an odd prime number $p \in A_E$. If $A_E = \{2, p\}$, then $p \in \Pi_E$ by Lemma 9, and condition (1) is satisfied. So, we assume that $\{2, p\} \neq A_E$ and find an odd prime number $q \in A_E \setminus \{2, p\}$. By Lemma 3, there is a number $x \in \mathbb{N}$ such that for the set $F = \{x, pq, 2pq\}$ we have $A_F = \{2, p, q\}$, $\Pi_F = \varnothing$, $\alpha_F(p) = \alpha_E(p)$ and $\alpha_F(q) = \alpha_E(q)$. Then $\mathcal{F}_E \subseteq \mathcal{F}_F$ by Lemma 5, and $\mathcal{F}_F \in \mathfrak{F} \setminus (\mathfrak{F}' \cup \{\mathcal{F}_\infty\})$ by Lemma 9. Now the maximality of the filter $\mathcal{F}_E$ implies that $\mathcal{F}_E = \mathcal{F}_F$ and hence $A_E = A_F = \{2, p, q\}$ and $\Pi_E \subseteq \Pi_F \cup \{2\} = \{2\}$, see Lemma 5.

To prove the "if" part, we consider two cases. First we assume that $A_E = \{2, p\}$ and $p \in \Pi_E$ for some odd prime number p. By Lemmas 8 and 9, $\mathcal{F}_E \in \mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$. To prove that $\mathcal{F}_E$ is a maximal element of $\mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$, take any finite set $F \subseteq \mathbb{Z}^\bullet$ such that $\mathcal{F}_E \subseteq \mathcal{F}_F \in \mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$. Lemma 6 implies that $\min\{|E|, |F|\} \geq 2$ and then by Lemmas 5 and 9, we have $A_F = \{2, p\}$, $\Pi_F \setminus \{2\} \subseteq \{p\}$ and $\alpha_E{\restriction}A_F \setminus \{p\} = \alpha_F{\restriction}A_F \setminus \{p\}$. Now notice that $p \in \Pi_F$ since otherwise $\mathcal{F}_F \in \mathfrak{F}'$ by Lemma 9. By using again Lemma 5 we get $\mathcal{F}_F = \mathcal{F}_E$ which means that $\mathcal{F}_E \in \mathfrak{F}''$.

Now assume that there are two distinct odd prime numbers p, q such that $A_E = \{2, p, q\}$ and $\Pi_E \subseteq \{2\}$. By Lemmas 8 and 9, $\mathcal{F}_E \in \mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$. To prove that $\mathcal{F}_E$ is a maximal element of $\mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$, take any finite set $F \subseteq \mathbb{Z}^\bullet$ such that $\mathcal{F}_E \subseteq \mathcal{F}_F \in \mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$. Lemma 5 implies that $A_F \subseteq \{2, p, q\}$, $\Pi_F \subseteq \{2\}$ and $\alpha_E{\restriction}A_F \setminus \Pi_E = \alpha_F{\restriction}A_F \setminus \Pi_E$. Taking into account that $\mathcal{F}_F \notin \mathfrak{F}' \cup \{\mathcal{F}_\infty\}$ and $\Pi_F \subseteq \{2\}$, we can apply Lemmas 9, 8 and conclude that $A_F = \{2, p, q\}$. We therefore know that $A_F = A_E$, $\Pi_E \cup \{2\} = \Pi_F \cup \{2\}$ and $\alpha_F{\restriction}A_E \setminus \Pi_F = \alpha_E{\restriction}A_E \setminus \Pi_F$. By Lemma 5, $\mathcal{F}_E = \mathcal{F}_F$ and hence $\mathcal{F}_E \in \mathfrak{F}''$. □

Lemma 12. *For any homeomorphism* h *of the Kirch space and any odd prime number* p *we have*

$$\tilde{h}(\mathcal{F}_{\{p,2p\}}) = \mathcal{F}_{\{p,2p\}}.$$

Proof. By Proposition 1, the homeomorphism h induces an order isomorphism $\tilde{h}$ of the superconnecting poset $\mathfrak{F}$ on the Kirch space. Then $\tilde{h}[\mathfrak{F}'] = \mathfrak{F}'$ and $\tilde{h}[\mathfrak{F}''] = \mathfrak{F}''$.

By Lemmas 11 and 3, $\mathfrak{F}'' = \mathfrak{F}''_2 \cup \mathfrak{F}''_3$ where

$$\mathfrak{F}''_2 = \left\{\mathcal{F}_{\{p,2p\}} : p \in \Pi \setminus \{2\}\right\} \qquad \text{and}$$
$$\mathfrak{F}''_3 = \left\{\mathcal{F}_{\{x,pq,2pq\}} : p, q \in \Pi \setminus \{3\},\ x \in \{0, \dots, pq-1\} \setminus (p\mathbb{Z} \cup q\mathbb{Z})\right\}.$$

By Lemmas 5 and 9, for every filter $\mathcal{F}_{\{p,2p\}} \in \mathfrak{F}''_2$ the set

$$\uparrow\!\mathcal{F}_{\{p,2p\}} = \left\{\mathcal{F} \in \mathfrak{F}' : \mathcal{F}_{\{p,2p\}} \subset \mathcal{F}_E\right\}$$

coincides with the set $\left\{\mathcal{F}_{\{a,p,2p\}} : a \in \{1, \dots, p-1\}\right\}$ and hence has cardinality $p-1$.

On the other hand, for any filter $\mathcal{F}_{\{x,pq,2pq\}} \in \mathfrak{F}''_3$, the set

$$\uparrow\!\mathcal{F}_{\{x,pq,2pq\}} = \left\{\mathcal{F} \in \mathfrak{F}' : \mathcal{F}_{\{x,pq,2pq\}} \subset \mathcal{F}\right\}$$

coincides with the doubleton $\left\{\mathcal{F}_{\{x,p,2p\}}, \mathcal{F}_{\{x,q,2q\}}\right\}$.

These order properties uniquely determine the filters $\mathcal{F}_{\{p,2p\}}$ for $p \in \Pi \setminus \{3\}$ and ensure that $\tilde{h}(\mathcal{F}_{\{p,2p\}}) = \mathcal{F}_{\{p,2p\}}$ for every $p \in \Pi \setminus \{3\}$.

Next, observe that $\mathcal{F}_{\{3,6\}}$ is a unique element $\mathcal{F}$ of $\mathfrak{F}''$ such that

$$\uparrow\mathcal{F} \cap \bigcup_{p \in \Pi\setminus\{3\}} \uparrow\mathcal{F}_{\{p,2p\}} = \varnothing.$$

This uniqueness order property of $\mathcal{F}_{\{3,6\}}$ ensures that $\tilde{h}(\mathcal{F}_{\{3,6\}}) = \mathcal{F}_{\{3,6\}}$. □

Lemma 13. *Let $E \subseteq \mathbb{Z}^\bullet$ be a finite subset such that $A_E = \{2, p\}$ for some odd prime number $p \notin \Pi_E$. Then $A_{h[E]} = \{2, p\}$.*

Proof. By Lemma 9, $\mathcal{F}_E \in \mathfrak{F}'$. Consider the doubleton $\{p, 2p\}$ which has $A_{\{p,2p\}} = \{2, p\}$ and $\Pi_{\{p,2p\}} = \{p\}$. By Lemma 5, $\mathcal{F}_{\{p,2p\}} \subseteq \mathcal{F}_E$ and by Lemma 12,

$$\mathcal{F}_{\{p,2p\}} = \tilde{h}(\mathcal{F}_{\{p,2p\}}) = \mathcal{F}_{\{h(p),h(2p)\}} \subseteq \mathcal{F}_{h[E]}.$$

By Lemma 5, $A_{h[E]} \subseteq A_{\{p,2p\}} = \{2, p\}$. By Lemma 8, $A_{h[E]} = \{2, p\}$. □

Definition 1. A homeomorphism h of the Kirch space $(\mathbb{Z}^\bullet, \tau_K)$ is called *positive* if $h(1) > 0$.

Lemma 14. *Every positive homeomorphism h of the Kirch space has $h(x) = x$ for any $x \in \{\pm 2^n, n \in \omega\}$*

Proof. Consider the graph $\Gamma_2 = (V_2, \mathcal{E})$ with the set of vertices $V_2 = \{\pm 2^n : n \in \omega\}$ and the set of edges

$$\mathcal{E} = \big\{\{2^n, 2^{n+1}\}, \{-2^n, -2^{n+1}\}, \{-2^n, 2^n\} : n \in \omega\big\}.$$

Observe that 1 and -1 are the unique vertices of Γ_2 that have order 2. Any other vertices have order 3. This ensures that $h(1) = \pm 1$. The positivity of h implies that $h(1) = 1$. Then $h(-1) = -1$, $h(2) = 2$. Hence $h(\pm 2^n) = \pm 2^n$ for all $n \in \omega$. □

Lemmas 14 and 12 imply

Lemma 15. *For any positive homeomorphism h of the Kirch space and any odd prime number p we have*

$$\tilde{h}(\mathcal{F}_{\{1,p,2p\}}) = \mathcal{F}_{\{1,p,2p\}} \quad \textit{and} \quad \tilde{h}(\mathcal{F}_{\{2,p,2p\}}) = \mathcal{F}_{\{2,p,2p\}}.$$

Lemma 16. *For an integer number $x \in \mathbb{Z}^\bullet \setminus \{-2, -1, 1, 2\}$ and an odd prime p, the following conditions are equivalent:*

(1) $p \in \Pi_x$;
(2) $\mathcal{F}_{\{1,x\}} \subseteq \mathcal{F}_{\{1,p,2p\}}$ *and* $\mathcal{F}_{\{2,x\}} \subseteq \mathcal{F}_{\{2,p,2p\}}$.

Proof. If $p \in \Pi_x$, then $A_{\{1,p,2p\}} = \{2, p\} \subseteq A_{\{1,x\}}$, $\Pi_{\{1,x\}} \cup \{2\} = \{2\} = \Pi_{\{1,p,2p\}} \cup \{2\}$ and $\alpha_{\{1,x\}}(p) = 1 = \alpha_{\{1,p,2p\}}(p)$. By Lemma 5, $\mathcal{F}_{\{1,x\}} \subseteq \mathcal{F}_{\{1,p,2p\}}$. By analogy we can prove that $\mathcal{F}_{\{2,x\}} \subseteq \mathcal{F}_{\{2,p,2p\}}$.

Conversely, assume $\mathcal{F}_{\{1,x\}} \subseteq \mathcal{F}_{\{1,p,2p\}}$ and $\mathcal{F}_{\{2,x\}} \subseteq \mathcal{F}_{\{2,p,2p\}}$. By Lemmas 5 and 2, we have

$$\{2,p\} = A_{\{1,p,2p\}} \subseteq A_{\{1,x\}} = \Pi_x \cup \Pi_{x-1} \text{ and } \{2,p\} = A_{\{2,p,2p\}} \subseteq A_{\{2,x\}} = \{2\} \cup \Pi_x \cup \Pi_{x-2}$$

and hence $p \in (\Pi_x \cup \Pi_{x-1}) \cap (\Pi_x \cup \Pi_{x-2}) \setminus \{2\} \subseteq \Pi_x$. □

Proposition 1 and Lemmas 14, 15, 16 imply

Lemma 17. *For every homeomorphism h of the Kirch space and any number $x \in \mathbb{N}$ we have*

$$\Pi_x \cup \{2\} = \Pi_{h(x)} \cup \{2\}.$$

For every prime number p consider the set

$$V_p = \{\pm 2^{n-1} p^m : n, m \in \mathbb{N}\}$$

of numbers $x \in \mathbb{N}$ such that $p \in \Pi_x \subseteq \{2, p\}$. Lemmas 14 and 17 imply that $h[V_p] = V_p$ for every homeomorphism h of the Kirch space.

Consider the graph $\Gamma_p = (V_p, \mathcal{E}_p)$ on the set V_p with the set of edges

$$\mathcal{E}_p := \{E \in [V_p]^2 : A_E = \{2, p\}\}.$$

Lemma 18. *For every prime number p and every homeomorphism h of the Kirch space, the restriction of h to V_p is an isomorphism of the graph Γ_p.*

Proof. Let $E \in \mathcal{E}_p$. Since $p \in \Pi_E$, we can apply Lemma 11 and conclude that $\mathcal{F}_E \in \mathfrak{F}''$. Using fact that $\tilde{h}$ is isomorphism of $\mathfrak{F}$ we get $\mathcal{F}_{h[E]} = \tilde{h}(\mathcal{F}_E) \in \mathfrak{F}''$. Since $h[E] \subseteq h[V_p] = V_p$, we obtain $p \in \Pi_{h[E]}$. Using Lemma 11 once more, we obtain that $A_{h[E]} = \{2, p\}$, which means that $h[E] \in \mathcal{E}_p$. By analogical reasoning we can prove that $h^{-1}[E] \in \mathcal{E}_p$ for every $E \in \mathcal{E}_p$. This means that $h{\restriction}V_p$ is an isomorphism of the graph Γ_p. $\square$

The structure of the graph Γ_p depends on properties of the prime number p.
A prime number p is called

- *Fermat prime* if $p = 2^n + 1$ for some $n \in \mathbb{N}$;
- *Mersenne prime* if $p = 2^n - 1$ for some $n \in \mathbb{N}$;
- *Fermat–Mersenne* if p is either Fermat prime or Mersenne prime.

It is known (and easy to see) that for any Fermat prime number $p = 2^n + 1$ the exponent n is a power of 2, and for any Mersenne prime number $p = 2^n - 1$ the power n is a prime number. It is not known whether there are infinitely many Fermat–Mersenne prime numbers. All known Fermat prime numbers are the numbers $2^{2^n} + 1$ for $0 \le n \le 4$ (see `oeis.org/A019434` in [9]). At the moment only 51 Mersenne prime numbers are known, see the sequence `oeis.org/A000043` in [9].

Lemma 19. *Let p be an odd prime number, $p \neq 3$.*

(1) *If $p = 3$, then the set $\mathcal{E}_p$ of the edges of the graph Γ_p coincides with the set of doubletons* $\{\varepsilon 2^{a-1}3^b, \varepsilon 2^{a-1}3^{b+1}\}$, $\{\varepsilon 2^{a-1}3^b, 2\varepsilon^{a-1}3^{b+2}\}$, $\{\varepsilon 2^{a-1}3^b, \varepsilon 2^a 3^b\}$, $\{\varepsilon 2^{a-1}3^b, \varepsilon 2^{a+1}3^b\}$, $\{\varepsilon 2^{a-1}3^{b+1}, \varepsilon 2^{a+1}3^b\}$, $\{\varepsilon 2^{a+1}3^b, \varepsilon 2^a 3^{b+1}\}$, $\{\varepsilon 2^{a+3}3^b, \varepsilon 2^a 3^{b+2}\}$, $\{\varepsilon 2^{a-1}3^b, -\varepsilon 2^{a-1}3^{b+1}\}$, $\{\varepsilon 2^{a-1}3^b, -\varepsilon 2^a 3^b\}$, $\{\varepsilon 2^{a-1}3^b, -\varepsilon 2^{a+2}3^b\}$, $\{\varepsilon 2^{a-1}3^b, -\varepsilon 2^{a-1}3^b\}$ *for some* $a, b \in \mathbb{N}$, $\varepsilon \in \{-1, 1\}$.

(2) *If $p = 2^m + 1 > 3$ is Fermat prime, then*

$$\mathcal{E}_p = \big\{\{\varepsilon 2^{a-1}p^b, \varepsilon 2^{a-1}p^{b+1}\}, \{\varepsilon 2^{a-1}p^b, \varepsilon 2^a p^b\}, \{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a+m-1}p^b\},$$
$$\{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}, \{\varepsilon 2^{m+a-1}p^b, \varepsilon 2^{a-1}p^{b+1}\} : a, b \in \mathbb{N},\ \varepsilon \in \{-1, 1\}\big\}.$$

(3) *If* $p = 2^m - 1 > 3$ *is a Mersenne prime, then*

$$\begin{aligned}\mathcal{E}_p =\big\{&\{\varepsilon 2^a p^b, \varepsilon 2^{a-1}p^b\}, \{\varepsilon 2^{a-1}p^b, \varepsilon 2^{m+a-1}p^b\}, \{\varepsilon 2^{a-1}p^{b+1}, \varepsilon 2^{m+a-1}p^b\},\\ &\{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}, \{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^{b+1}\} : a, b \in \mathbb{N},\ \varepsilon \in \{-1, 1\}\big\}.\end{aligned}$$

(4) *If* p *is not Fermat–Mersenne, then*

$$\mathcal{E}_p = \big\{\{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}, \{\varepsilon 2^{a-1}p^b, \varepsilon 2^a p^b\} : a, b \in \mathbb{N},\ \varepsilon \in \{-1, 1\}\big\}.$$

Proof. Proof of Lemma 19 in each of cases (1)–(4) will be similar. The edges of graph Γ_p are 2-element subsets of set V_p such that $A_E = \{2, p\}$. Since the vertices of graph Γ_p are numbers of the form $\pm 2^{n-1}p^m$, where $n, m \in \mathbb{N}$, we can apply Lemma 2 and conclude that a doubleton $\{x, y\} \subset V_p$ belongs to $\mathcal{E}_p$ if and only if $\{2, p\} = \Pi_x \cup \Pi_y \cup \Pi_{x-y}$. In subsequent proofs, we will intensively use the Mihăilescu Theorem 4 saying that $2^3, 3^2$ is a unique pair of consecutive powers.

1. First we consider the case of $p = 3$. It is easy to see that the doubletons $\{x, y\}$ written in statement (1) have $\Pi_x \cup \Pi_y \cup \Pi_{x-y} \subseteq \{2, 3\}$, which implies that $\{x, y\} \in \mathcal{E}_3$. It remains to show that every doubleton $\{x, y\} \in \mathcal{E}_3$ is of the form indicated in statement (1). Write $\{x, y\}$ as $\{\varepsilon 2^{a-1}3^b, \delta 2^{c-1}3^d\}$ for some $a, b, c, d \in \mathbb{N}$, $\varepsilon, \delta \in \{-1, 1\}$ such that $2^{a-1}3^b \le 2^{c-1}3^d$.

If $a = c$ and $b = d$, then $\varepsilon \ne \delta$ and $\{x, y\} = \{\varepsilon 2^{a-1}3^b, -\varepsilon 2^{a-1}3^b\}$.

If $a = c$, then $b \le d$ and the inclusion $\Pi_{x-y} \subseteq \{2, 3\}$ implies that $\Pi_{3^{d-b}-\varepsilon/\delta} \subseteq \{2, 3\}$ and hence $3^{d-b} - \varepsilon/\delta$ is a power of 2. If $\varepsilon/\delta = 1$ then by the Mihăilescu Theorem 4, $d - b \in \{1, 2\}$, which means that $\{x, y\}$ is equal to $\{\varepsilon 2^{a-1}3^b, \varepsilon 2^{a-1}3^{b+1}\}$ or $\{\varepsilon 2^{a-1}3^b, \varepsilon 2^{a-1}3^{b+2}\}$. If $\varepsilon/\delta = -1$ then by the Mihăilescu Theorem 4, $d - b \in \{0, 1\}$, which means that $\{x, y\}$ is equal to $\{\varepsilon 2^{a-1}3^b, -\varepsilon 2^{a-1}3^b\}$ or $\{\varepsilon 2^{a-1}3^b, -\varepsilon 2^{a-1}3^{b+1}\}$.

If $b = d$, then $a \le c$ and the inclusion $\Pi_{x-y} \subseteq \{2, 3\}$ implies that $\Pi_{2^{c-a}-\varepsilon/\delta} \subseteq \{2, 3\}$ and hence $2^{c-a} - \varepsilon/\delta$ is either 2 or a power of 3. If $\varepsilon/\delta = 1$ then by the Mihăilescu Theorem 4, $c - a \in \{1, 2\}$, which means that $\{x, y\}$ is equal to $\{\varepsilon 2^{a-1}3^b, \varepsilon 2^a 3^b\}$ or $\{\varepsilon 2^{a-1}3^b, \varepsilon 2^{a+1}3^b\}$. If $\varepsilon/\delta = -1$ then by the Mihăilescu Theorem 4, $c - a \in \{0, 1, 3\}$, which means that $\{x, y\}$ is equal to $\{\varepsilon 2^{a-1}3^b, -\varepsilon 2^{a-1}3^b\}$, $\{\varepsilon 2^{a-1}3^b, -\varepsilon 2^a 3^b\}$ or $\{\varepsilon 2^{a-1}3^b, -\varepsilon 2^{a+2}3^b\}$.

So, we assume that $a \ne c$ and $b \ne d$. In this case we should consider four subcases.

If $a < c$ and $b < d$, then $\Pi_{x-y} \subseteq \{2, 3\}$ implies that each prime divisor of $2^{c-a}3^{d-b} - \varepsilon/\delta$ is equal to 2 or 3, which is not possible.

If $a < c$ and $b > d$, then $\Pi_{x-y} \subseteq \{2, 3\}$ and $2^{a-1}3^b \le 2^{c-1}3^d$ imply that $2^{c-a} - (\varepsilon/\delta)3^{b-d} = 1$ which implies that $\varepsilon = \delta$. Hence $c - a = 2$ and $b - d = 1$ by the Mihăilescu Theorem 4. In this case $\{x, y\} = \{\varepsilon 2^{a-1}3^{d+1}, \varepsilon 2^{a+1}3^d\}$.

If $a > c$ and $b < d$, then $\Pi_{x-y} \subseteq \{2, 3\}$ and $2^{a-1}3^b \le 2^{c-1}3^d$ imply that $3^{d-b} - (\varepsilon/\delta)2^{a-c} = 1$. This implies that $\varepsilon/\delta = 1$ and hence $\langle d - b, a - c\rangle \in \{\langle 1, 1\rangle, \langle 2, 3\rangle\}$ by the Mihăilescu Theorem 4. In this case $\{x, y\}$ is equal to $\{2^{c+1}3^b, 2^c 3^{b+1}\}$ or $\{2^{c+3}3^b, 2^c 3^{b+2}\}$.

The subcase $a > c$ and $b > d$ is forbidden by the inequality $2^{a-1}3^b \le 2^{c-1}3^d$.

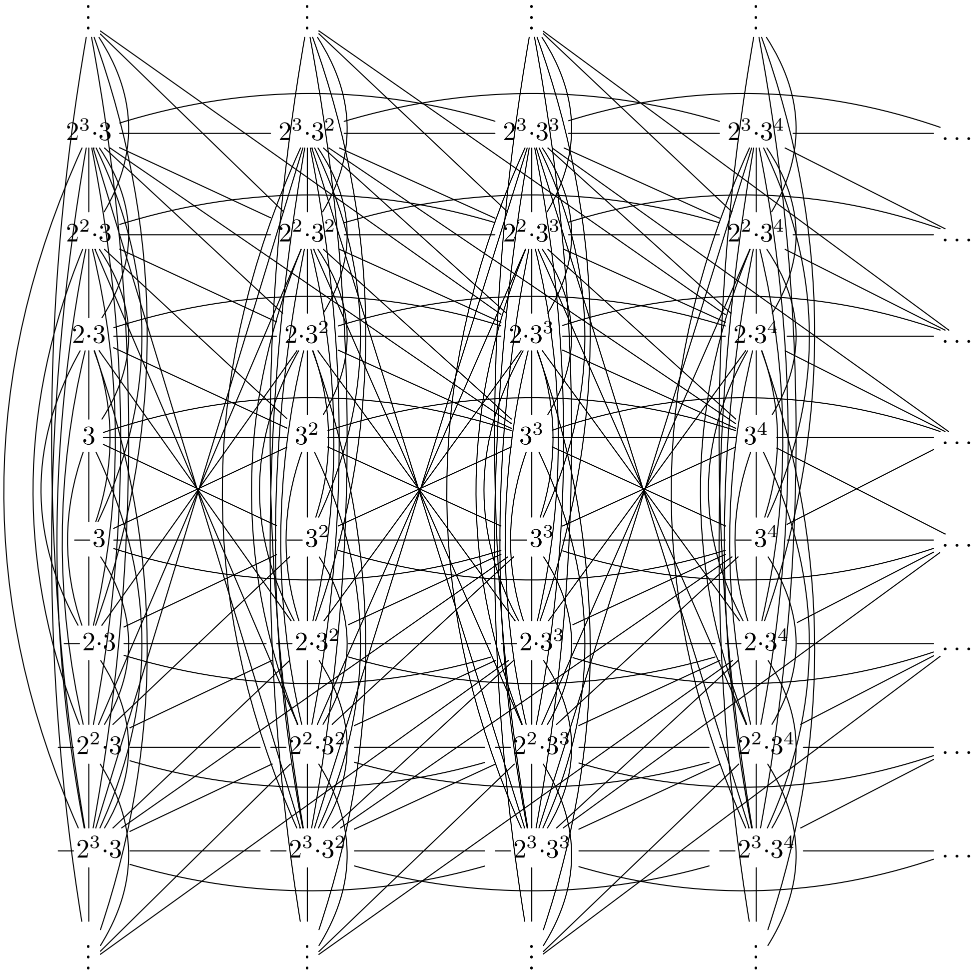

FIGURE 1. The graph Γ_3

2. Assume that $p = 2^m + 1 > 3$ is a Fermat prime. In this case $m > 1$. Since $p > 3$, p is not Mersenne prime. It is easy to check that every doubleton

$$\{x, y\} \in \big\{\{\varepsilon 2^{a-1}p^b, \varepsilon 2^{a-1}p^{b+1}\}, \{\varepsilon 2^{a-1}p^b, \varepsilon 2^a p^b\}, \{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a+m-1}p^b\},$$
$$\{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}, \{\varepsilon 2^{m+a-1}p^b, \varepsilon 2^{a-1}p^{b+1}\} : a, b \in \mathbb{N},\ \varepsilon \in \{-1, 1\}\big\}$$

has $A_{\{x,y\}} = \Pi_x \cup \Pi_y \cup \Pi_{x-y} = \{2, p\}$ and hence $\{x, y\} \in \mathcal{E}_p$.

Now assume that $\{x, y\} \in \mathcal{E}_p$ is an edge of the graph Γ_p. Then

$$\Pi_x \cup \Pi_y \cup \Pi_{x-y} = A_{\{x,y\}} = \{2, p\}$$

and $\{x,y\}$ can be written as $\{\varepsilon 2^{a-1}p^b, \delta 2^{c-1}p^d\}$ for some $a,b,c,d \in \mathbb{N}$, $\varepsilon,\delta \in \{-1,1\}$ with $2^{a-1}p^b \le 2^{c-1}p^d$.

If $a = c$, $b = d$ and $\varepsilon = -\delta$ then $\Pi_{\varepsilon 2^{a-1}p^b - \delta 2^{a-1}p^b} = \Pi_{\varepsilon 2^a p^b} \subset \{2,p\}$. In this case $\{x,y\} = \{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}$.

If $a = c$, then $b \le d$ and the inclusion $\Pi_{x-y} \subseteq \{2,p\}$ implies that $\Pi_{p^{d-b}-\varepsilon/\delta} \subseteq \{2,p\}$ and hence $p^{d-b} - \varepsilon/\delta$ is a power of 2. By the Mihăilescu Theorem 4, $d-b \in \{0,1\}$. If $d-b = 0$, then $\varepsilon = -\delta$ and $\{x,y\} = \{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}$ by the preceding case. So, we assume that $d-b = 1$. Since p is not Mersenne prime, we conclude that $\varepsilon = \delta$, and hence $\{x,y\}$ is equal to $\{\varepsilon 2^{a-1}p^b, \varepsilon 2^{a-1}p^{b+1}\}$.

If $b = d$, then $a \le c$ and the inclusion $\Pi_{x-y} \subseteq \{2,p\}$ implies that $\Pi_{2^{c-a}-\varepsilon/\delta} \subseteq \{2,p\}$ and hence $2^{c-a} - \varepsilon/\delta$ is a power of p. By the Mihăilescu Theorem 4, $2^{c-a} - \varepsilon/\delta \in \{1,p\} = \{1, 2^m+1\}$. If $\varepsilon = \delta$ then $c - a = 1$, which means that $\{x,y\}$ is equal to $\{\varepsilon 2^{a-1}p^b, \varepsilon 2^a p^b\}$. If $\varepsilon = -\delta$ then $c - a = m$ and $\{x,y\} = \{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a+m-1}p^b\}$.

So, we assume that $a \ne c$ and $b \ne d$. In this case we should consider four subcases.

If $a < c$ and $b < d$, then $\Pi_{x-y} \subseteq \{2,p\}$ implies that each prime divisor of $2^{c-a}p^{d-b} - \varepsilon/\delta$ is equal to 2 or p, which is not possible.

If $a < c$ and $b > d$, then $\Pi_{x-y} \subseteq \{2,p\}$ implies that $2^{c-a} - (\varepsilon/\delta)p^{b-d} = 1$ and hence $\varepsilon = \delta$. In this case the Mihăilescu Theorem 4 ensures that $b - d = 1$ and hence $2^{c-a} = p + 1 = 2^m + 2$ which is not possible (as $m > 1$).

If $a > c$ and $b < d$, then $\Pi_{x-y} \subseteq \{2,p\}$ implies that $p^{d-b} - (\varepsilon/\delta)2^{a-c} = 1$, which implies that $\varepsilon = \delta$. The Mihăilescu Theorem 4 implies that $d - b = 1$ and hence $2^{a-c} = p - 1 = 2^m$ and $a - c = m$. In this case $\{x,y\} = \{\varepsilon 2^{c+m-1}2^b, \varepsilon 2^{c-1}p^{b+1}\}$.

The subcase $a > c$ and $b > d$ is forbidden by the inequality $2^{a-1}p^b \le 2^{c-1}p^d$.

3. Assume that $p = 2^m - 1 > 3$ is Mersenne prime. In this case $m > 2$ and p is not Fermat. It is easy to check that every doubleton

$$\begin{aligned}\{x,y\} \in \big\{&\{\varepsilon 2^a p^b, \varepsilon 2^{a-1}p^b\}, \{\varepsilon 2^{a-1}p^b, \varepsilon 2^{m+a-1}p^b\}, \{\varepsilon 2^{a-1}p^{b+1}, \varepsilon 2^{m+a-1}p^b\},\\ &\{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}, \{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^{b+1}\} : a,b \in \mathbb{N},\ \varepsilon \in \{-1,1\}\big\}\end{aligned}$$

has $A_{\{x,y\}} = \Pi_x \cup \Pi_y \cup \Pi_{x-y} = \{2,p\}$ and hence $\{x,y\} \in \mathcal{E}_p$.

Now assume that $\{x,y\} \in \mathcal{E}_p$ is an edge of the graph Γ_p. Then $\Pi_x \cup \Pi_y \cup \Pi_{x-y} = A_{\{x,y\}} = \{2,p\}$ and $\{x,y\}$ can be written as $\{\varepsilon 2^{a-1}p^b, \delta 2^{c-1}p^d\}$ for some $a,b,c,d \in \mathbb{N}$, $\varepsilon,\delta \in \{-1,1\}$ with $2^{a-1}p^b \le 2^{c-1}p^d$.

If $a = c$, $b = d$, then $\varepsilon = -\delta$ and $\{x,y\} = \{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}$.

If $a = c$, then $b \le d$ and the inclusion $\Pi_{x-y} \subseteq \{2,p\}$ implies that $\Pi_{p^{d-b}-\varepsilon/\delta} \subseteq \{2,p\}$ and hence $p^{d-b} - \varepsilon/\delta$ is a power of 2. By the Mihăilescu Theorem 4, $d - b \in \{0,1\}$. If $d - b = 0$, then $\{x,y\} = \{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}$ by the preceding case. So, we assume that $d - b = 1$. If $\varepsilon = \delta$, then $p^{d-b} - \varepsilon/\delta = p - 1 = 2^m - 2$ is a power of 2, which is not true as $m > 2$. Therefore $\varepsilon = -\delta$ and $\{x,y\}$ is equal to $\{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^{b+1}\}$

If $b = d$, then $a \le c$ and the inclusion $\Pi_{x-y} \subseteq \{2,p\}$ implies that $\Pi_{2^{c-a}-\varepsilon/\delta} \subseteq \{2,p\}$ and hence $2^{c-a} - \varepsilon/\delta$ is a power of p. By the Mihăilescu Theorem 4, $2^{c-a} - \varepsilon/\delta \in \{1,p\} = \{1, 2^m - 1\}$, which implies that $\varepsilon = \delta$ and $c - a \in \{1, m\}$. Therefore $\{x,y\}$ is equal to $\{\varepsilon 2^{a-1}p^b, \varepsilon 2^a p^b\}$ or $\{\varepsilon 2^{a-1}p^b, \varepsilon 2^{m+a-1}p^b\}$.

So, we assume that $a \ne c$ and $b \ne d$. By analogy with the case of Fermat primes, we can show that the subcases ($a < c$ and $b < d$) and ($a > c$ and $b > d$) are impossible.

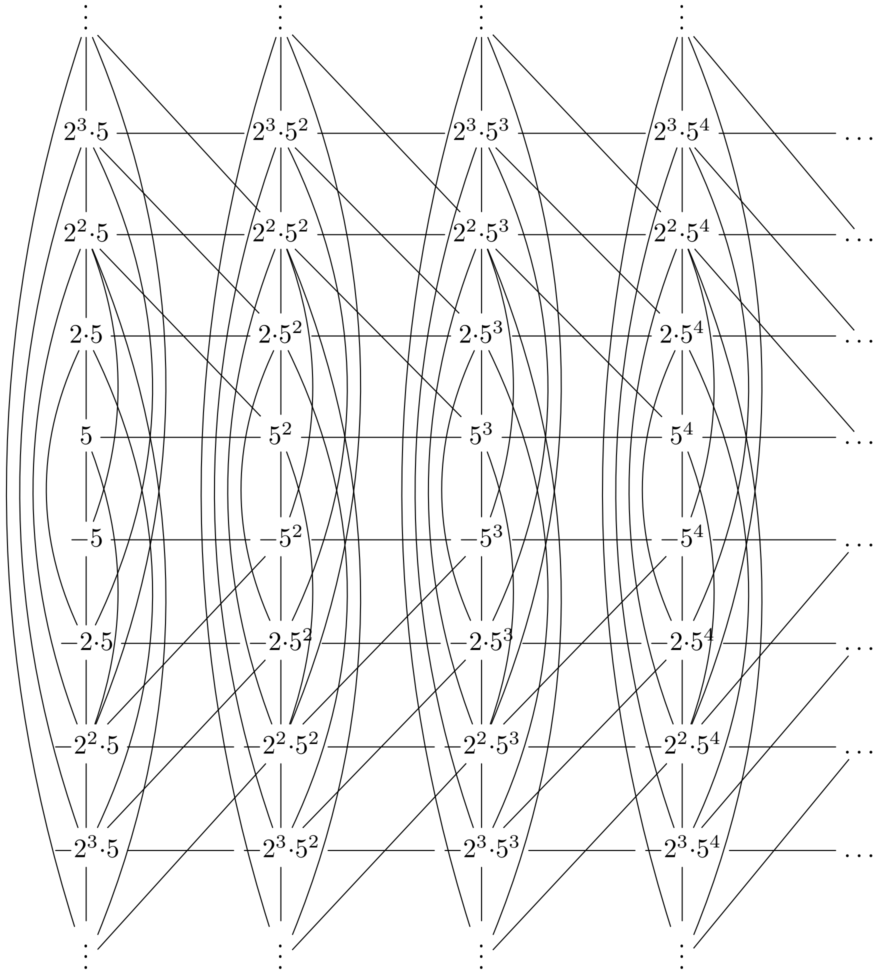

FIGURE 2. The graph Γ_5

If $a < c$ and $b > d$, then $\Pi_{x-y} \subseteq \{2, p\}$ implies that $2^{c-a} - (\varepsilon/\delta)p^{b-d} = 1$, and hence $\varepsilon/\delta = 1$. Then the Mihăilescu Theorem 4 ensures that $b - d = 1$ and hence $2^{c-a} = p + 1 = 2^m$ and $c - a = m$. In this case $\{x, y\} = \{\varepsilon 2^{a-1}p^{d+1}, \varepsilon 2^{a+m-1}p^d\}$.

If $a > c$ and $b < d$, then $\Pi_{x-y} \subseteq \{2, p\}$ implies that $p^{d-b} - (\varepsilon/\delta)2^{a-c} = 1$ and hence $\varepsilon/\delta = 1$. Then Mihăilescu Theorem 4 implies that $d - b = 1$ and hence $2^{a-c} = p - 1 = 2^m - 2$, which is not possible as $m > 2$.

4. Assume that p is not Fermat–Mersenne. It is easy to check that every doubleton

$$\{x, y\} \in \big\{\{\varepsilon 2^{a-1}p^b, -\varepsilon 2^{a-1}p^b\}, \{\varepsilon 2^{a-1}p^b, \varepsilon 2^a p^b\} : a, b \in \mathbb{N},\ \varepsilon \in \{-1, 1\}\big\}$$

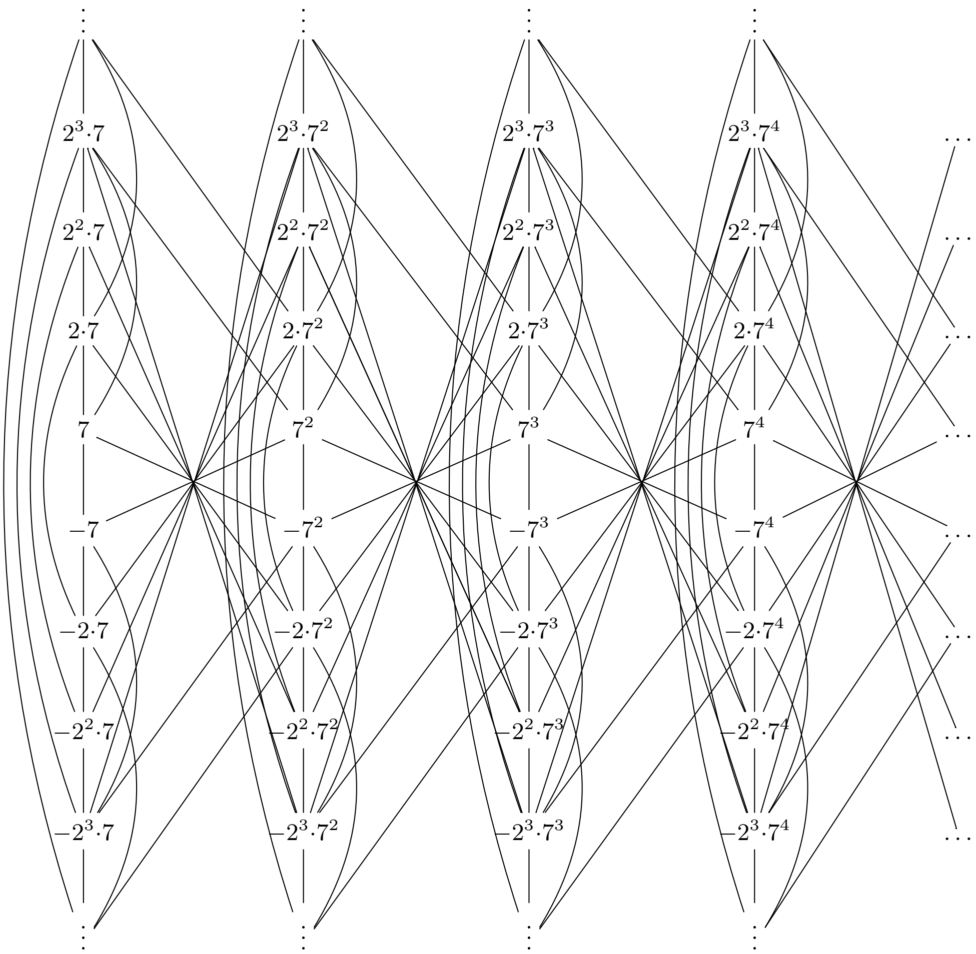

FIGURE 3. The graph Γ_7

has $A_{\{x,y\}} = \Pi_x \cup \Pi_y \cup \Pi_{x-y} = \{2, p\}$ and hence $\{x, y\} \in \mathcal{E}_p$.

Now assume that $\{x, y\} \in \mathcal{E}_p$ is an edge of the graph Γ_p. Then $\Pi_x \cup \Pi_y \cup \Pi_{x-y} = A_{\{x,y\}} = \{2, p\}$ and $\{x, y\}$ can be written as $\{\varepsilon 2^{a-1}p^b, \delta 2^{c-1}p^d\}$ for some $a, b, c, d \in \mathbb{N}$, $\varepsilon, \delta \in \{-1, 1\}$ with $2^{a-1}p^b \le 2^{c-1}p^d$.

If $a = c$ and $b = d$, then $\varepsilon \ne \delta$ and $\{x, y\} = \{2^{a-1}p^b, -2^{a-1}p^b\}$.

If $a = c$, then $b \le d$ and the inclusion $\Pi_{x-y} \subseteq \{2, p\}$ implies that $\Pi_{p^{d-b}-\varepsilon/\delta} \subseteq \{2, p\}$ and hence $p^{d-b} - \varepsilon/\delta$ is a power of 2. By the Mihăilescu Theorem 4, $d - b = 1$ and hence p is either Fermat prime or Mersenne prime which is not true.

If $b = d$, then $a \le c$ and the inclusion $\Pi_{x-y} \subseteq \{2, p\}$ implies that $\Pi_{2^{c-a}-\varepsilon/\delta} \subseteq \{2, p\}$ and hence $2^{c-a} - \varepsilon/\delta$ is a power of p. By the Mihăilescu Theorem 4, $2^{c-a} - \varepsilon/\delta \in \{1, p\}$.

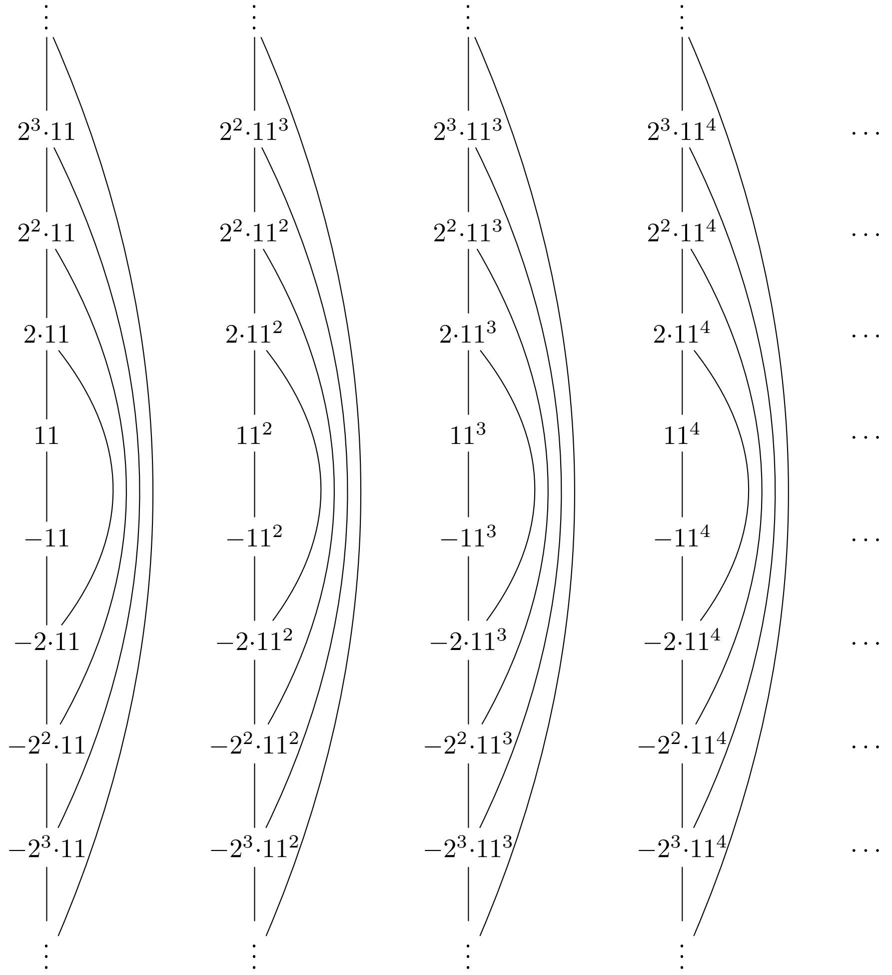

FIGURE 4. The graph Γ_{11}

Taking into account that p is neither Fermat nor Mersenne prime, we conclude that if $\varepsilon = \delta$, $2^{c-a} - 1 = 1$ and hence $c - a = 1$. Then $\{x, y\} = \{\varepsilon 2^{a-1} p^b, \varepsilon 2^a p^b\}$.

So, we assume that $a \neq c$ and $b \neq d$. By analogy with the case of Fermate primes, we can show that the subcases ($a < c$ and $b < d$) and ($a > c$ and $b > d$) are impossible.

If $a < c$ and $b > d$, then $\Pi_{x-y} \subseteq \{2, p\}$ implies that $2^{c-a} - (\varepsilon/\delta)p^{b-d} = 1$. By the Mihăilescu Theorem 4 $b - d = 1$ and hence $p = 2^{c-a} - 1$ is a Mersenne prime, which is not true.

If $a > c$ and $b < d$, then $\Pi_{x-y} \subseteq \{2, p\}$ implies that $p^{d-b} - (\varepsilon/\delta)2^{a-c} = 1$. By the Mihăilescu Theorem 4 $d - b = 1$ and hence $p = 1 + 2^{a-c}$ is a Fermat prime, which is not true. $\square$

In Figures 1, 2, 3, 4 we draw the graphs Γ_p for p equal to $3, 5, 7, 11$. Observe that 3 is both Fermat and Mersenne prime, 5 is Fermat prime, 7 is Mersenne prime and 11 is not Fermat–Mersenne.

Lemma 20. *Let p be an odd prime number and h be a positive homeomorphism of the Kirch space.*

(1) *If p is Fermat-Mersenne, then $h(p) = p$;*
(2) *If p is not Fermat-Mersenne, then $h(p) = p^n$ for some $n \in \mathbb{N}$.*

Proof. 1. Lemma 19(1) implies that the degree of ± 3 in the graph Γ_3 is equal to 8 but the other vertices have degree at least 9. Hence $h(3) = \pm 3$. Assume that $h(3) = -3$. Then by Lemma 14 and by Lemma 13

$$\{2,3\} = A_{\{2,3\}} = A_{h(\{2,3\})} = A_{\{2,-3\}} = \{2,3,5\}$$

but this is not true and hence $h(3) = 3$.

Assume that $p > 3$ is either Fermat or Mersenne prime. Lemma 19(2,3) implies that the degree of $\pm p$ in the graph Γ_p is 4 but the other vertices have degree at least 5. Hence $h(p) = \pm p$. Assume that $h(p) = -p$. By Lemma 14, $A_{\{1,p\}} = A_{\{1,h(p)\}} = A_{\{1,-p\}}$, so $\{p\} \cup \Pi_{p-1} = \{p\} \cup \Pi_{p+1}$, according to Lemma 2. This implies that $\Pi_{p-1} = \Pi_{p+1} = \{2\}$. Hence p is both Fermate and Mersenne which is possible iff $p = 3$ and this contradicts our assumption. Therefore $h(p) = p$.

2. Let p be an odd prime number, which is not Fermat–Mersenne. Lemma 19(4) implies that the set $\pm p^{\mathbb{N}} = \{\varepsilon p^n : n \in \mathbb{N}, \varepsilon \in \{-1,1\}\}$ coincides with the set of vertices of order 2 in the graph Γ_p. Taking into account that $h{\restriction}V_p$ is an isomorphism of the graph Γ_p, we conclude that $h(p) = \pm p^n$ for some $n \in \mathbb{N}$. Assume that $h(p) = -p^n$. Then $h(\{-1,p\}) = \{-1,-p^n\}$. By Lemma 13, $A_{\{-1,p\}} = A_{\{-1,-p^n\}}$, so $\{p\} \cup \Pi_{p+1} = \{p\} \cup \Pi_{p^n-1}$, according to Lemma 2. Since $\{p\}$ does not intersect Π_{p+1} and Π_{p^n-1} we conclude that $\Pi_{p+1} = \Pi_{p^n-1}$. Hence we get the inclusion $\Pi_{p-1} \subseteq \Pi_{p^n-1} = \Pi_{p+1}$. If some prime number d divides $p-1$ then the inclusion $\Pi_{p-1} \subseteq \Pi_{p+1}$ implies that d divides $p+1$, consequently d divides the difference $(p+1)-(p-1) = 2$ and hence $d = 2$. As a consequence, $\Pi_{p-1} = \{2\}$ and $p - 1 = 2^m$ for some $m \in \mathbb{N}$ which contradicts the assumption that p is not Fermat prime. Hence $h(p) = p^n$. □

Lemma 21. *For any positive homeomorphism h of the Kirch space and any prime number p we have $h(p) = p$.*

Proof. If $p = 2$, then $h(p) = p$ by Lemma 14. If p is Fermat–Mersenne, then $h(p) = p$ by Lemma 20. So, we assume p is not Fermat–Mersenne. By Lemma 20, $h(p) = p^n$ for some $n \in \mathbb{N}$. By Lemmas 2, 14 and 13,

$$\{p\} \cup \Pi_{p-1} = A_{\{1,p\}} = A_{\{1,h(p)\}} = A_{\{1,p^n\}} = \{p\} \cup \Pi_{p^n-1}$$

and hence $\Pi_{p^n-1} = \Pi_{p-1}$. Since p is not Mersenne prime, Zsigmondy Theorem 5 guarantees that $n = 1$ and hence $h(p) = p^1 = p$. □

Lemma 22. *The positive homeomorphism group of the Kirch space is trivial.*

Proof. To derive a contradiction, assume that the Kirch space admits a homeomorphism h such that $h(x) \neq x$ for some number x. By the Hausdorff property of the Kirch space and the continuity of h, there exists a neighborhood O_x of x in the Kirch topology such that $h[O_x] \cap O_x = \varnothing$. By the definition of the Kirch topology, there exists a square-free number b such that $\Pi_b \cap \Pi_x = \varnothing$ and $x + b\mathbb{Z} \subseteq O_x$. By the Dirichlet Theorem 3, the arithmetic progression $x+b\mathbb{N} \subseteq x+b\mathbb{Z}$ contains some prime number p. Then $h[O_x]\cap O_x = \varnothing$ implies $h(p) \neq p$, which contradicts Lemma 21. □

Our final lemma completes the proof of Theorem 1.

Lemma 23. *Any homeomorphism h of the Kirch space $\mathbb{Z}^\bullet$ is equal to $i : \mathbb{Z}^\bullet \to \mathbb{Z}^\bullet$, $i : x \mapsto x$ or to $j : \mathbb{Z}^\bullet \to \mathbb{Z}^\bullet, j : x \mapsto -x$.*

Proof. If h is positive, then $h = i$ by previous Lemma. If h is not positive then $h(1) < 0$ and $j \circ h(1) > 0$. Then the homeomorphism $j \circ h$ is positive and equals i by the preceding case. This implies that

$$h = i \circ h = (j \circ j) \circ h = j \circ (j \circ h) = j \circ i = j.$$

□

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ГОМЕОМОРФІЗМИ ПРОСТОРУ НЕНУЛЬОВИХ ЦІЛИХ ЧИСЕЛ З ТОПОЛОГІЄЮ КІРХА

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Топологія Голомба (відповідно Кірха) на множині $\mathbb{Z}^\bullet$ ненульових цілих чисел породжується базою, що складається з арифметичних прогресій $a + b\mathbb{Z} = \{a + bn : n \in \mathbb{Z}\}$, де $a \in \mathbb{Z}^\bullet$ і b – взаємно просте з a число, (що не ділиться на квадрат жодного простого числа). У 2019 році Даріо Спіріто довів, що простір ненульових цілих чисел з топологією Голомба допускає лише два автогомеоморфізми. Ми доводимо аналогічний факт для простору ненульових цілих, наділеного топологією Кірха: він також має рівно два автогомеоморфізми.

Ключові слова: топологія Кірха, суперзв'язний простір, суперзв'язуюча частково впорядкована множина.