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THE MONOID OF MONOTONE INJECTIVE PARTIAL SELFMAPS OF THE POSET $(\mathbb{N}^3, \leqslant)$ WITH COFINITE DOMAINS AND IMAGES

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Let n be a positive integer $\geqslant 2$ and $\mathbb{N}^n_\leqslant$ be the n-th power of positive integers with the product order of the usual order on $\mathbb{N}$. In the paper we study the semigroup of injective partial monotone selfmaps of $\mathbb{N}^n_\leqslant$ with cofinite domains and images. We show that the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$ is isomorphic to the group $\mathscr{S}_n$ of permutations of an n-element set, and describe the subsemigroup of idempotents of $\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$. Also in the case $n = 3$ we describe the property of elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ as partial bijections of the poset $\mathbb{N}^3_\leqslant$ and Green's relations on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$. In particular we show that $\mathscr{D} = \mathscr{J}$ in $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$.

Key words: semigroup of partial bijections, monotone partial map, idempotent, Green's relations.

1. Introduction and preliminaries

We shall follow the terminology of [19] and [44].

In this paper we shall denote the cardinality of the set A by $|A|$. We shall identify all sets X with their cardinality $|X|$. For an arbitrary positive integer n by $\mathscr{S}_n$ we denote the group of permutations of an n-elements set. Also, for infinite subsets A and B of an infinite set X we shall write $A \subseteq^* B$ if and only if there exists a finite subset A_0 of A such that $A \setminus A_0 \subseteq B$.

An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$.

If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup then $E(S)$ is closed under multiplication and we shall refer to

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$E(S)$ as a *band* (or the *band of* S). If the band $E(S)$ is a non-empty subset of S then the semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order.

If S is a semigroup, then we shall denote Green's relations on S by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ (see [22] or [19, Section 2.1]):

$$\begin{aligned} a\mathscr{R}b &\text{ if and only if } aS^1 = bS^1;\\ a\mathscr{L}b &\text{ if and only if } S^1a = S^1b;\\ a\mathscr{J}b &\text{ if and only if } S^1aS^1 = S^1bS^1;\\ \mathscr{D} &= \mathscr{L}\circ\mathscr{R} = \mathscr{R}\circ\mathscr{L};\\ \mathscr{H} &= \mathscr{L}\cap\mathscr{R}. \end{aligned}$$

The $\mathscr{R}$-class (resp., $\mathscr{L}$-, $\mathscr{H}$-, $\mathscr{D}$- or $\mathscr{J}$-class) of the semigroup S which contains an element a of S will be denoted by R_a (resp., L_a, H_a, D_a or J_a).

If $\alpha\colon X \rightharpoonup Y$ is a partial map, then by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ we denote the domain and the range of α, respectively.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of an infinite set X of cardinality λ together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha\colon y\alpha \in \operatorname{dom}\beta\}$, for $\alpha, \beta \in \mathscr{I}_\lambda$. The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the set X (see [19, Section 1.9]). The symmetric inverse semigroup was introduced by Wagner [48] and it plays a major role in the semigroup theory. An element $\alpha \in \mathscr{I}_\lambda$ is called *cofinite*, if the sets $\lambda\setminus\operatorname{dom}\alpha$ and $\lambda\setminus\operatorname{ran}\alpha$ are finite.

If X is a non-empty set and $\leqslant$ is a reflexive, antisymmetric, transitive binary relation on X then $\leqslant$ is called a *partial order* on X and $(X, \leqslant)$ is said to be a *partially ordered set* or shortly a *poset*.

Let $(X, \leqslant)$ be a partially ordered set. A non-empty subset A of $(X, \leqslant)$ is called:

- a *chain* if the induced partial order from $(X, \leqslant)$ onto A is linear, i.e., any two elements from A are comparable in $(X, \leqslant)$;
- an ω-*chain* if A is order isomorphic to the set of negative integers with the usual order $\leq$;
- an *anti-chain* if any two distinct elements from A are incomparable in $(X, \leqslant)$.

For an arbitrary $x \in X$ and non-empty $A \subseteq X$ we denote

$$\uparrow x = \{y \in X\colon x \leqslant y\}, \quad \downarrow x = \{y \in X\colon y \leqslant x\}, \quad \uparrow A = \bigcup_{x\in A} \uparrow x \quad \text{and} \quad \downarrow A = \bigcup_{x\in A} \downarrow x.$$

We shall say that a partial map $\alpha\colon X \rightharpoonup X$ is *monotone* if $x \leqslant y$ implies $(x)\alpha \leqslant (y)\alpha$ for $x, y \in \operatorname{dom}\alpha$.

Let $\mathbb{N}$ be the set of positive integers with the usual linear order $\leq$ and $n \geqslant 2$ be an arbitrary positive integer. On the Cartesian power $\mathbb{N}^n = \underbrace{\mathbb{N}\times\cdots\times\mathbb{N}}_{n\text{-times}}$ we define the product partial order, i.e.,

$$(i_1, \ldots, i_n) \leqslant (j_1, \ldots, j_n) \qquad \text{if and only if} \qquad (i_k \leqslant j_k) \quad \text{for all} \quad k = 1, \ldots, n.$$

Later the set $\mathbb{N}^n$ with this partial order will be denoted by $\mathbb{N}^n_{\leqslant}$.

For an arbitrary positive integer $n \geqslant 2$ by $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ we denote the *semigroup of injective partial monotone selfmaps of* $\mathbb{N}^n_{\leqslant}$ *with cofinite domains and images*. Obviously, $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ is a submonoid of the semigroup $\mathscr{I}_\omega$ and $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ is a countable semigroup.

Furthermore, we shall denote the identity of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ by $\mathbb{I}$ and the group of units of $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ by $H(\mathbb{I})$.

The *bicyclic semigroup* (or the *bicyclic monoid*) $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q, subject only to the condition $pq = 1$. The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathscr{C}(p,q)$ under h is a cyclic group (see [19, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example a well-known Andersen's result [1] states that a (0–)simple semigroup with an idempotent is completely (0–)simple if and only if it does not contain an isomorphic copy of the bicyclic semigroup. Semigroup topologizations and shift-continuous topologizations of generalizations of the bicyclic monoid, they embedding into compact-like topological semigroups was studied in [5]–[9], [11, 14, 18, 20, 21], [24]–[28], [34, 35, 43, 46] and [2, 3, 4, 10, 12, 33, 42], respectively.

The bicyclic monoid is isomorphic to the semigroup of all bijections between uppersets of the poset $(\mathbb{N}, \leq)$ (see: see Exercise IV.1.11(ii) in [47]). So, the semigroup of injective isotone partial selfmaps with cofinite domains and images of positive integers is a generalization of the bicyclic semigroup. Hence, it is a natural problem to describe semigroups of injective isotone partial selfmaps with cofinite domains and images of posets with ω-chain.

The semigroups $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ of injective isotone partial selfmaps with cofinite domains and images of positive integers and integers, respectively, are studied in [34] and [35]. It was proved that the semigroups $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ have similar properties to the bicyclic semigroup: they are bisimple and every non-trivial homomorphic image $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is a group, and moreover the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ has $\mathbb{Z}(+)$ as a maximal group image and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ has $\mathbb{Z}(+) \times \mathbb{Z}(+)$, respectively.

In the paper [36] algebraic properties of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ of cofinite partial bijections of an infinite cardinal λ are studied. It is shown that $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a bisimple inverse semigroup and that for every non-empty chain L in $E(\mathscr{I}_\lambda^{\mathrm{cf}})$ there exists an inverse subsemigroup S of $\mathscr{I}_\lambda^{\mathrm{cf}}$ such that S is isomorphic to the bicyclic semigroup and $L \subseteq E(S)$, Green's relations on $\mathscr{I}_\lambda^{\mathrm{cf}}$ are described and it is proved that every non-trivial congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a group congruence. Also, the structure of the quotient semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$, where σ is the least group congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$, is described.

In the paper [32] the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ of monotone injective partial selfmaps of the set of $L_n \times_{\mathrm{lex}} \mathbb{Z}$ having cofinite domain and image, where $L_n \times_{\mathrm{lex}} \mathbb{Z}$ is the lexicographic product of n-elements chain and the set of integers with the usual linear order is studied. Green's relations on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ are described and it is shown that the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is bisimple and its projective congruences are established. Also, in [32] it is proved that $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is finitely generated, every automorphism of $\mathscr{IO}_\infty(\mathbb{Z})$ is inner, and it is shown that in the case $n \geqslant 2$ the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ has non-inner automorphisms. In [32] we proved that for every positive integer n the quotient semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\operatorname{lex}})/\sigma$, where σ is a least group congruence on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\operatorname{lex}})$, is isomorphic to the direct power $(\mathbb{Z}(+))^{2n}$. The structure of the sublattice of congruences on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\operatorname{lex}})$ which are contained in the least group congruence is described in [29].

In the paper [30] algebraic properties of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$ are studied. The properties of elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$ as monotone partial bijection of $\mathbb{N}^2_{\leqslant}$ are described and showed that the group of units of $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$ is isomorphic to the cyclic group of order two. Also in [30] the subsemigroup of idempotents of $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$ and Green's relations on $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$ are described. In particular, it is proved that $\mathscr{D}=\mathscr{J}$ in $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$. In [31] the natural partial order $\preccurlyeq$ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$ is described and it is shown that it coincides with the natural partial order the induced from symmetric inverse monoid over the set $\mathbb{N}\times\mathbb{N}$ onto the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$. Also, it is proved that the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$ is isomorphic to the semidirect product $\mathscr{PO}^+_\infty(\mathbb{N}^2_{\leqslant})\rtimes\mathbb{Z}_2$ of the monoid $\mathscr{PO}^+_\infty(\mathbb{N}^2_{\leqslant})$ of orientation-preserving monotone injective partial selfmaps of $\mathbb{N}^2_{\leqslant}$ with cofinite domains and images by the cyclic group $\mathbb{Z}_2$ of order two. It is described the congruence σ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$, which is generated by the natural order $\preccurlyeq$ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})$: $\alpha\sigma\beta$ if and only if α and β are comparable in $\big(\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant}),\preccurlyeq\big)$. It is proved that the quotient semigroup $\mathscr{PO}^+_\infty(\mathbb{N}^2_{\leqslant})/\sigma$ is isomorphic to the free commutative monoid $\mathfrak{AM}_\omega$ over an infinite countable set and it is shown that the quotient semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_{\leqslant})/\sigma$ is isomorphic to the semidirect product of the free commutative monoid $\mathfrak{AM}_\omega$ by the group $\mathbb{Z}_2$.

In the paper [38] the semigroup $\mathbf{I}\mathbb{N}_\infty$ of all partial co-finite isometries of positive integers is studied. The semigroup $\mathbf{I}\mathbb{N}_\infty$ is some generalization of the bicyclic monoid and it is a submonoid of $\mathscr{I}^{\nearrow}_\infty(\mathbb{N})$. Green's relations on the semigroup $\mathbf{I}\mathbb{N}_\infty$ and its band are described there and it is proved that $\mathbf{I}\mathbb{N}_\infty$ is a simple E-unitary F-inverse semigroup. Also there is described the least group congruence $\mathfrak{C}_{\mathbf{mg}}$ on $\mathbf{I}\mathbb{N}_\infty$ and it is proved that the quotient semigroup $\mathbf{I}\mathbb{N}_\infty/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the additive group of integers. An example of a non-group congruence on the semigroup $\mathbf{I}\mathbb{N}_\infty$ is presented. Also, it is proved that a congruence on the semigroup $\mathbf{I}\mathbb{N}_\infty$ is a group congruence if and only if its restriction onto an isomorphic copy of the bicyclic semigroup in $\mathbf{I}\mathbb{N}_\infty$ is a group congruence.

In the paper [39] submonoids of the monoid $\mathscr{I}^{\nearrow\!\!\!\!\urcorner}_\infty(\mathbb{N})$ of almost monotone injective co-finite partial selfmaps of positive integers $\mathbb{N}$ is established. Let $\mathscr{C}_\mathbb{N}$ be the subsemigroup $\mathscr{I}^{\nearrow\!\!\!\!\urcorner}_\infty(\mathbb{N})$ which is generated by the partial shift $n\mapsto n+1$ and its inverse partial map. In [39] it was shown that every automorphism of a full inverse subsemigroup of $\mathscr{I}^{\nearrow}_\infty(\mathbb{N})$ which contains the semigroup $\mathscr{C}_\mathbb{N}$ is the identity map. Also there is constructed a submonoid $\mathbf{I}\mathbb{N}^{[1]}_\infty$ of $\mathscr{I}^{\nearrow\!\!\!\!\urcorner}_\infty(\mathbb{N})$ with the following property: if S is an inverse submonoid of $\mathscr{I}^{\nearrow\!\!\!\!\urcorner}_\infty(\mathbb{N})$ such that S contains $\mathbf{I}\mathbb{N}^{[1]}_\infty$ as a submonoid, then every non-identity congruence $\mathfrak{C}$ on S is a group congruence. Also, it is proved that if S is an inverse submonoid of $\mathscr{I}^{\nearrow\!\!\!\!\urcorner}_\infty(\mathbb{N})$ such that S contains $\mathscr{C}_\mathbb{N}$ as a submonoid then S is simple and the quotient semigroup $S/\mathfrak{C}_{\mathbf{mg}}$, where $\mathfrak{C}_{\mathbf{mg}}$ is minimum group congruence on S, is isomorphic to the additive group of integers.

We observe that the semigroups of all partial co-finite isometries of integers are studied in [15, 16, 37].

The monoid $\mathbf{I}\mathbb{N}_{\infty}^{n}$ of cofinite partial isometries of the n-th power of the set of positive integers $\mathbb{N}$ with the usual metric for a positive integer $n \geqslant 2$ is studied in [40]. The semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$ is a submonoid of $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ for any positive integer $n \geqslant 2$. In [40] it is proved that for any integer $n \geqslant 2$ the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$ is isomorphic to the semidirect product $\mathscr{S}_n \ltimes_{\mathfrak{h}} (\mathscr{P}_{\infty}(\mathbb{N}^n), \cup)$ of the free semilattice with the unit $(\mathscr{P}_{\infty}(\mathbb{N}^n), \cup)$ by the symmetric group $\mathscr{S}_n$.

Later in this paper we shall assume that n is an arbitrary positive integer $\geqslant 2$.

In this paper we study the semigroup of injective partial monotone selfmaps of the poset $\mathbb{N}^n_{\leqslant}$ with cofinite domains and images. We show that the group of units $H(\mathbb{I})$ of the monoid $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ is isomorphic to the group $\mathscr{S}_n$ and describe the subgroup of idempotents of $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$. Also in the case $n = 3$ we describe the property of elements of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ as partial bijections of the poset $\mathbb{N}^n_{\leqslant}$ and Green's relations on the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^3_{\leqslant})$. In particular we show that $\mathscr{D} = \mathscr{J}$ in $\mathscr{PO}_{\infty}(\mathbb{N}^3_{\leqslant})$.

2. Properties of elements of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ as monotone partial permutations

In this short section we describe properties of elements of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ as monotone partial transformations of the poset $\mathbb{N}^n_{\leqslant}$.

It is obvious that the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ consists of exactly all order isomorphisms of the poset $\mathbb{N}^n_{\leqslant}$ and hence Theorem 2.8 of [28] implies the following

Theorem 1. *For any positive integer n the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ is isomorphic to the group $\mathscr{S}_n$ of permutations of an n-elements set. Moreover, every element of $H(\mathbb{I})$ permutates coordinates of elements of $\mathbb{N}^n$, and only these permutations are elements of $H(\mathbb{I})$.*

Since every $\alpha \in \mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ is a cofinite monotone partial transformation of the poset $\mathbb{N}^n_{\leqslant}$ the following statement holds.

Lemma 1. *If $(1.\ldots,1) \in \operatorname{dom}\alpha$ for some $\alpha \in \mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ then $(1.\ldots,1)\alpha = (1.\ldots,1)$.*

For an arbitrary $i = 1,\ldots,n$ define

$$\mathscr{K}_i = \big\{(1,\ldots,\underbrace{m}_{i\text{th}},\ldots,1) \in \mathbb{N}^n \colon m \in \mathbb{N}\big\}$$

and by $\mathfrak{pr}_i \colon \mathbb{N}^n \to \mathbb{N}^n$ denote the projection onto the i-th coordinate, i.e., for every $(m_1,\ldots,m_i,\ldots,m_n) \in \mathbb{N}^n$ put

$$(m_1,\ldots,\underbrace{m_i}_{i\text{th}},\ldots,m_n)\mathfrak{pr}_i = (1,\ldots,\underbrace{m_i}_{i\text{th}},\ldots,1).$$

Lemma 2. *Let $\{\overline{x}_1,\ldots,\overline{x}_k\}$ be a set of points in $\mathbb{N}^n \setminus \{(1,\ldots,1)\}$, $k \in \mathbb{N}$. Then the set $\mathbb{N}^n \setminus ({\uparrow}\overline{x}_1 \cup \ldots \cup {\uparrow}\overline{x}_k)$ is finite if and only if $k \geqslant n$ and for every $\mathscr{K}_i$, $i = 1,\ldots,n$, there exists $\overline{x}_j \in \{\overline{x}_1,\ldots,\overline{x}_k\}$ such that $\overline{x}_j \in \mathscr{K}_i$.*

Proof. $(\Leftarrow)$ Without loss of generality we may assume that $\overline{x}_j \in \mathscr{K}_j$ for every positive integer $j \leqslant n$. Then simple verifications imply that the set $\mathbb{N}^n \setminus ({\uparrow}\overline{x}_1 \cup \ldots \cup {\uparrow}\overline{x}_n)$ is finite, and hence so is the set $\mathbb{N}^n \setminus ({\uparrow}\overline{x}_1 \cup \ldots \cup {\uparrow}\overline{x}_k)$.

($\Rightarrow$) Suppose to the contrary that there exist a subset $\{\overline{x}_1,\ldots,\overline{x}_k\}\subseteq\mathbb{N}^n\setminus\{(1,\ldots,1)\}$ and an integer $i\in\{1,\ldots,n\}$ such that $\mathbb{N}^n\setminus({\uparrow}\overline{x}_1\cup\ldots\cup{\uparrow}\overline{x}_k)$ is finite and $\overline{x}_j\notin\mathscr{K}_i$ for any $j\in\{1,\ldots,k\}$.

The definition of $\mathscr{K}_i$ $(i=1,\ldots,n)$ implies that $\mathscr{K}_i$ with the induced partial order from $\mathbb{N}^n_{\leqslant}$ is an ω-chain such that ${\downarrow}\mathscr{K}_i=\mathscr{K}_i$. Hence, for any $\overline{x}\in\mathbb{N}^n$ we have that either $\mathscr{K}_i\setminus{\uparrow}\overline{x}$ is finite or $\mathscr{K}_i\cap{\uparrow}\overline{x}=\varnothing$. Then by our assumption we get that the set $\mathbb{N}^n\setminus({\uparrow}\overline{x}_1\cup\ldots\cup{\uparrow}\overline{x}_n)$ is infinite, a contradiction. The inequality $k\geqslant n$ follows from the above arguments. □

Later for an arbitrary non-empty subset A of $\mathbb{N}^n$ by ε_A we shall denote the identity map of the set $\mathbb{N}^n\setminus A$. It is obvious that the following lemma holds.

Lemma 3. *For an arbitrary non-empty subset A of $\mathbb{N}^n$, ε_A is an element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$, and hence so are $\varepsilon_A\alpha$, $\alpha\varepsilon_A$, and $\varepsilon_A\alpha\varepsilon_A$ for any $\alpha\in\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$.*

Proposition 1. *For an arbitrary element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ there exists a unique permutation $\mathfrak{s}\colon\{1,\ldots,n\}\to\{1,\ldots,n\}$ such that $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\subseteq\mathscr{K}_{(i)\mathfrak{s}}$ for any $i=1,\ldots,n$.*

Proof. Lemma 3 implies that without loss of generality we may assume that $(1,\ldots,1)\notin\operatorname{dom}\alpha$ and $(1,\ldots,1)\notin\operatorname{ran}\alpha$.

Since for any $i=1,\ldots,n$ the set $\mathscr{K}_i$ with the induced order from the poset $\mathbb{N}^n_{\leqslant}$ is an ω-chain, the set $\mathscr{K}_i\cap\operatorname{dom}\alpha$ contains the least element $\bar{l}_i^\alpha$. By Lemma 2 the set $\mathbb{N}^n\setminus\big({\uparrow}\bar{l}_1^\alpha\cup\cdots\cup{\uparrow}\bar{l}_n^\alpha\big)$ is finite and hence so is $\operatorname{dom}\alpha\setminus\big({\uparrow}\bar{l}_1^\alpha\cup\cdots\cup{\uparrow}\bar{l}_n^\alpha\big)$. Since α is a cofinite partial bijection of $\mathbb{N}^n$, we have that

$$\big({\uparrow}\bar{l}_1^\alpha\cup\cdots\cup{\uparrow}\bar{l}_n^\alpha\big)\alpha=\big({\uparrow}\bar{l}_1^\alpha\big)\alpha\cup\cdots\cup\big({\uparrow}\bar{l}_n^\alpha\big)\alpha$$

and the set $\mathbb{N}^n\setminus\big(\big({\uparrow}\bar{l}_1^\alpha\big)\alpha\cup\cdots\cup\big({\uparrow}\bar{l}_n^\alpha\big)\alpha\big)$ is finite. Also, since α is a monotone partial bijection of the poset $\mathbb{N}^n_{\leqslant}$ we obtain that $\big({\uparrow}\bar{l}_i^\alpha\big)\alpha\subseteq{\uparrow}\big(\bar{l}_i^\alpha\big)\alpha$ for all $i=1,\ldots,n$. Then by Lemma 2 there exists a permutation $\mathfrak{s}\colon\{1,\ldots,n\}\to\{1,\ldots,n\}$ such that $(\bar{l}_i^\alpha)\alpha\in\mathscr{K}_{(i)\mathfrak{s}}$ for any $i=1,\ldots,n$, because

$$\mathbb{N}^n\setminus\big({\uparrow}\big(\bar{l}_1^\alpha\big)\alpha\cup\cdots\cup\big({\uparrow}\bar{l}_n^\alpha\big)\alpha\big)\subseteq\mathbb{N}^n\setminus\big(\big({\uparrow}\bar{l}_1^\alpha\big)\alpha\cup\cdots\cup\big({\uparrow}\bar{l}_n^\alpha\big)\alpha\big)$$

and the set $\mathbb{N}^n\setminus\big({\uparrow}\big(\bar{l}_1^\alpha\big)\alpha\cup\cdots\cup\big({\uparrow}\bar{l}_n^\alpha\big)\alpha\big)$ is finite. This implies that $(\overline{x})\alpha\in\mathscr{K}_{(i)\mathfrak{s}}$ for all $\overline{x}\in\mathscr{K}_i\cap\operatorname{dom}\alpha$ and any $i=1,\ldots,n$.

The proof of uniqueness of the permutation $\mathfrak{s}$ for $\alpha\in\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ is trivial. This completes the proof of the proposition. □

Theorem 1 and Proposition 1 imply the following corollary.

Corollary 1. *For every element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ there exists a unique element σ of the group of units $H(\mathbb{I})$ of $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ such that $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\sigma\subseteq\mathscr{K}_i$ and $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\sigma^{-1}\alpha\subseteq\mathscr{K}_i$ for all $i=1,\ldots,n$.*

Lemma 4. *There is no a finite family $\{L_1,\ldots,L_k\}$ of chains in the poset $\mathbb{N}^2_{\leqslant}$ such that $\mathbb{N}^2=L_1\cup\cdots\cup L_k$. Moreover, every co-finite subset in $\mathbb{N}^2_{\leqslant}$ has this property.*

Proof. Suppose to the contrary that there exists a positive integer k such that $\mathbb{N}^2 = L_1 \cup \cdots \cup L_k$ and L_i is a chain for each $i = 1, \ldots, k$. Then

$$\{(1, k+1), (2, k), \ldots, (k, 2), (k+1, 1)\}$$

is an anti-chain in the poset $\mathbb{N}^2_{\leqslant}$ which contains exactly $k+1$ elements. Without loss of generality we may assume that $L_i \cap L_j = \varnothing$ for $i \neq j$. Since $\mathbb{N}^2 = L_1 \sqcup \cdots \sqcup L_k$, by the pigeonhole principle (or by the Dirichlet drawer principle, see [13, Section 7.3]) there exists a chain L_i, $i = 1, \ldots, k$, which contains at least two distinct elements of the set $\{(1, k+1), (2, k), \ldots, (k, 2), (k+1, 1)\}$, a contradiction.

Assume that A is a co-finite subset of $\mathbb{N}^2_{\leqslant}$ such that $A = \mathbb{N}^2 \setminus \{x_1, \ldots, x_p\}$ for some positive integer p. For every $i = 1, \ldots, p$ we put $L_{k+i} = \{x_i\}$. Then for every finite partition $\{L_1, \ldots, L_k\}$ of A such that L_i is a chain for each $i = 1, \ldots, k$ the family $\{L_1, \ldots, L_k, L_{k+1} \ldots, L_{k+p}\}$ is a finite partition of the poset $\mathbb{N}^2_{\leqslant}$ such that L_i is a chain for each $i = 1, \ldots, k+p$. This contradicts the above part of the proof, and hence the second statement of the lemma holds. □

For any distinct $i, j \in \{1, \ldots, n\}$ we denote

$$\mathscr{K}_{i,j} = \{(x_1, \ldots, x_n) \in \mathbb{N}^n \colon x_k = 1 \text{ for all } k \in \{1, \ldots, n\} \setminus \{i, j\}\}$$

and

$$\mathscr{K}^{\circ}_{i,j} = \mathscr{K}_{i,j} \setminus (\mathscr{K}_i \cup \mathscr{K}_j)$$

Lemma 5. *Let n be a positive integer $\geqslant 3$. Let $\overline{x}_i$ be an arbitrary element of $\mathscr{K}_i \setminus \{1, \ldots, 1\}$ for $i = 3, \ldots, n$ and $\overline{y}_{1,2}$ be an arbitrary element of $\mathscr{K}^{\circ}_{1,2}$. Then there exists a finite family $\{L_1, \ldots, L_k\}$ of chains in the poset $\mathbb{N}^n_{\leqslant}$ such that*

$$L_1 \cup \cdots \cup L_k = \mathbb{N}^n \setminus \left(\uparrow\overline{y}_{1,2} \cup \uparrow\overline{x}_3 \cup \cdots \cup \uparrow\overline{x}_n\right).$$

Proof. Let $\overline{x}_i = (1, 1, \ldots, \underbrace{x_i}_{i\text{th}}, \ldots, 1)$ for $i = 3, \ldots, n$ and $\overline{y}_{1,2} = (y_1, y_2, 1 \ldots, 1)$. Then for any element $\overline{a} = (a_1, \ldots, a_n)$ of the set $\mathbb{N}^n \setminus \left(\uparrow\overline{y}_{1,2} \cup \uparrow\overline{x}_3 \cup \cdots \cup \uparrow\overline{x}_n\right)$ the following conditions hold:

(*i*) $a_i < x_i$ for any $i = 3, \ldots, n$;
(*ii*) if $a_1 \geqslant y_1$ then $a_2 < y_2$;
(*iii*) if $a_2 \geqslant y_2$ then $a_1 < y_1$.

These conditions imply that

$$\mathbb{N}^n \setminus \left(\uparrow\overline{y}_{1,2} \cup \uparrow\overline{x}_3 \cup \cdots \cup \uparrow\overline{x}_n\right) = \bigcup \{S(k_3, \ldots, k_n) \colon k_3 < x_3, \ldots, k_n < x_n\},$$

where

$$S(k_3, \ldots, k_n) = \bigcup \{L_i(k_3, \ldots, k_n) \colon i = 1, \ldots y_1 - 1\} \cup \\ \cup \bigcup \{R_j(k_3, \ldots, k_n) \colon j = 1, \ldots y_2 - 1\},$$

with

$$L_i(k_3, \ldots, k_n) = \{(i, p, k_3, \ldots, k_n) \in \mathbb{N}^n \colon p \in \mathbb{N}\}$$

and

$$R_j(k_3, \ldots, k_n) = \{(p, j, k_3, \ldots, k_n) \in \mathbb{N}^n \colon p \in \mathbb{N}\}.$$

We observe that for arbitrary positive integers i,j, $k_3,\ldots,k_n$ the sets $L_i(k_3,\ldots,k_n)$ and $R_j(k_3,\ldots,k_n)$ are chains in the poset $\mathbb{N}^n_{\leqslant}$. Since the set $\mathbb{N}^n\setminus\big(\uparrow\overline{y}_{1,2}\cup\uparrow\overline{x}_3\cup\cdots\cup\uparrow\overline{x}_n\big)$ is the union of finitely many sets of the form $S(k_3,\ldots,k_n)$ the above arguments imply the required statement of the lemma. □

Proposition 2. *Let α be an element of $\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$ such that $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\subseteq\mathscr{K}_i$ for all $i=1,\ldots,n$. Then $(\mathscr{K}_{i_1,i_2}\cap\operatorname{dom}\alpha)\alpha\subseteq\mathscr{K}_{i_1,i_2}$ for all distinct $i_1,i_2=1,\ldots,n$.*

Proof. Suppose to the contrary that there exists $\overline{x}\in\mathscr{K}_{i_1,i_2}\cap\operatorname{dom}\alpha$ such that $(\overline{x})\alpha\notin\mathscr{K}_{i_1,i_2}$. By Theorem 1 without loss of generality we may assume that $i_1=1$ and $i_2=2$, i.e., $\overline{x}\in\mathscr{K}_{1,2}$ and $(\overline{x})\alpha\notin\mathscr{K}_{1,2}$. By Lemma 1, $\overline{x}\neq(1,\ldots,1)$.

For every $i=3,\ldots,n$ we let $\overline{x}_i^\alpha=(1,1,\ldots,\underbrace{x_i^\alpha}_{i\text{th}},\ldots,1)\in\operatorname{dom}\alpha$ be the smallest element of $\mathscr{K}_i$ such that $(\overline{x}_i^\alpha)\alpha\neq(1,\ldots,1)$. There exists $\overline{x}_{1,2}^\alpha=(x_1^\alpha,x_2^\alpha,1,\ldots,1)\in\operatorname{dom}\alpha\cap\mathscr{K}_{1,2}^\circ$ such that $\overline{x}\leqslant\overline{x}_{1,2}^\alpha$. Since $\alpha\in\mathscr{PO}_\infty(\mathbb{N}^n_{\leqslant})$, $(\overline{x})\alpha\leqslant(\overline{x}_{1,2}^\alpha)\alpha\notin\mathscr{K}_{1,2}$.

Now, the monotonicity of α implies that $\big(\uparrow\overline{x}_{1,2}^\alpha\big)\alpha\subseteq\uparrow\big(\overline{x}_{1,2}^\alpha\big)\alpha$ and $(\uparrow\overline{x}_i^\alpha)\alpha\subseteq\uparrow(\overline{x}_i^\alpha)\alpha$ for any $i=3,\ldots,n$. By our assumption we have that

$$\mathscr{K}_{1,2}\cap\operatorname{ran}\alpha\subseteq\big(\mathbb{N}^n_{\leqslant}\setminus\big(\uparrow\overline{x}_{1,2}^\alpha\cup\uparrow\overline{x}_3^\alpha\cup\cdots\cup\uparrow\overline{x}_n^\alpha\big)\big)\alpha.$$

Since the partial transformation α preserves chains in the poset $\mathbb{N}^n_{\leqslant}$, Lemma 5 implies that the set $\mathscr{K}_{1,2}\cap\operatorname{ran}\alpha$ is a union of finitely many chains, which contradicts Lemma 4. The obtained contradiction implies the assertion of the proposition. □

Theorem 2. *Let α be an element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_{\leqslant})$ such that $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\subseteq\mathscr{K}_i$ for all $i=1,2,3$. Then the following assertions hold:*

- (i) *if $(x_1,x_2,x_3)\in\operatorname{dom}\alpha$ and $(x_1,x_2,x_3)\alpha=(x_1^\alpha,x_2^\alpha,x_3^\alpha)$ then $x_1^\alpha\leq x_1$, $x_2^\alpha\leq x_2$ and $x_3^\alpha\leq x_3$ and hence $(\overline{x})\alpha\leqslant\overline{x}$ for any $\overline{x}\in\operatorname{dom}\alpha$;*
- (ii) *there exists a smallest positive integer n_α such that $(x_1,x_2,x_3)\alpha=(x_1,x_2,x_3)$ for all $(x_1,x_2,x_3)\in\operatorname{dom}\alpha\cap\uparrow(n_\alpha,n_\alpha,n_\alpha)$.*

Proof. (i) We shall prove the inequality $x_1^\alpha\leq x_1$ by induction. The proofs of the inequalities $x_2^\alpha\leq x_2$ and $x_3^\alpha\leq x_3$ are similar.

By Proposition 2 we have that if $x_1=1$ then $x_1^\alpha=1$, as well.

Next we shall show that the following statement holds:

if for some positive integer $p>1$ the inequality $x_1<p$ implies $x_1^\alpha\leq x_1$ then the equality $x_1=p$ implies $x_1^\alpha\leq x_1$, too.

Suppose to the contrary that there exists $(x_1,x_2,x_3)\in\operatorname{dom}\alpha$ such that

$$x_1=p=(x_1,x_2,x_3)\mathfrak{pr}_1,\quad (x_1,x_2,x_3)\alpha=(x_1^\alpha,x_2^\alpha,x_3^\alpha)\quad\text{and}\quad x_1+1\leqslant x_1^\alpha.$$

We define a partial map $\varpi\colon\mathbb{N}^3\rightharpoonup\mathbb{N}^3$ with $\operatorname{dom}\varpi=\mathbb{N}^3\setminus(\{1\}\times L(x_2)\times L(x_2))$ and $\operatorname{ran}\varpi=\mathbb{N}^3$ by the formula

$$(i_1,i_2,i_3)\varpi=\begin{cases}(i_1-1,i_2,i_3), & \text{if } i_2\in L(x_2) \text{ and } i_3\in L(x_2);\\ (i_1,i_2,i_3), & \text{otherwise,}\end{cases}$$

where $L(x_2)=\{1,\ldots,x_2\}$ and $L(x_3)=\{1,\ldots,x_3\}$. It is obvious that $\varpi\in\mathscr{PO}_\infty(\mathbb{N}^3_{\leqslant})$, and hence $\gamma\varpi^k\in\mathscr{PO}_\infty(\mathbb{N}^3_{\leqslant})$ for any positive integer k and any $\gamma\in\mathscr{PO}_\infty(\mathbb{N}^3_{\leqslant})$. This observation implies that without loss of generality we may assume that $x_1^\alpha=x_1+1$. Then the assumption of the theorem implies that there exists the smallest element $(i_{\mathrm{m}},1,1)$ of $\mathscr{K}_1$ such that $i_{\mathrm{m}}^{\alpha} > x_1^{\alpha}+1$, where $(i_{\mathrm{m}}^{\alpha},1,1)=(i_{\mathrm{m}},1,1)\alpha$. Since $({\uparrow}(i_{\mathrm{m}},1,1))\alpha \subseteq {\uparrow}(i_{\mathrm{m}}^{\alpha},1,1)$, $({\uparrow}(x_1,x_2,x_3))\alpha \subseteq {\uparrow}(x_1^{\alpha},x_2^{\alpha},x_3^{\alpha})$ and the set $\mathbb{N}^3\setminus \operatorname{ran}\alpha$ is finite, our assumption implies that the set

$$\mathscr{S}_{x_1}(\alpha)=\{(x_1,p_2,p_3)\in \operatorname{dom}\alpha\colon p_2,p_3\in\mathbb{N}\}$$

is a union of finitely many subchains of the poset $(\mathbb{N}^3,\leqslant)$. This contradicts Lemma 4 because the set $\mathscr{S}_{x_1}(\alpha)$ with the induced partial order from $\mathbb{N}^3_{\leqslant}$ is order isomorphic to a cofinite subset of the poset $\mathbb{N}^2_{\leqslant}$. The obtained contradiction implies the requested inequality $x_1^{\alpha}\leq x_1$ and hence we have that statement (i) holds.

The last assertion of (i) follows from the definition of the poset $\mathbb{N}^3_{\leqslant}$.

(ii) Fix an arbitrary $\alpha\in\mathscr{PO}_{\infty}(\mathbb{N}^3_{\leqslant})$ such that $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\subseteq\mathscr{K}_i$ for all $i=1,2,3$. Suppose to the contrary that for any positive integer n there exists

$$(x_1,x_2,x_3)\in\operatorname{dom}\alpha\cap{\uparrow}(n,n,n)$$

such that $(x_1,x_2,x_3)\alpha\neq(x_1,x_2,x_3)$. We put $\mathsf{N}_{\operatorname{dom}\alpha}=\left|\mathbb{N}^3\setminus\operatorname{dom}\alpha\right|+1$ and

$$\mathsf{M}_{\operatorname{dom}\alpha}=\max\big\{\,\{x_1\colon(x_1,x_2,x_3)\notin\operatorname{dom}\alpha\},\{x_2\colon(x_1,x_2,x_3)\notin\operatorname{dom}\alpha\},\\ \{x_3\colon(x_1,x_2,x_3)\notin\operatorname{dom}\alpha\}\,\big\}+1.$$

The definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^3_{\leqslant})$ implies that the positive integers $\mathsf{N}_{\operatorname{dom}\alpha}$ and $\mathsf{M}_{\operatorname{dom}\alpha}$ are well defined. Put $n_0=\max\{\mathsf{N}_{\operatorname{dom}\alpha},\mathsf{M}_{\operatorname{dom}\alpha}\}$. Then our assumption implies that there exists $(x_1,x_2,x_3)\in\operatorname{dom}\alpha\cap{\uparrow}(n_0,n_0,n_0)$ such that

$$(x_1,x_2,x_3)\alpha=(x_1^{\alpha},x_2^{\alpha},x_3^{\alpha})\neq(x_1,x_2,x_3).$$

By statement (i) we have that $(x_1^{\alpha},x_2^{\alpha},x_3^{\alpha})<(x_1,x_2,x_3)$. We consider the case when $x_1^{\alpha}<x_1$. In the cases when $x_2^{\alpha}<x_2$ or $x_3^{\alpha}<x_3$ the proofs are similar. We assume that $x_1\leqslant x_2$ and $x_1\leqslant x_3$. By statement (i) the partial bijection α maps the set $S=\big\{(x,y,z)\in\mathbb{N}^3\colon x,y,z\leqslant x_1-1\big\}$ into itself. Also, by the definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^3_{\leqslant})$ the partial bijection α maps the set

$$\{(x_1,1,1),\ldots,(x_1,1,x_1),(x_1,2,1),\ldots,(x_1,2,x_1),\ldots,(x_1,x_1,1),\ldots,(x_1,x_1,x_1)\}$$

into S, too. Then our construction implies that

$$|S\setminus\operatorname{dom}\alpha|=\left|\mathbb{N}^3\setminus\operatorname{dom}\alpha\right|=\mathsf{N}_{\operatorname{dom}\alpha}-1$$

and

$$|\{(x_1,1,1),\ldots,(x_1,1,x_1),(x_1,2,1),\ldots,(x_1,2,x_1),\ldots,(x_1,x_1,1),\ldots,(x_1,x_1,x_1)\}|\geqslant\mathsf{N}_{\operatorname{dom}\alpha},$$

a contradiction. In the case when $x_2\leqslant x_1$ and $x_2\leqslant x_3$ or $x_3\leqslant x_1$ and $x_3\leqslant x_2$ we get contradictions in similar ways. This completes the proof of existence of such a positive integer n_α for any $\alpha\in\mathscr{PO}_{\infty}(\mathbb{N}^3_{\leqslant})$. The existence of such minimal positive integer n_α follows from the fact that the set of all positive integers with the usual order $\leqslant$ is well-ordered. □

Theorem 2(iii) and Proposition 1 imply the following corollary.

Corollary 2. *For an arbitrary element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ there exist elements σ_1, σ_2 of the group of units $H(\mathbb{I})$ of $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ and a smallest positive integer n_α such that*

$$(x_1, x_2, x_3)\sigma_1\alpha = (x_1, x_2, x_3)\alpha\sigma_2 = (x_1, x_2, x_3)$$

for each $(x_1, x_2, x_3) \in \operatorname{dom}\alpha \cap {\uparrow}(n_\alpha, n_\alpha, n_\alpha)$.

Corollary 2 implies

Corollary 3. $\left|\mathbb{N}^3 \setminus \operatorname{ran}\alpha\right| \leqslant \left|\mathbb{N}^3 \setminus \operatorname{dom}\alpha\right|$ *for an arbitrary $\alpha \in \mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$.*

3. Algebraic properties of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$

Proposition 3. *Let X be a non-empty set and let $\mathscr{PB}(X)$ be a semigroup of partial bijections of X with the usual composition of partial self-maps. Then an element α of $\mathscr{PB}(X)$ is an idempotent if and only if α is an identity partial self-map of X.*

Proof. The implication $(\Leftarrow)$ is trivial.

$(\Rightarrow)$ Let an element α be an idempotent of the semigroup $\mathscr{PB}(X)$. Then for every $x \in \operatorname{dom}\alpha$ we have that $(x)\alpha\alpha = (x)\alpha$ and hence we get that $\operatorname{dom}\alpha^2 = \operatorname{dom}\alpha$ and $\operatorname{ran}\alpha^2 = \operatorname{ran}\alpha$. Also since α is a partial bijective self-map of X we conclude that the previous equalities imply that $\operatorname{dom}\alpha = \operatorname{ran}\alpha$. Fix an arbitrary $x \in \operatorname{dom}\alpha$ and suppose that $(x)\alpha = y$. Then $(x)\alpha = (x)\alpha\alpha = (y)\alpha = y$. Since α is a partial bijective self-map of the set X, we have that the equality $(y)\alpha = y$ implies that the full preimage of y under the partial map α is equal to y. Similarly the equality $(x)\alpha = y$ implies that the full preimage of y under the partial map α is equal to x. Thus we get that $x = y$ and our implication holds. □

Proposition 3 implies the following corollary.

Corollary 4. *An element α of $\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$ is an idempotent if and only if α is an identity partial self-map of $\mathbb{N}^n_\leqslant$ with the cofinite domain.*

Corollary 4 implies the following proposition.

Proposition 4. *Let n be a positive integer $\geqslant 2$. The subset of idempotents $E(\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant))$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$ is a commutative submonoid of $\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$ and moreover $E(\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant))$ is isomorphic to the free semilattice with unit $(\mathscr{P}^*(\mathbb{N}^n), \cup)$ over the set $\mathbb{N}^n$ under the map $(\varepsilon)\mathfrak{h} = \mathbb{N}^n \setminus \operatorname{dom}\varepsilon$.*

Later we shall need the following technical lemma.

Lemma 6. *Let X be a non-empty set, $\mathscr{PB}(X)$ be the semigroup of partial bejections of X with the usual composition of partial self-maps and $\alpha \in \mathscr{PB}(X)$. Then the following assertions hold:*

- *(i)* *$\alpha = \gamma\alpha$ for some $\gamma \in \mathscr{PB}(X)$ if and only if the restriction $\gamma|_{\operatorname{dom}\alpha} \colon \operatorname{dom}\alpha \to X$ is an identity partial map;*
- *(ii)* *$\alpha = \alpha\gamma$ for some $\gamma \in \mathscr{PB}(X)$ if and only if the restriction $\gamma|_{\operatorname{ran}\alpha} \colon \operatorname{ran}\alpha \to X$ is an identity partial map.*

Proof. (i) The implication $(\Leftarrow)$ is trivial.

$(\Rightarrow)$ Suppose that $\alpha = \gamma\alpha$ for some $\gamma \in \mathscr{PB}(X)$. Then $\operatorname{dom}\alpha \subseteq \operatorname{dom}\gamma$ and $\operatorname{dom}\alpha \subseteq \operatorname{ran}\gamma$. Since $\gamma\colon X \rightharpoonup X$ is a partial bijection, the above arguments imply that $(x)\gamma = x$ for each $x \in \operatorname{dom}\alpha$. Indeed, if $(x)\gamma = y \neq x$ for some $y \in \operatorname{dom}\alpha$ then since $\alpha\colon X \rightharpoonup X$ is a partial bijection we have that either

$$(x)\alpha = (x)\gamma\alpha = (y)\alpha \neq (x)\alpha, \qquad \text{if} \quad y \in \operatorname{dom}\alpha,$$

or $(y)\alpha$ is undefined. This completes the proof of the implication.

The proof of (ii) is similar to that of (i). □

Lemma 6 implies the following corollary.

Corollary 5. *Let n be a positive integer $\geqslant 2$ and α be an element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$. Then the following assertions hold:*

- (i) *$\alpha = \gamma\alpha$ for some $\gamma \in \mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$ if and only if the restriction $\gamma|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha \to \mathbb{N}^n$ is an identity partial map;*
- (ii) *$\alpha = \alpha\gamma$ for some $\gamma \in \mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$ if and only if the restriction $\gamma|_{\operatorname{ran}\alpha}\colon \operatorname{ran}\alpha \to \mathbb{N}^n$ is an identity partial map.*

The following theorem describes Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{D}$ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$.

Theorem 3. *Let α and β be elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$. Then the following assertions hold:*

- (i) *$\alpha\mathscr{L}\beta$ if and only if $\alpha = \mu\beta$ for some $\mu \in H(\mathbb{I})$;*
- (ii) *$\alpha\mathscr{R}\beta$ if and only if $\alpha = \beta\nu$ for some $\nu \in H(\mathbb{I})$;*
- (iii) *$\alpha\mathscr{H}\beta$ if and only if $\alpha = \mu\beta = \beta\nu$ for some $\mu, \nu \in H(\mathbb{I})$;*
- (iv) *$\alpha\mathscr{D}\beta$ if and only if $\alpha = \mu\beta\nu$ for some $\mu, \nu \in H(\mathbb{I})$.*

Proof. (i) The implication $(\Leftarrow)$ is trivial.

$(\Rightarrow)$ Suppose that $\alpha\mathscr{L}\beta$ in the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$. Then there exist $\gamma, \delta \in \mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $\alpha = \gamma\beta$ and $\beta = \delta\alpha$. The last equalities imply that $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

Next, we consider the following cases:

- (i_1) $(\mathscr{K}_i \cap \operatorname{dom}\alpha)\alpha \subseteq \mathscr{K}_i$ and $(\mathscr{K}_j \cap \operatorname{dom}\beta)\beta \subseteq \mathscr{K}_j$ for all $i, j = 1, 2, 3$;
- (i_2) $(\mathscr{K}_i \cap \operatorname{dom}\alpha)\alpha \subseteq \mathscr{K}_i$ for all $i = 1, 2, 3$ and $(\mathscr{K}_j \cap \operatorname{dom}\beta)\beta \nsubseteq \mathscr{K}_j$ for some $j = 1, 2, 3$;
- (i_3) $(\mathscr{K}_i \cap \operatorname{dom}\alpha)\alpha \nsubseteq \mathscr{K}_i$ for some $i = 1, 2, 3$ and $(\mathscr{K}_j \cap \operatorname{dom}\beta)\beta \subseteq \mathscr{K}_j$ for all $j = 1, 2, 3$;
- (i_4) $(\mathscr{K}_i \cap \operatorname{dom}\alpha)\alpha \nsubseteq \mathscr{K}_i$ and $(\mathscr{K}_j \cap \operatorname{dom}\beta)\beta \nsubseteq \mathscr{K}_j$ for some $i, j = 1, 2, 3$.

Suppose that case (i_1) holds. Then Proposition 1 and the equalities $\alpha = \gamma\beta$ and $\beta = \delta\alpha$ imply that

$$(1) \qquad (\mathscr{K}_i \cap \operatorname{dom}\gamma)\gamma \subseteq \mathscr{K}_i \qquad \text{and} \qquad (\mathscr{K}_j \cap \operatorname{dom}\delta)\delta \subseteq \mathscr{K}_j, \quad \text{for all} \quad i, j = 1, 2, 3,$$

and moreover we have that $\alpha = \gamma\delta\alpha$ and $\beta = \delta\gamma\beta$. Hence by Lemma 6 we have that the restrictions $(\gamma\delta)|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha \rightharpoonup \mathbb{N}^3$ and $(\delta\gamma)|_{\operatorname{dom}\beta}\colon \operatorname{dom}\beta \rightharpoonup \mathbb{N}^3$ are identity partial maps. Then by condition (1) we obtain that the restrictions $\gamma|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha \rightharpoonup \mathbb{N}^3$ and $\delta|_{\operatorname{dom}\beta}\colon \operatorname{dom}\beta \rightharpoonup \mathbb{N}^3$ are identity partial maps, as well. Indeed, otherwise there exists

$\overline{x} \in \operatorname{dom}\alpha$ such that either $(\overline{x})\gamma \nleqslant \overline{x}$ or $(\overline{x})\delta \nleqslant \overline{x}$, which contradicts Theorem 2(ii). Thus, the above arguments imply that in case (i_1) we have the equality $\alpha = \beta$.

Suppose that case (i_2) holds. By Corollary 1 there exists an element μ of the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $(\mathscr{K}_j \cap \operatorname{dom}\beta)\mu\beta \subseteq \mathscr{K}_j$ for all $j = 1,2,3$, and, since $\alpha\mathscr{L}\beta$, we have that

$$\alpha = \gamma\beta = \gamma\mathbb{I}\beta = \gamma(\mu^{-1}\mu)\beta = (\gamma\mu^{-1})(\mu\beta)$$

and $\mu\beta = (\mu\delta)\alpha$. Hence we get that $\alpha\mathscr{L}(\mu\beta)$, $(\mathscr{K}_i \cap \operatorname{dom}\alpha)\alpha \subseteq \mathscr{K}_i$ and $(\mathscr{K}_j \cap \operatorname{dom}\beta)\mu\beta \subseteq \mathscr{K}_j$ for all $i,j = 1,2,3$. Then we apply case (i_1) for the elements α and $\mu\beta$ and obtain the equality $\alpha = \mu\beta$, where μ is the above determined element of the group of units $H(\mathbb{I})$.

In case (i_3) the proof of the equality $\alpha = \mu\beta$ is similar to case (i_2).

Suppose that case (i_4) holds. By Corollary 1 there exist elements μ_α and μ_β of the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $(\mathscr{K}_j \cap \operatorname{dom}\alpha)\mu_\alpha\alpha \subseteq \mathscr{K}_j$ and $(\mathscr{K}_j \cap \operatorname{dom}\beta)\mu_\beta\beta \subseteq \mathscr{K}_j$ for all $i,j = 1,2,3$, and, since $\alpha\mathscr{L}\beta$, we have that

$$\alpha = \gamma\beta = \gamma\mathbb{I}\beta = \gamma(\mu_\beta^{-1}\mu_\beta)\beta = (\gamma\mu_\beta^{-1})(\mu_\beta\beta)$$

and

$$\beta = \delta\alpha = \delta\mathbb{I}\alpha = \delta(\mu_\alpha^{-1}\mu_\alpha)\alpha = (\delta\mu_\alpha^{-1})(\mu_\alpha\alpha).$$

Hence we get that

$$\mu_\alpha\alpha = (\mu_\alpha\gamma\mu_\beta^{-1})(\mu_\beta\beta) \qquad \text{and} \qquad \mu_\beta\beta = (\mu_\beta\delta\mu_\alpha^{-1})(\mu_\alpha\alpha).$$

The last two equalities imply that $(\mu_\beta\beta)\mathscr{L}(\mu_\alpha\alpha)$ and by above part of the proof we have that $(\mathscr{K}_j \cap \operatorname{dom}\alpha)\mu_\alpha\alpha \subseteq \mathscr{K}_j$ and $(\mathscr{K}_j \cap \operatorname{dom}\beta)\mu_\beta\beta \subseteq \mathscr{K}_j$ for all $i,j = 1,2,3$. Then we apply case (i_1) for the elements $\mu_\alpha\alpha$ and $\mu_\beta\beta$ and obtain the equality $\mu_\alpha\alpha = \mu_\beta\beta$. Hence $\alpha = \mu_\alpha^{-1}\mu_\alpha\alpha = \mu_\alpha^{-1}\mu_\beta\beta$. Since $\mu_\alpha, \mu_\alpha \in H(\mathbb{I})$, $\mu = \mu_\alpha^{-1}\mu_\beta \in H(\mathbb{I})$ as well.

The proof of assertion (ii) is dual to that of (i).

Assertion (iii) follows from (i) and (ii).

(iv) Suppose that $\alpha\mathscr{D}\beta$ in $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$. Then there exists $\gamma \in \mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$. By statements (i) and (ii) there exist $\mu, \nu \in H(\mathbb{I})$ such that $\alpha = \mu\gamma$ and $\gamma = \beta\nu$ and hence $\alpha = \mu\beta\nu$. Converse, suppose that $\alpha = \mu\beta\nu$ for some $\mu, \nu \in H(\mathbb{I})$. Then by (i), (ii), we have that $\alpha\mathscr{L}(\beta\nu)$ and $(\beta\nu)\mathscr{R}\beta$, and hence $\alpha\mathscr{D}\beta$ in $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$. □

Theorem 3 implies Corollary 6 which gives the inner characterization of Green's relations $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ as partial permutations of the poset $\mathbb{N}^3_\leqslant$.

Corollary 6. (i) *Every $\mathscr{L}$-class of $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ contains exactly* 6 *distinct elements.*

(ii) *Every $\mathscr{R}$-class of $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ contains exactly* 6 *distinct elements.*

(iii) *Every $\mathscr{H}$-class of $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ contains at most* 6 *distinct elements.*

Proof. Statements (i), (ii) and (iii) are trivial and they follow from the corresponding statements of Theorem 3. □

Lemma 7. *Let α, β and γ be elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $\alpha = \beta\alpha\gamma$. Then the following statements hold:*

(*i*) *if* $(\mathscr{K}_i\cap\operatorname{dom}\beta)\beta\subseteq\mathscr{K}_i$ *for any* $i=1,2,3$, *then the restrictions* $\beta|_{\operatorname{dom}\alpha}\colon\operatorname{dom}\alpha\rightharpoonup\mathbb{N}^3$ *and* $\gamma|_{\operatorname{ran}\alpha}\colon\operatorname{ran}\alpha\rightharpoonup\mathbb{N}^3$ *are identity partial maps;*

(*ii*) *if* $(\mathscr{K}_i\cap\operatorname{dom}\gamma)\gamma\subseteq\mathscr{K}_i$ *for any* $i=1,2,3$, *then the restrictions* $\beta|_{\operatorname{dom}\alpha}\colon\operatorname{dom}\alpha\rightharpoonup\mathbb{N}^3$ *and* $\gamma|_{\operatorname{ran}\alpha}\colon\operatorname{ran}\alpha\rightharpoonup\mathbb{N}^3$ *are identity partial maps;*

(*iii*) *there exist elements* σ_β *and* σ_γ *of the group of units* $H(\mathbb{I})$ *of* $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ *such that* $\alpha=\sigma_\beta\alpha\sigma_\gamma$.

Proof. (*i*) Assume that the inclusion $(\mathscr{K}_i\cap\operatorname{dom}\beta)\beta\subseteq\mathscr{K}_i$ holds for any $i=1,2,3$. Then one of the following cases holds:

(1) $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\subseteq\mathscr{K}_i$ for any $i=1,2,3$;
(2) there exists $i\in\{1,2,3\}$ such that $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\nsubseteq\mathscr{K}_i$.

If case (1) holds then the equality $\alpha=\beta\alpha\gamma$ and Proposition 1 imply that $(\mathscr{K}_i\cap\operatorname{dom}\gamma)\gamma\subseteq\mathscr{K}_i$ for any $i=1,2,3$. Suppose that $(\overline{x})\beta<\overline{x}$ for some $\overline{x}\in\operatorname{dom}\alpha$. Then by Theorem 2(*i*) we have that

$$(\overline{x})\alpha=(\overline{x})\beta\alpha\gamma<(\overline{x})\alpha\gamma\leqslant(\overline{x})\alpha,$$

which contradicts the equality $\alpha=\beta\alpha\gamma$. The obtained contradiction implies that the restriction $\beta|_{\operatorname{dom}\alpha}\colon\operatorname{dom}\alpha\rightharpoonup\mathbb{N}^3$ is an identity partial map. This and the equality $\alpha=\beta\alpha\gamma$ imply that the restriction $\gamma|_{\operatorname{ran}\alpha}\colon\operatorname{ran}\alpha\rightharpoonup\mathbb{N}^3$ is an identity partial map too.

Suppose that case (2) holds. Then by Corollary 1 there exists an element σ of the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\sigma\subseteq\mathscr{K}_i$ for any $i=1,2,3$. Now, the equality $\alpha=\beta\alpha\gamma$ implies that

$$\alpha\sigma=\beta\alpha\gamma\sigma=\beta\alpha\mathbb{I}\gamma\sigma=\beta\alpha(\sigma\sigma^{-1})\gamma\sigma=\beta(\alpha\sigma)(\sigma^{-1}\gamma\sigma).$$

By case (1) we have that the restrictions $\beta|_{\operatorname{dom}\alpha}\colon\operatorname{dom}\alpha\rightharpoonup\mathbb{N}^3$ is an identity partial map, which implies that $\beta\alpha=\alpha$. Then we have that $\alpha=\beta\alpha\gamma=\alpha\gamma$ and hence by Corollary 5 the restriction $\gamma|_{\operatorname{ran}\alpha}\colon\operatorname{ran}\alpha\rightharpoonup\mathbb{N}^3$ is an identity partial map, which completes the proof of statement (*i*).

(*ii*) The proof of this statement is dual to (*i*). Indeed, assume that the inclusion $(\mathscr{K}_i\cap\operatorname{dom}\gamma)\gamma\subseteq\mathscr{K}_i$ holds for any $i=1,2,3$. Then one of the following cases holds:

(1) $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\subseteq\mathscr{K}_i$ for any $i=1,2,3$;
(2) there exists $i\in\{1,2,3\}$ such that $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\alpha\nsubseteq\mathscr{K}_i$.

If case (1) holds then the equality $\alpha=\beta\alpha\gamma$ and Proposition 1 imply that $(\mathscr{K}_i\cap\operatorname{dom}\beta)\beta\subseteq\mathscr{K}_i$ for any $i=1,2,3$. Similarly as in the proof of statement (*i*) Theorem 2(*i*) implies that the restrictions $\beta|_{\operatorname{dom}\alpha}\colon\operatorname{dom}\alpha\rightharpoonup\mathbb{N}^3$ and $\gamma|_{\operatorname{ran}\alpha}\colon\operatorname{ran}\alpha\rightharpoonup\mathbb{N}^3$ are identity partial maps.

Suppose that case (2) holds. Then by Corollary 1 there exists an element σ of the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $(\mathscr{K}_i\cap\operatorname{dom}\alpha)\sigma\alpha\subseteq\mathscr{K}_i$ for any $i=1,2,3$. Now, the equality $\alpha=\beta\alpha\gamma$ implies that

$$\sigma\alpha=\sigma\beta\alpha\gamma=\sigma\beta\mathbb{I}\alpha\gamma=\sigma\beta(\sigma^{-1}\sigma)\alpha\gamma=(\sigma\beta\sigma^{-1})(\sigma\alpha)\gamma.$$

By case (1) we have that the restriction $\gamma|_{\operatorname{ran}\alpha}\colon\operatorname{ran}\alpha\rightharpoonup\mathbb{N}^3$ is an identity partial map, which implies that $\alpha=\alpha\gamma$. Then we have that $\alpha=\beta\alpha\gamma=\beta\alpha$ and hence by Corollary 5 the restriction $\beta|_{\operatorname{dom}\alpha}\colon\operatorname{dom}\alpha\rightharpoonup\mathbb{N}^3$ is an identity partial map as well, which completes the proof of statement (*ii*).

(iii) Assume that $\alpha = \beta\alpha\gamma$. By the Lagrange Theorem (see: [41, Section 1.5]) for every element σ of the group of permutations $\mathscr{S}_n$ the order of σ divides the order of $\mathscr{S}_n$. This, Proposition 1 and the equality $\alpha = \beta\alpha\gamma$ imply that

(2) $(\mathscr{K}_i \cap \operatorname{dom}\beta^6)\beta^6 \subseteq \mathscr{K}_i$ and $(\mathscr{K}_i \cap \operatorname{dom}\gamma^6)\gamma^6 \subseteq \mathscr{K}_i$, for any $i = 1, 2, 3$.

Also, the equality $\alpha = \beta\alpha\gamma$ implies that

$$\alpha = \beta\alpha\gamma = \beta(\beta\alpha\gamma)\gamma = \beta^2\alpha\gamma^2 = \ldots = \beta^6\alpha\gamma^6.$$

Then statements (i), (ii) and conditions (2) imply that the restrictions $\beta^6|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha \rightharpoonup \mathbb{N}^3$ and $\gamma^6|_{\operatorname{ran}\alpha}\colon \operatorname{ran}\alpha \rightharpoonup \mathbb{N}^3$ are identity partial maps. By Corollary 1 there exist unique elements $\sigma_\beta, \sigma_\gamma \in H(\mathbb{I})$ such that $(\mathscr{K}_i \cap \operatorname{dom}\beta)\beta\sigma_\beta^{-1} \subseteq \mathscr{K}_i$, $(\mathscr{K}_i \cap \operatorname{dom}\beta)\sigma_\beta\beta \subseteq \mathscr{K}_i$, $(\mathscr{K}_i \cap \operatorname{dom}\alpha)\gamma\sigma_\gamma^{-1} \subseteq \mathscr{K}_i$ and $(\mathscr{K}_i \cap \operatorname{dom}\gamma)\sigma_\gamma\gamma \subseteq \mathscr{K}_i$ for all $i = 1, 2, 3$. Then we have that

$$\begin{aligned}\beta^6 &= (\beta\mathbb{I}\beta)(\beta\mathbb{I}\beta)(\beta\mathbb{I}\beta)\\ &= (\beta\sigma_\beta^{-1}\sigma_\beta\beta)(\beta\sigma_\beta^{-1}\sigma_\beta\beta)(\beta\sigma_\beta^{-1}\sigma_\beta\beta)\\ &= (\beta\sigma_\beta^{-1})(\sigma_\beta\beta)(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)\end{aligned} \tag{3}$$

and

$$\begin{aligned}\gamma^6 &= (\gamma\mathbb{I}\gamma)(\gamma\mathbb{I}\gamma)(\gamma\mathbb{I}\gamma)\\ &= (\gamma\sigma_\gamma^{-1}\sigma_\gamma\gamma)(\gamma\sigma_\gamma^{-1}\sigma_\gamma\gamma)(\gamma\sigma_\gamma^{-1}\sigma_\gamma\gamma)\\ &= (\gamma\sigma_\gamma^{-1})(\sigma_\gamma\gamma)(\gamma\sigma_\gamma^{-1})(\sigma_\gamma\gamma)(\gamma\sigma_\gamma^{-1})(\sigma_\gamma\gamma).\end{aligned} \tag{4}$$

We claim that $(\overline{x})(\beta\sigma_\beta^{-1}) = \overline{x}$ for any $\overline{x} \in \operatorname{dom}\alpha$. Assume that $(\overline{x})(\beta\sigma_\beta^{-1}) \neq \overline{x}$ for some $\overline{x} \in \operatorname{dom}\alpha$. Then the choice of the element $\sigma_\beta \in H(\mathbb{I})$, Theorem 2$(i)$ and (3) imply that

$$\begin{aligned}(\overline{x})\beta^6 &= (\overline{x})(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)\\ &< (\overline{x})(\sigma_\beta\beta)(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)\\ &\leqslant (\overline{x})(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)\\ &< (\overline{x})(\sigma_\beta\beta)(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)\\ &\leqslant (\overline{x})(\beta\sigma_\beta^{-1})(\sigma_\beta\beta)\\ &< (\overline{x})(\sigma_\beta\beta)\\ &\leqslant \overline{x},\end{aligned}$$

which contradicts the fact that the restriction $\beta^6|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha \rightharpoonup \mathbb{N}^3$ is an identity partial map. Hence we have that $(\overline{x})(\beta\sigma_\beta^{-1}) = \overline{x}$ for any $\overline{x} \in \operatorname{dom}\alpha$, which implies that the equality $(\overline{x})\beta = (\overline{x})\sigma_\beta$ holds for any $\overline{x} \in \operatorname{dom}\alpha$.

Using (4) as in the above we prove the equality $(\overline{x})\gamma = (\overline{x})\sigma_\gamma$ holds for any $\overline{x} \in \operatorname{ran}\alpha$.

The obtained equalities and the definition of the composition of partial maps imply statement (iii). □

Lemma 8. *Let α and β be elements of the semigroup $\mathscr{P\!O}_\infty(\mathbb{N}^3_\leqslant)$ and A be a cofinite subset of $\mathbb{N}^3$. If the restriction $(\alpha\beta)|_A\colon A \rightharpoonup \mathbb{N}^3$ is an identity partial map then there exists an element σ of the group of units $H(\mathbb{I})$ of $\mathscr{P\!O}_\infty(\mathbb{N}^3_\leqslant)$ such that $(\overline{x})\alpha = (\overline{x})\sigma$ and $(\overline{y})\beta = (\overline{y})\sigma^{-1}$ for all $\overline{x} \in A$ and $\overline{y} \in (A)\alpha$.*

Proof. We observe that one of the following cases holds:

(1) $(\mathscr{K}_i \cap A)\alpha \subseteq \mathscr{K}_i$ for any $i = 1, 2, 3$;
(2) there exists $i \in \{1, 2, 3\}$ such that $(\mathscr{K}_i \cap A)\alpha \nsubseteq \mathscr{K}_i$.

If case (1) holds then the assumption of the lemma and Proposition 1 imply that $(\mathscr{K}_i \cap (A)\alpha)\beta \subseteq \mathscr{K}_i$ for any $i = 1, 2, 3$. Suppose that $(\overline{x})\alpha < \overline{x}$ for some $\overline{x} \in A$. Then by Theorem 2(i) we have that

$$(\overline{x})\alpha\beta < (\overline{x})\beta \leqslant \overline{x},$$

which contradicts the assumption of the lemma. Similarly we show that the case $(\overline{y})\beta < \overline{y}$ for some $\overline{y} \in (A)\alpha$ does not hold. The obtained contradiction implies that $(\overline{x})\alpha = \overline{x}$ and $(\overline{x})\beta = \overline{x}$ for all $\overline{x} \in A$.

Suppose that case (2) holds. Then by Corollary 1 there exists an element σ of the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $(\mathscr{K}_i \cap \operatorname{dom}\alpha)\alpha\sigma \subseteq \mathscr{K}_i$ for any $i = 1, 2, 3$. Now, the assumption of the lemma implies that

$$(\overline{x})\alpha\beta = (\overline{x})\alpha\mathbb{I}\beta = (\overline{x})\alpha\sigma\sigma^{-1}\beta = \overline{x},$$

and hence by the above part of the proof we get that $(\overline{x})\alpha\sigma = \overline{x}$ and $(\overline{y})\sigma^{-1}\beta = \overline{x}$ for all $\overline{y} \in (A)\alpha$. The obtained equalities and the definition of the composition of partial maps imply the statement of the lemma. □

Lemma 9. *Let α, β, γ and δ be elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $\alpha = \gamma\beta\delta$. Then there exist $\gamma^*, \delta^* \in \mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $\alpha = \gamma^*\beta\delta^*$, $\operatorname{dom}\gamma^* = \operatorname{dom}\alpha$, $\operatorname{ran}\gamma^* = \operatorname{dom}\beta$, $\operatorname{dom}\delta^* = \operatorname{ran}\beta$ and $\operatorname{ran}\delta^* = \operatorname{ran}\alpha$.*

Proof. For a cofinite subset A of $\mathbb{N}^3$ by ι_A we denote the identity map of A. It is obvious that $\iota_A \in \mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ for any cofinite subset A of $\mathbb{N}^3$. This implies that $\alpha = \iota_{\operatorname{dom}\alpha}\alpha\iota_{\operatorname{ran}\alpha}$ and $\beta = \iota_{\operatorname{dom}\beta}\beta\iota_{\operatorname{ran}\beta}$, and hence we have that

$$\alpha = \iota_{\operatorname{dom}\alpha}\alpha\iota_{\operatorname{ran}\alpha} = \iota_{\operatorname{dom}\alpha}\gamma\beta\delta\iota_{\operatorname{ran}\alpha} = \iota_{\operatorname{dom}\alpha}\gamma\iota_{\operatorname{dom}\beta}\beta\iota_{\operatorname{ran}\beta}\delta\iota_{\operatorname{ran}\alpha}.$$

We put $\gamma^* = \iota_{\operatorname{dom}\alpha}\gamma\iota_{\operatorname{dom}\beta}$ and $\delta^* = \iota_{\operatorname{ran}\beta}\delta\iota_{\operatorname{ran}\alpha}$. The above two equalities and the definition of the semigroup operation of $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ imply that $\operatorname{dom}\gamma^* \subseteq \operatorname{dom}\alpha$, $\operatorname{ran}\gamma^* \subseteq \operatorname{dom}\beta$, $\operatorname{dom}\delta^* \subseteq \operatorname{ran}\beta$ and $\operatorname{ran}\delta^* \subseteq \operatorname{ran}\alpha$. Similar arguments and the equality $\alpha = \gamma^*\beta\delta^*$ imply the converse inclusions which implies the statement of the lemma. □

Theorem 4. *$\mathscr{D} = \mathscr{J}$ in $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$.*

Proof. The inclusion $\mathscr{D} \subseteq \mathscr{J}$ is trivial.

Fix any $\alpha, \beta \in \mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $\alpha\mathscr{J}\beta$. Then there exist $\gamma_\alpha, \delta_\alpha, \gamma_\beta, \delta_\beta \in \mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $\alpha = \gamma_\alpha\beta\delta_\alpha$ and $\beta = \gamma_\beta\alpha\delta_\beta$ (see [22] or [23, Section II.1]). By Lemma 9 without loss of generality we may assume that

$$\operatorname{dom}\gamma_\alpha = \operatorname{dom}\alpha, \qquad \operatorname{ran}\gamma_\alpha = \operatorname{dom}\beta, \qquad \operatorname{dom}\delta_\alpha = \operatorname{ran}\beta, \qquad \operatorname{ran}\delta_\alpha = \operatorname{ran}\alpha$$

and

$$\operatorname{dom}\gamma_\beta = \operatorname{dom}\beta, \qquad \operatorname{ran}\gamma_\beta = \operatorname{dom}\alpha, \qquad \operatorname{dom}\delta_\beta = \operatorname{ran}\alpha, \qquad \operatorname{ran}\delta_\beta = \operatorname{ran}\beta.$$

Hence we have that $\alpha = \gamma_\alpha\gamma_\beta\alpha\delta_\beta\delta_\alpha$ and $\beta = \gamma_\beta\gamma_\alpha\beta\delta_\alpha\delta_\beta$. Then only one of the following cases holds:

(1) $(\mathscr{K}_i \cap \operatorname{dom}(\gamma_\alpha\gamma_\beta))\gamma_\alpha\gamma_\beta \subseteq \mathscr{K}_i$ for any $i = 1, 2, 3$;
(2) there exists $i \in \{1, 2, 3\}$ such that $(\mathscr{K}_i \cap \operatorname{dom}(\gamma_\alpha\gamma_\beta))\gamma_\alpha\gamma_\beta \nsubseteq \mathscr{K}_i$.

If case (1) holds then Lemma 7(i) implies that $(\gamma_\alpha\gamma_\beta)\colon \operatorname{dom}\alpha \rightharpoonup \mathbb{N}^3$ and $(\delta_\beta\delta_\alpha)\colon \operatorname{ran}\alpha \rightharpoonup \mathbb{N}^3$ are identity partial maps. Now by Lemma 8 there exist elements σ_α and σ_β of the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $(\overline{x})\gamma_\alpha = (\overline{x})\sigma_\alpha$, $(\overline{y})\gamma_\beta = (\overline{y})\sigma_\alpha^{-1}$, $(\overline{u})\delta_\beta = (\overline{u})\sigma_\beta$ and $(\overline{v})\delta_\alpha = (\overline{v})\sigma_\beta^{-1}$, for all $\overline{x} \in \operatorname{dom}\alpha$, $\overline{y} \in (\operatorname{dom}\alpha)\gamma_\alpha = \operatorname{ran}\gamma_\alpha = \operatorname{dom}\beta$, $\overline{u} \in \operatorname{ran}\alpha$ and $\overline{v} \in (\operatorname{ran}\alpha)\delta_\beta = \operatorname{ran}\delta_\beta = \operatorname{ran}\beta$. Then the above arguments imply that $\alpha = \sigma_\alpha\beta\sigma_\beta^{-1}$ and hence by Theorem 3(iv) we get that $\alpha\mathscr{D}\beta$ in $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$.

If case (2) holds then we have that

$$\alpha = \gamma_\alpha\gamma_\beta\alpha\delta_\beta\delta_\alpha = (\gamma_\alpha\gamma_\beta)^2\alpha(\delta_\beta\delta_\alpha)^2 = \ldots = (\gamma_\alpha\gamma_\beta)^6\alpha(\delta_\beta\delta_\alpha)^6$$

and

$$\beta = \gamma_\beta\gamma_\alpha\beta\delta_\alpha\delta_\beta = (\gamma_\beta\gamma_\alpha)^2\beta(\delta_\alpha\delta_\beta)^2 = \ldots = (\gamma_\beta\gamma_\alpha)^6\beta(\delta_\alpha\delta_\beta)^6.$$

We put

$$\gamma_\beta^\circ = \gamma_\beta(\gamma_\alpha\gamma_\beta)^5 \qquad \text{and} \qquad \delta_\beta^\circ = \delta_\beta(\delta_\alpha\delta_\beta)^5.$$

Lemma 7(i) implies that $(\gamma_\alpha\gamma_\beta^\circ)\colon \operatorname{dom}\alpha \rightharpoonup \mathbb{N}^3$ and $(\delta_\beta^\circ\delta_\alpha)\colon \operatorname{ran}\alpha \rightharpoonup \mathbb{N}^3$ are identity partial maps. Now by Lemma 8 there exist elements σ_α and σ_β of the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ such that $(\overline{x})\gamma_\alpha = (\overline{x})\sigma_\alpha$, $(\overline{y})\gamma_\beta^\circ = (\overline{y})\sigma_\alpha^{-1}$, $(\overline{u})\delta_\beta^\circ = (\overline{u})\sigma_\beta$ and $(\overline{v})\delta_\alpha = (\overline{v})\sigma_\beta^{-1}$, for all $\overline{x} \in \operatorname{dom}\alpha$, $\overline{y} \in (\operatorname{dom}\alpha)\gamma_\alpha = \operatorname{ran}\gamma_\alpha = \operatorname{dom}\beta$, $\overline{u} \in \operatorname{ran}\alpha$ and $\overline{v} \in (\operatorname{ran}\alpha)\delta_\beta^\circ = \operatorname{ran}\delta_\beta^\circ = \operatorname{ran}\beta$. Then the above arguments imply that $\alpha = \sigma_\alpha\beta\sigma_\beta^{-1}$ and hence by Theorem 3(iv) we get that $\alpha\mathscr{D}\beta$ in $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$. □

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МОНОЇД МОНОТОННИХ ІН'ЄКТИВНИХ ЧАСТКОВИХ ПЕРЕТВОРЕНЬ ЧАСТКОВО ВПОРЯДКОВАНОЇ МНОЖИНИ $(\mathbb{N}^3, \leqslant)$ З КОСКІНЧЕННИМИ ОБЛАСТЯМИ ВИЗНАЧЕННЯ ТА ЗНАЧЕНЬ

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Нехай n – натуральне число $\geqslant 2$ і $\mathbb{N}^n_{\leqslant}$ — n-ий степінь множини натуральних чисел $\mathbb{N}$ з частковим порядком добутку звичайного лінійного порядку на $\mathbb{N}$.
Часткове перетворення $\alpha\colon X_{\leqslant} \rightharpoonup X_{\leqslant}$ частково впорядкованої множини $X_{\leqslant}$ називається *монотонним*, якщо з $x \leqslant y$ випливає нерівність $x\alpha \leqslant y\alpha$, для $x, y \in X_{\leqslant}$.
Досліджено структурні властивості моноїда $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ часткових монотонних перетворень частково впорядкованої множини $\mathbb{N}^n_{\leqslant}$ з коскінченними областями визначення та значень. Доведено, що група одиниць $H(\mathbb{I})$ напівгрупи $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ ізоморфна групі $\mathscr{S}_n$ підстановок n-елементної множини та описано піднапівгрупу ідемпотентів напівгрупи $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$. Також, у випадку $n = 3$ описано властивості елементів напівгрупи $\mathscr{PO}_{\infty}(\mathbb{N}^3_{\leqslant})$ як часткових бієкцій частково впорядкованої множини $\mathbb{N}^3_{\leqslant}$, і відношення Ґріна на напівгрупі $\mathscr{PO}_{\infty}(\mathbb{N}^3_{\leqslant})$. Зокрема доведено, шо відношення Ґріна $\mathscr{D}$ і $\mathscr{J}$ на моноїді $\mathscr{PO}_{\infty}(\mathbb{N}^3_{\leqslant})$ збігаються.

Ключові слова: напівгрупа часткових бієкцій, монотонне часткове відображення, ідемпотент, відношення Ґріна.