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BRENDLE'S PROOF OF THE CONSISTENCY OF $\mathfrak{b} < \mathfrak{a}$, WITHOUT RANKS, GAMES, AND COHEN REALS

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We present a simplified version of the proof of one of the main results of [3].

Key words: Mad family, filter, Mathias forcing.

1. Introduction

Our goal is to give a proof of the following result. We remind the reader of the definitions of notions involved in it at the beginning of the next section.

Theorem 1 (Brendle 98)**.** (GCH) *Let κ be an uncountable regular cardinal. Then there exists a ccc poset $\mathbb{P}$ which forces $\mathfrak{b} = \kappa < \mathfrak{a} = \kappa^+ = \mathfrak{c}$.*

We will follow the same strategy as in [3], the main technical ingredient thereof being simplified. More precisely, $\mathbb{P} = \mathbb{P}_{\kappa^+}$ comes from a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa^+ \rangle$ of ccc posets. The poset $\mathbb{Q}_0$ forces $\mathfrak{b} = \kappa = 2^\omega$ (e.g., one can take as $\mathbb{Q}_0$ the poset adding κ-many Hechler reals over V). Fix a dominating family $B = \{b_\xi : \xi < \kappa\} \subset \omega^{\uparrow\omega} \cap V^{\mathbb{Q}_0}$ such that $b_\xi \leqslant^* b_\eta$ for all $\xi < \eta$. If α has cofinality $< \kappa$, then $\dot{\mathbb{Q}}_\alpha$ is a name for a *partial* Hechler forcing producing a $\leqslant^*$-bound for certain $X_\alpha \subset \omega^{\uparrow\omega} \cap V^{\mathbb{P}_\alpha}$ of size $|X_\alpha| < \kappa$, supplied by a bookkeeping function fixed in advance. The purpose of these $\dot{\mathbb{Q}}_\alpha$'s is to make sure that $\mathfrak{b} = \mathfrak{c} = \kappa$ holds in $V^{\mathbb{P}_\gamma}$ for any γ of cofinality κ. Moreover, since the partial Hechler posets $\dot{\mathbb{Q}}_\alpha$ have size $< \kappa$, they preserve the unboundedness of B (it is well-known and easy to check that no poset of size $< \kappa$ can force B to be bounded), provided that $\mathbb{P}_\alpha$ did so, and the latter will be arranged with the help of Propositions 2 and 1 below. At stage γ of cofinality κ our bookkeeping function gives us a ($\mathbb{P}_\gamma$-name for an) almost disjoint family $\mathcal{A}_\gamma$. The poset $\dot{\mathbb{Q}}_\gamma$ forces $\mathcal{A}_\gamma$ to be non-maximal and preserves the unboundedness of B.

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In order to prove Theorem 1 it is enough to accomplish the natural scenario discussed above. Propositions 2 and 1 along with a standard bookkeeping allow us to do this.

Proposition 2 is analogous to [3, 3.1. Theorem]. However, unlike in the proof of the latter result, in our proof of Proposition 2 we use neither auxiliary Cohen reals, nor tricky arguments involving ranks, which hopefully makes our proof somewhat more straightforward.

The proof given in [3] has inspired yet another construction of a model of $\mathfrak{b} < \mathfrak{a}$, see [6]. Their proof is rather different from the one we present in this note: They use countably closed non-ccc iterands which "forces" them to use countable supports and hence gives $\mathfrak{c} = \omega_2$, as well as they use some variants of games on filters considered in [8, 9].

There have been more attempts to simplify or to modify Brendle's proof from [3], see, e.g., [4]. Also, O. Guzmán has informed us in private communication that he knows how to eliminate Cohen reals. Moreover, Guzmán and Kalajdzievski have recently proved in [7] the consistency of $\omega_1 = \mathfrak{u} < \mathfrak{a} = \omega_2$. This yields $\mathfrak{b} < \mathfrak{a}$ since $\mathfrak{b} \leqslant \mathfrak{u}$ in ZFC and their posets do not add Cohen reals as these destroy ground model basis of ultrafilters. Nonetheless we believe that our approach might be still of some interest.

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2. Proofs

As usually, $\omega = \{0, 1, 2, \ldots\}$ denotes the set of natural numbers and $\omega^{\uparrow\omega}$ stands for non-decreasing elements of ω^ω. A family $\mathcal{A} \subset [\omega]^\omega$ is called *almost disjoint* if $A_0 \cap A_1$ is finite for any distinct $A_0, A_1 \in \mathcal{A}$. An infinite almost disjoint family $\mathcal{A}$ is called a *mad family* if $\mathcal{A} \cup \{X\}$ fails to be almost disjoint for any $X \in [\omega]^\omega \setminus \mathcal{A}$. The minimal cardinality of a mad family is denoted by $\mathfrak{a}$.

For $x, y \in \omega^\omega$ notation $x \leqslant^* y$ means that the set $\{n \in \omega : x(n) > y(n)\}$ is finite. $\mathfrak{b}$ denotes the minimal cardinality of $B \subset \omega^\omega$ which is unbounded with respect to $\leqslant^*$. It is known that $\omega_1 \leqslant \mathfrak{b} \leqslant \mathfrak{a}$, see [2, 10] for the information about $\mathfrak{a}, \mathfrak{b}$, and other combinatorial cardinal characteristics of the reals.

In what follows D denotes an unbounded subset of $\omega^{\uparrow\omega}$ which is σ-directed, i.e., for every $D_0 \in [D]^\omega$ there exists $g \in D$ such that $d \leqslant^* g$ for all $d \in D_0$. For instance, the dominating set B of Hechler generic reals mentioned above is like this.

The following fact follows from [1, Lemma 6.5.7].

Proposition 1. *Let δ be a limit ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ be a finite support iteration of ccc posets such that $\Vdash_{\mathbb{P}_\alpha}$"$D$ is unbounded" for all $\alpha < \delta$. Then $\Vdash_{\mathbb{P}_\delta}$"$D$ is unbounded".*

A subset $\mathcal{F}$ of $[\omega]^\omega$ is called a *filter* if $\mathcal{F}$ contains all co-finite sets, is closed under finite intersections of its elements, and under taking supersets. Every filter $\mathcal{F}$ gives rise to a natural forcing notion $\mathbb{M}_\mathcal{F}$ introducing a generic subset $X \in [\omega]^\omega$ such that $X \subset^* F$ for all $F \in \mathcal{F}$ as follows: $\mathbb{M}_\mathcal{F}$ consists of pairs $\langle s, F \rangle$ such that $s \in [\omega]^{<\omega}$, $F \in \mathcal{F}$, and $\max s < \min F$. A condition $\langle s, F \rangle$ is stronger than $\langle t, G \rangle$ if $F \subset G$, s is an end-extension of t, and $s \setminus t \subset G$. $\mathbb{M}_\mathcal{F}$ is usually called *Mathias forcing associated with $\mathcal{F}$*.

Every almost disjoint family $\mathcal{A}$ generates a filter

$$\mathcal{F}(\mathcal{A}) = \left\{ F \subset \omega \colon \exists \mathcal{B} \in [\mathcal{A}]^{<\omega} \ \left(\omega \setminus \bigcup \mathcal{B} \subset^* F \right) \right\}.$$

It is clear that any forcing producing an infinite pseudointersection of $\mathcal{F}(\mathcal{A})$ (or any other bigger filter) ruins the maximality of $\mathcal{A}$.

The next proposition yields the poset used at stages of iteration with cofinality κ.

Proposition 2. *($\mathfrak{b} = \mathfrak{c} = \kappa$.) Let $\mathcal{A}$ be an almost disjoint family. Then there exists a filter $\mathcal{U} \supset \mathcal{F}(\mathcal{A})$ such that $\mathbb{M}_{\mathcal{U}}$ preserves D unbounded.*

We shall need several auxiliary results. First of all, we shall assume in the sequel that $\mathcal{F}(\mathcal{A})$ is not contained in any filter $\mathcal{U}$ which is a union of $< \kappa$ many compacts, as otherwise $\mathcal{U}$ is as required: Any union of $< \mathfrak{b}$ many compacts has all of its continuous images under maps into $\omega^{\uparrow\omega}$ bounded, and $\mathbb{M}_{\mathcal{U}}$ preserves all ground model unbounded sets for any filters like that, see [5, Theorem 1.4].

For $\mathcal{X} \subset [\omega]^\omega$ and $Z \subset \omega$ we denote by $\mathcal{X} \upharpoonright Z$ the family $\{X \cap Z : X \in \mathcal{X}\}$. Also, $\mathcal{X}^+$ standardly stands for $\{Y \subset \omega : \forall X \in \mathcal{X}\ (|X \cap Y| = \omega)\}$.

Lemma 1. *$\mathcal{A} \cap \mathcal{U}^+$ is infinite for every filter $\mathcal{U} \subset \mathcal{F}(\mathcal{A})^+$ which is a union of $< \kappa$ many compacts.*

Proof. Suppose on the contrary that $\mathcal{A}' = \mathcal{A} \cap \mathcal{U}^+$ is finite and set $F = \omega \setminus \cup\mathcal{A}' \in \mathcal{F}(\mathcal{A})$. Then $\mathcal{F}(\mathcal{A}) \upharpoonright F \subset \mathcal{U} \upharpoonright F$. Indeed, $\mathcal{F}(\mathcal{A}) \upharpoonright F$ is the filter on F generated by $\{(\omega \setminus A) \cap F : A \in \mathcal{A} \setminus \mathcal{A}'\}$ and $\omega \setminus A \in \mathcal{U}$ for every $A \in \mathcal{A} \setminus \mathcal{A}'$. Thus $\mathcal{F}(\mathcal{A})$ is contained in a filter on ω which is a union of $< \kappa$ many compacts (namely $\{X : \exists U \in \mathcal{U}(F \cap U \subset^* X)\}$), which contradicts our assumption on $\mathcal{A}$. □

In what follows the family of filters $\mathcal{U}$ on ω which are unions of $< \kappa$ many compacts will be denoted by $\mathcal{C}_\kappa$. Let us denote by $\mathcal{E}$ the family of all subsets E of $\textsc{fin} := [\omega]^{<\omega} \setminus \{\varnothing\}$ such that for every $n \in \omega$ there exists $e \in E$ with $\min e > n$. For any $A \subset \textsc{fin}$ we denote by $\mathcal{K}(A)$ the family $\{X \subset \omega : X \cap a \neq \varnothing$ for all $a \in A\}$. It is clear that $\mathcal{K}(A)$ is compact for all A as above, and $\mathcal{K}(E) \subset [\omega]^\omega$ if $E \in \mathcal{E}$. We shall call $E \in \mathcal{E}$ *centered* if so is $\mathcal{K}(E)$, where a family $\mathcal{X} \subset [\omega]^\omega$ is called centered if $\cap\mathcal{X}' \in [\omega]^\omega$ for all $\mathcal{X}' \in [\mathcal{X}]^{<\omega}$.

For a filter $\mathcal{F}$ on ω we denote by $\mathcal{F}^{<\omega}$ the filter on $\textsc{fin}$ generated by $\{\mathcal{P}(F) \cap \textsc{fin} : F \in \mathcal{F}\}$ as a base. Note that this notation is unusual since $\mathcal{F}^{<\omega}$ "should" denote the family of all finite sequences of elements of $\mathcal{F}$, which is not the object we have defined in the previous sentence. However, we shall use this notation since it is standard in the current literature.

Observation 1. *Let $E \in \mathcal{E}$. Then $X \in \mathcal{K}(E)^+$ iff for every $n \in \omega$ there exists $e \in E$, $\min e \geqslant n$, such that $e \subset X$.*

In particular, for a filter $\mathcal{F}$ on ω, $\{\uparrow e : e \in E\}$ covers $\mathcal{F}$ iff $\mathcal{F} \subset \mathcal{K}(E)^+$ iff $\mathcal{K}(E) \subset \mathcal{F}^+$ iff $E \in (\mathcal{F}^{<\omega})^+$. (Here $\uparrow X = \{Y \subset \omega : X \subset Y\}$ for any $X \subset \omega$.)

Proof. The "if" part is obvious. For the "only if" one, assume to the contrary that $X \not\supset e$ for any $e \in E$ with $\min e \geqslant n$. For every $e \in E$ select $n_e \in e$ as follows: if $e \cap n \neq \varnothing$, pick $n_e \in e \cap n$, and otherwise pick $n_e \in e \setminus X$. Then $Y = \{n_e : e \in E\} \in \mathcal{K}(E)$ and $Y \cap X \subset n$ thus contradicting our assumption that $X \in \mathcal{K}(E)^+$. □

Lemma 2. *Let $\mathcal{R} \in \mathcal{C}_\kappa$ be such that $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}$ is centered. Suppose that $\langle E_n : n \in \omega\rangle \in \mathcal{E}^\omega$ is a decreasing sequence such that $E_n \subset \mathcal{P}(\omega \setminus n)$ and $\mathcal{K}(E_n) \subset \langle\mathcal{F}(\mathcal{A}) \cup \mathcal{R}\rangle^+$ for all n. Then one of the following two options holds:*

(*i*) *There exists $n \in \omega$ and $X \in \mathcal{K}(E_n)$ such that $\{\omega \setminus X\} \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R}$ is centered. In particular, for any filter $\mathcal{U}$ containing the latter union, $E_n \notin (\mathcal{U}^{<\omega})^+$.*

(*ii*) *There exists $g \in D$ such that letting $H' = \bigcup_{n\in\omega} E_n \cap \mathcal{P}(g(n))$, we have that $\mathcal{F}(\mathcal{A}) \cup \mathcal{R} \cup \mathcal{K}(H')$ is centered. In particular, for any filter $\mathcal{U}$ containing the latter union, $H' \in (\mathcal{U}^{<\omega})^+$, i.e., for every $U \in \mathcal{U}$ there exists $e \in H'$ such that $e \subset U$.*

Proof. Suppose that (*i*) fails. It follows that

$$\bigcup_{n\in\omega} \mathcal{K}(E_n) \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R} \text{ is centered.} \tag{1}$$

Indeed, otherwise there exists $n \in \omega$ such that $\mathcal{K}(E_n) \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R}$ is not centered because the sequence $\langle E_n : n \in \omega\rangle \in \mathcal{E}^\omega$ is decreasing. Since $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}$ is centered by our assumption, there are $F \in \langle \mathcal{F}(\mathcal{A}) \cup \mathcal{R}\rangle$ and $\{X_0, \dots, X_m\} \in \mathcal{K}(E_n)$ such that $\bigcap_{i\leqslant m} X_i \cap F = \varnothing$. Thus $F \subset \bigcup_{i\leqslant m}(\omega \setminus X_i)$, which implies $\omega \setminus X_i \in \langle \mathcal{F}(\mathcal{A}) \cup \mathcal{R}\rangle^+$ for some $i \leqslant m$, i.e., (*i*) takes place, which contradicts our assumption. Thus Equation (1) is true.

Applying now Lemma 1 and Observation 1 to $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{K}(E_n) \cup \mathcal{R}$, we can find mutually distinct $\{A_i : i \in \omega\} \subset \mathcal{A}$ such that for every $n, i \in \omega$, $X \in \mathcal{R}$, and $\vec{Y} = \langle Y_j : j < n\rangle \in \mathcal{K}(E_n)^n$ there exists $e \in E_n$ such that $e \subset X \cap \bigcap_{j<n} Y_j \cap A_i$. Since $\mathcal{K}(E_n)$ is compact, there exists $k \in \omega$ such that for every $\vec{Y} \in \mathcal{K}(E_n)^n$ and $i \leqslant n$ there exists $e \in E_n$ such that $e \subset X \cap \bigcap_{j<n} Y_j \cap A_i \cap k$. Let $k_{X,n}$ be the minimal k with this property.

Claim 1. *Let $X \in \mathcal{R}$ and $n \in \omega$. Then for every $\vec{Z} \in \mathcal{K}(E_n \cap \mathcal{P}(k_{X,n}))^n$ and $i \leqslant n$ there exists $e \in E_n$ such that $e \subset X \cap \bigcap_{j<n} Z_j \cap A_i \cap k_{X,n}$.*

Proof. Suppose that the claim is wrong and pick $i < n$ and $\vec{Z} \in \mathcal{K}(E_n \cap \mathcal{P}(k_{X,n}))^n$ witnessing its failure. For every $e \in E_n \setminus \mathcal{P}(k_{X,n})$ select $n_e \in e \setminus k_{X,n}$ and set $Y_j = Z_j \cup \{n_e : e \in E_n \setminus \mathcal{P}(k_{X,n})\}$. It follows that $Y_j \in \mathcal{K}(E_n)$ for all $j < n$ and there is no $e \in E_n$ such that $e \subset X \cap \bigcap_{j<n} Y_j \cap A_i \cap k_{X,n}$, a contradiction to our choice of $k_{X,n}$. □

Observe that the map $\mathcal{R} \ni X \mapsto \langle k_{X,n} : n \in \omega\rangle$ is continuous, and consequently there exists $f \in \omega^\omega$ such that for all X and all but finitely many $n \in \omega$ we have $k_{X,n} < f(n)$, because $\mathcal{R} \in \mathcal{C}_\kappa$ and $\kappa = \mathfrak{b}$.

Claim 2. *$\mathcal{F}(\mathcal{A}) \cup \mathcal{R} \cup \mathcal{K}(H_I)$ is centered for any $I \in [\omega]^\omega$, where $H_I = \bigcup_{n\in I} E_n \cap \mathcal{P}(f(n))$.*

Proof. Let us fix $\mathcal{A}' \in [\mathcal{A}]^{<\omega}$, $n \in \omega$, $I \in [\omega]^\omega$, $X \in \mathcal{R}$, and $\langle Y_j : j < n\rangle \in \mathcal{K}(H_I)^n$. It suffices to prove that $(\omega \setminus \cup\mathcal{A}') \cap X \cap \bigcap_{j<n} Y_j \setminus n \neq \varnothing$. Let us fix $i \in \omega$ such that $A_i \notin \mathcal{A}'$ and let $m \in I \setminus \max\{i, n\}$ be such that $A_i \cap (\cup\mathcal{A}') \subset m$, and $k_{X,m} < f(m)$. Note that all but finitely many $m \in I$ are as above. By Claim 1 there exists $e \in E_m$ such that

$$e \subset X \cap \bigcap_{j<n} Y_j \cap A_i \cap f(m),$$

and hence also $e \subset \omega \setminus \cup\mathcal{A}'$ because $\omega \setminus \cup\mathcal{A}' \supset A_i \setminus m$ by our choice of m and $\min e \geqslant m$ for all $e \in E_m$. □

Now let $g \in D$ be such that $[f < g] := \{n \in \omega : f(n) < g(n)\}$ is infinite. It suffices to note that H' defined in item (*ii*) of the formulation contains $H_{[f<g]}$ and hence $\mathcal{F}(\mathcal{A}) \cup \mathcal{R} \cup \mathcal{K}(H')$ is centered because so is $\mathcal{F}(\mathcal{A}) \cup \mathcal{R} \cup \mathcal{K}(H_{[f<g]})$ by Claim 2. □

We shall also need the following result proved in [6] (it is Proposition 1 there, stated in a slightly different terminology) which allows us to work in the proof of Proposition 2 directly with a filter instead of working with the Mathias forcing associated to it.

Теорема 1 (Guzmán–Hrušák–Martínez, 2014). *Let $\mathcal{F}$ be a filter and $D \subset \omega^{\uparrow\omega}$ be unbounded and σ-directed. Then $\mathbb{M}_{\mathcal{F}}$ preserves the unboundedness of D iff for every decreasing sequence $\langle E_n : n \in \omega\rangle$ of elements of $(\mathcal{F}^{<\omega})^+$ there exists $g \in D$ such that $\bigcup_{n\in\omega}(E_n \cap \mathcal{P}(g(n))) \in (\mathcal{F}^{<\omega})^+$. Moreover, in this characterization we may assume that $E_n \subset \mathcal{P}(\omega \setminus n)$ for all $n \in \omega$.*

We are in a position now to present the

Proof of Proposition 2. Let $\{\langle E_n^\alpha : n \in \omega\rangle : \alpha \in \kappa\}$ be an enumeration of all decreasing sequences $\langle E_n : n \in \omega\rangle \in \mathcal{E}^\omega$ such that $E_n \subset \mathcal{P}(\omega \setminus n)$ for all n. Set $\mathcal{R}^0 = \{\omega\}$ and assume that for some $\alpha \in \kappa$ we have already constructed $\mathcal{R}^\alpha \in \mathcal{C}_\kappa$ such that $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha$ is centered. Now consider the sequence $\langle E_n^\alpha : n \in \omega\rangle$. Three cases are possible.

1. There exists $n_\alpha \in \omega$ such that $\mathcal{K}(E^\alpha_{n_\alpha}) \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha$ is not centered. Given any ultrafilter $\mathcal{G}$ containing $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha$, we can find $X_\alpha \in \mathcal{K}(E^\alpha_{n_\alpha})$ such that $X_\alpha \notin \mathcal{G}$, and therefore $\{\omega \setminus X_\alpha\} \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha$ is centered being a subset of $\mathcal{G}$. Now we set $\mathcal{R}^{\alpha+1} = \langle \mathcal{R}^\alpha \cup \{\omega \setminus X_\alpha\}\rangle$.

2. For $\mathcal{R} := \mathcal{R}^\alpha$ and $\langle E_n : n \in \omega\rangle := \langle E_n^\alpha : n \in \omega\rangle$, item (i) from Lemma 2 takes place. This means that there exist $n_\alpha \in \omega$ and $X_\alpha \in \mathcal{K}(E^\alpha_{n_\alpha})$ such that $\{\omega\setminus X_\alpha\}\cup\mathcal{F}(\mathcal{A})\cup \mathcal{R}^\alpha$ is centered. As in item 1 we set $\mathcal{R}^{\alpha+1} = \langle \mathcal{R}^\alpha \cup \{\omega \setminus X_\alpha\}\rangle$.

3. For $\mathcal{R} := \mathcal{R}^\alpha$ and $\langle E_n : n \in \omega\rangle := \langle E_n^\alpha : n \in \omega\rangle$, item (ii) from Lemma 2 takes place. Then there exists $g_\alpha \in D$ such that letting $H_\alpha = \bigcup_{n\in\omega} E_n^\alpha \cap \mathcal{P}(g_\alpha(n))$, we have that $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha \cup \mathcal{K}(H_\alpha)$ is centered. In this case we set $\mathcal{R}^{\alpha+1} = \langle \mathcal{R}^\alpha \cup \mathcal{K}(H_\alpha)\rangle$.

This completes our inductive construction of the sequence $\langle \mathcal{R}^\alpha : \alpha < \kappa\rangle$. Set $\mathcal{R}^\kappa = \bigcup_{\alpha<\kappa}\mathcal{R}^\alpha$ and let $\mathcal{U}$ be the filter generated by $\mathcal{F}(\mathcal{A})\cup\mathcal{R}^\kappa$. We claim that $\mathcal{U}$ is as required. Indeed, consider any $\langle E_n^\alpha : n \in \omega\rangle$. If in the construction of $\mathcal{R}^{\alpha+1}$ one of the first two alternatives took place, we know that $E^\alpha_{n_\alpha} \notin (\mathcal{U}^{<\omega})^+$ as witnessed by $\omega\setminus X_\alpha \in \mathcal{U}$. So let us assume that the third alternative took place. Then $H_\alpha = \bigcup_{n\in\omega} E_n^\alpha \cap \mathcal{P}(g_\alpha(n)) \in (\mathcal{U}^{<\omega})^+$ by the definition of $\mathcal{U}$. It remains to use Theorem 1. □

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ДОВЕДЕННЯ БРЕНДЛА НЕСУПЕРЕЧНОСТІ $\mathfrak{b} < \mathfrak{a}$, ЯКЕ НЕ ВИКОРИСТОВУЄ РАНГІВ, ІГОР І ЧИСЕЛ КОЕНА

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Наведено спрощене доведення головного результату статті [3]. На відміну від оригінального доведення, ми не використовуємо рангів і допоміжних чисел Коена. Також не використовуються ігри на ідеалах, які фігурують у інших відомих автору спрощеннях доведення вищезгаданого результату Брендла.

Ключові слова: максимальна майже диз'юнктна сім'я, фільтр, форсінг Матіаса.