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INVERSE SOURCE PROBLEM FOR SEMILINEAR TIME FRACTIONAL EQUATION

Halyna LOPUSHANSKA, Olga MYAUS

Ivan Franko National University of Lviv,
Universitetska Str., 1, 79000, Lviv, Ukraine
e-mail: lhp@ukr.net
Lviv Polytechnic National University,
Bandera Str., 12, 79013, Lviv, Ukraine
e-mail: myausolya2016@gmail.com

We study the inverse problem of the restoration dependent on time continuous factor in the right-hand side of a semilinear time fractional equation. We find sufficient conditions of uniqueness of both classical and generalized solution under integral type over-determination condition.

Key words: fractional derivative, inverse problem, over-determination condition, Green vector-function, integral equation.

Equations with fractional derivatives and inverse problems for them arise in many branches of science and engineering with memory being taken into account. Some inverse problems to diffusion-wave equations with different unknown functions or parameters (source, order of partial derivative, older or minor coefficient, boundary or initial data) were investigated, for example, in [1]-[11]. In particular, in [1, 3, 4, 6, 7] integral type over-determination conditions were used in the inverse source and coefficient problems for such equations.

In this paper, we find the conditions of uniqueness of the solution (u, g) of the inverse problem

$$D_t^{\alpha} u - \Delta u = g(t) F_0(x,t,u), \quad (x,t) \in \Omega \times (0,T], \tag{1}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T],$$

$$\begin{gathered} u(x,0) = F_1(x), \quad x \in \bar{\Omega}, \\ \int_{\Omega} u(x,t)\varphi_0(x)dx = F_2(t), \; t \in [0,T] \end{gathered} \tag{2}$$

2010 Mathematics Subject Classification: 35S15 with the Caputo derivative $D_t^\alpha u$ of order $\alpha \in (0,1)$ in the case of regular data and the inverse problem

$$\begin{gathered} u_t^{(\alpha)} - \Delta u = g(t)F_0(x,t,u), \quad (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = F_1(x), \;\; x \in \Omega, \end{gathered} \tag{3}$$

$$(u(\cdot,t), \varphi_0(\cdot)) = F_2(t), \quad t \in [0,T] \tag{4}$$

with the Riemann-Liouville derivative $u_t^{(\alpha)}$ in the case of given distribution F_1. Here Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$ ($n \geq 2$) with the boundary $\partial\Omega$ of the class C^∞, $(u(\cdot,t), \varphi_0(\cdot))$ stands for the value of an unknown distribution u on a given test function φ_0 for every $t \in [0,T]$.

We note that inverse problems for semilinear parabolic and ultraparabolic equations with derivatives of the integer orders were investigated, for example, in [12, 13]. The coefficient inverse problem for a semilinear time fractional telegraph equation in the case of regular data was studied in [6].

We shall use the method of Green's functions [14]-[18].

1. Uniqueness of the classical solution

Let $Q = \Omega \times (0,T]$, $f * g$ be the convolution of functions f and g,

$$f_\lambda(t) = \frac{\theta(t) t^{\lambda-1}}{\Gamma(\lambda)} \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad f_\lambda(t) = f'_{1+\lambda}(t) \quad \text{for} \quad \lambda \leqslant 0,$$

where $\Gamma(t)$ is the Gamma-function, $\theta(t)$ is the Heaviside function. Note that

$$f_\lambda * f_\mu = f_{\lambda+\mu},$$

the Riemann-Liouville derivative $v_t^{(\alpha)}(x,t)$ of order $\alpha > 0$ is defined by the formula

$$v_t^{(\alpha)}(x,t) = f_{-\alpha}(t) * v(x,t),$$

the Caputo fractional derivative of order $\alpha \in (0,1)$ is defined as follows [19, 20]

$$D_t^\alpha v(x,t) = \frac{1}{\Gamma(1-\alpha)} \Big[\frac{\partial}{\partial t} \int\limits_0^t \frac{v(x,\tau)}{(t-\tau)^\alpha} d\tau - \frac{v(x,0)}{t^\alpha} \Big] = v_t^{(\alpha)}(x,t) - f_{1-\alpha}(t) v(x,0).$$

Let $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_j \in \mathbb{Z}_+$, $j = 1, 2, \ldots, n$, $|\beta| = \beta_1 + \cdots + \beta_n$, $\gamma \in (0,1)$, $D^\beta \varphi(x) = \dfrac{\partial^{|\beta|} \varphi(x)}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}$, $C^\gamma(\Omega)$ ($C^\gamma(Q)$) be the space of bounded continuous functions on Ω (Q, respectively) satisfying the Hölder continuity condition (Hölder continuity condition with respect to space variables), $C^\gamma(\bar{\Omega}) = C^\gamma(\Omega) \cap C(\bar{\Omega})$,

$$C^{m+\gamma}(\Omega) = \left\{ \varphi \in C^m(\Omega) \colon D^\beta \varphi \in C^\gamma(\Omega) \text{ for } |\beta| = m \right\},$$

$$C^{m+\gamma}(Q) = \left\{ \varphi \in C^m(Q) \colon D^\beta \varphi \in C^\gamma(Q) \text{ for } |\beta| = m \right\}, \; m \in \mathbb{N},$$

$$C^{2,\alpha}(Q) = \left\{ v \in C^{2+\gamma}(Q) : D_t^\alpha v \in C^\gamma(Q) \right\}, \quad C^{2,\alpha}(\bar{Q}) = C^{2,\alpha}(Q) \cap C(\bar{Q}).$$

Definition 1. A pair of functions $(u, g) \in C^{2,\alpha}(\bar{Q}) \times C[0, T]$ satisfying the equation (1) on Q and the conditions (2) is called a *classical solution* of problem (1), (2).

The definition implies the consistency conditions

$$F_1|_{\partial\Omega} = 0, \quad \int\limits_{\Omega} F_1(x)\varphi_0(x)dx = F_2(0).$$

Definition 2. A vector-function $(G_0(x, t, y, \tau), G_1(x, t, y))$ is called the Green's vector-function of the problem

$$D_t^{\alpha} u(x, t) - \Delta u(x, t) = g_0(x, t), \quad (x, t) \in Q, \tag{5}$$

$$u|_{\partial\Omega\times[0,T]} = 0, \quad u(x, 0) = g_1(x), \quad x \in \bar{\Omega} \tag{6}$$

if under smooth finite g_0, g_1 the function

$$u(x, t) = \int\limits_0^t d\tau \int\limits_{\Omega} G_0(x, t, y, \tau)g_0(y, \tau)dy + \int\limits_{\Omega} G_1(x, t, y)g_1(y)dy, \quad (x, t) \in \overline{Q} \tag{7}$$

is the classical solution (in $C^{2,\alpha}(\bar{Q})$) of this problem.

It is known (see, for example, [15]) that such Green's vector-function exists and for bounded $g_0 \in C^{\gamma}(Q)$, $g_1 \in C^{\gamma}(\Omega)$ the unique solution $u \in C^{2,\alpha}(\bar{Q})$ of problem (5), (6) exists. This solution is defined by (7).

We pass to the inverse problem (1), (2).

Theorem 1. *Assume that* $\alpha \in (0, 1)$, $\varphi_0 \in C^{2+\gamma}(\bar{\Omega})$, $\varphi_0|_{\partial\Omega} = 0$, $F_0 \in C^{1+\gamma}(Q \times \mathbb{R})$,

$$R_v(t) = \int\limits_{\Omega} F_0(x, t, v(x, t))\varphi_0(x)dx \neq 0, \ \ \forall t \in [0, T], \ v \in C^{\gamma}(\bar{Q}). \tag{8}$$

Then the solution $(u, g) \in C^{2,\alpha}(\bar{Q}) \times C[0, T]$ *of problem* (1), (2) *is unique.*

Proof. Take two solutions $(u_1, g_1), (u_2, g_2) \in C^{2,\alpha}(\bar{Q}) \times C[0, T]$ of problem (1), (2) and substitute them into equation (1). Denoting $u = u_1 - u_2$, $g = g_1 - g_2$ we obtain the equation

$$D_t^{\alpha} u - \Delta u = g_1(t)F_0(x, t, u_1) - g_2(t)F_0(x, t, u_2).$$

By the Hadamard lemma

$$F_0(x, t, u_1) - F_0(x, t, u_2) = F_{01}(x, t, u_1, u_2)u(x, t)$$

with some known, dependent on u_1, u_2, function $F_{01} \in C^{\gamma}(Q)$. Then the previous equation gets a form

$$D_t^{\alpha} u - \Delta u = g_2(t)F_{01}(x, t, u_1, u_2)u + g(t)F_0(x, t, u_1(x, t)), \quad (x, t) \in Q. \tag{9}$$

It follows from (2) that

$$u|_{\partial\Omega\times[0,T]} = 0, \quad u(x, 0) = 0, \quad x \in \bar{\Omega}, \tag{10}$$

$$\int\limits_{\Omega} u(x, t)\varphi_0(x)dx = 0, \ t \in [0, T]. \tag{11}$$

Since

$$\int\limits_\Omega D_t^\alpha u(x,t)\varphi_0(x)dx = D_t^\alpha \int\limits_\Omega u(x,t)\varphi_0(x)dx = 0,$$

equation (9) and the over-determination condition (11) imply

$$\int\limits_\Omega u(x,t)\Delta\varphi_0(x)dx =$$

$$= g(t)\int\limits_\Omega F_0(x,t,u_1(x,t))\varphi_0(x)dx + g_2(t)\int\limits_\Omega F_{01}(x,t,u_1,u_2)u(x,t)\varphi_0(x)dx, \quad t\in[0,T].$$

From here, using assumption (8), we find

$$g(t)=\frac{1}{R_{u_1}(t)}\Big[\int\limits_\Omega u(z,t)\Delta\varphi_0(z)dz - g_2(t)\int\limits_\Omega F_{01}(z,t,u_1,u_2)u(z,t)\varphi_0(z)dz\Big], \ t\in[0,T]. \quad (12)$$

As in [16] and [6] we get that for each $g\in C[0,T]$ the function u is the solution of problem (9), (10) if and only if it satisfies (in $C^\gamma(Q)$) the integral equation

$$u(x,t) = \int\limits_0^t g_2(\tau)d\tau\int\limits_\Omega G_0(x,t,y,\tau)F_{01}(y,\tau,u_1,u_2)u(y,\tau)dy+$$
$$+\int\limits_0^t g(\tau)d\tau\int\limits_\Omega G_0(x,t,y,\tau)F_0(y,\tau,u_1(y,\tau))dy, \quad (x,t)\in\bar\Omega. \quad (13)$$

Substituting expression (12) for $g(t)$ in (13) we obtain the integral equation

$$u(x,t) = \int\limits_0^t d\tau\int\limits_\Omega\Bigg[\frac{K(x,t,\tau)d\tau}{R_{u_1}(\tau)}\Big(\Delta\varphi_0(z) - g_2(\tau)F_{01}(z,\tau,u_1,u_2)\varphi_0(z)\Big)+$$
$$+g_2(\tau)G_0(x,t,z,\tau)F_{01}(z,\tau,u_1,u_2)\Bigg]u(z,\tau)dz, \quad (x,t)\in\bar Q \quad (14)$$

where

$$K(x,t,\tau) = \int\limits_\Omega G_0(x,t,y,\tau)F_0(y,\tau,u_1(y,\tau))dy$$

is the known function. It follows from [15] that

$$|G_0(x,t,y,\tau)| \le \frac{C_0|x-y|^{2-n}}{t-\tau}e^{-c_0\left(\frac{|x-y|^2}{(t-\tau)^\beta}\right)^{\frac{1}{2-\beta}}}, \ (x,t),(y,\tau)\in Q, \ (x,t)\ne(y,\tau),$$

and from [21], for example, that $K(x,t,\tau)$ is the continuous function in $x\in\Omega$ and

$$|K(x,t,\tau)| \le C(t-\tau)^{\alpha-1}, \quad x\in\Omega, \ 0\le\tau<t\le T.$$

Hereinafter $C_0, c_0, C, \widehat{C}, C_i$ $(i\in\mathbb{N})$ are positive constants.

We get the homogeneous linear second type Volterra integral equation (14) with integrable kernel which has the unique solution $u(x,t)=0$, $(x,t)\in\bar Q$. Then from (12) we obtain $g(t)=0$, $t\in[0,T]$. □

2. Uniqueness of the generalized solution

Let

$$C^{m,(0)}[0,T] = \left\{\eta \in C^m[0,T] \colon \eta^{(k)}(T) = 0,\, k = 0, 1, \ldots, m\right\},$$
$$\mathcal{D}[0,T] = C^{\infty,(0)}[0,T], \qquad \mathcal{D}(\bar\Omega) = C^\infty(\bar\Omega),$$
$$\mathcal{D}(\bar Q) = \left\{\psi \in C^\infty(\bar Q) \colon \psi(x,\cdot) \in C^{\infty,(0)}[0,T]\right\}.$$

We denote by E' the space of linear continuous functionals (distributions) on E. The symbol (f,φ) stands for the value of the distribution f on the test function φ.

In [21], the linear problem

$$\begin{gathered} u^{(\alpha)}(x,t) - \Delta u(x,t) = g_0(x,t), \quad (x,t) \in Q, \\ u|_{\partial\Omega\times[0,T]} = 0, \quad u(x,0) = F_1(x), \quad x \in \bar\Omega \end{gathered} \tag{15}$$

was studied in the case $g_0 \in \mathcal{D}'(\bar Q)$, $F_1 \in \mathcal{D}'(\bar\Omega)$. The existence of its unique solution $u \in \mathcal{D}'(\bar Q)$ was established. It was shown that this solution may be given by

$$\big(u,\varphi\big) = \big(g_0(\cdot,\tau), (\widehat{\mathcal{G}}_0\varphi)(\cdot,\tau)\big) + \big(F_1(\cdot), (\widehat{\mathcal{G}}_1\varphi)(\cdot)\big) \quad \forall \varphi \in \mathcal{D}(\bar Q)$$

where

$$(\widehat{\mathcal{G}}_0\varphi)(y,\tau) = \int\limits_\tau^T dt \int\limits_\Omega G_0(x,t,y,\tau)\varphi(x,t)dx,$$
$$(\widehat{\mathcal{G}}_1\varphi)(y) = \int\limits_0^T dt \int\limits_\Omega G_1(x,t,y)\varphi(x,t)dx, \quad y \in \bar\Omega, \tau \in [0,T].$$

As in [22, p. 209], for arbitrary $u \in \mathcal{D}'(\bar Q)$, $\varphi \in \mathcal{D}(\bar\Omega)$ we use the distribution $\big(u(x,\cdot),\varphi(x)\big) \in \mathcal{D}'[0,T]$ acting by the formula

$$\big((u(x,t),\varphi(x)),\eta(t)\big) = \big(u(x,t),\varphi(x)\eta(t)\big) \quad \forall \eta \in \mathcal{D}[0,T]$$

and say that the distribution $u \in \mathcal{D}'(\bar Q)$ is continuous at the variable $t \in [0,T]$ if $\big(u(x,\cdot),\varphi(x)\big) \in C[0,T]$ for each test function φ. We introduce the space of such distributions

$$\mathcal{D}'_C(\bar Q) = \left\{v \in \mathcal{D}'(\bar Q) \colon (v(\cdot,t),\varphi(\cdot)) \in C[0,T] \quad \forall \varphi \in \mathcal{D}(\bar\Omega)\right\}.$$

In [23], for the linear problem (15) with $g_0(x,t) = g(t)F(x)$, $g \in C[0,T]$, $F, F_1 \in \mathcal{D}'(\bar\Omega)$ the existence and uniqueness of the solution $u \in \mathcal{D}'_C(\bar Q)$ was proved. This solution has the representation

$$\big(u(\cdot,t),\varphi(\cdot)\big) = \int_0^t g(\tau)\left(F(y), \int_\Omega G_0(x,t,y,\tau)\varphi(x)dx\right)d\tau +$$
$$+\left(F_1(y), \int_\Omega G_1(x,t,y)\varphi(x)dx\right) \qquad \forall \varphi \in \mathcal{D}(\bar\Omega).$$

Let $\mathcal{Z}(\bar\Omega) \subset C^\gamma(\bar\Omega)$ be the Banach space with the norm $||\cdot||_{\mathcal{Z}(\bar\Omega)}$,

$$\mathcal{Z}_0(\bar\Omega) = \left\{\psi \in \mathcal{Z}(\bar\Omega) \colon \psi|_{\partial\Omega} = 0\right\},$$

$$\mathcal{Z}(\bar{Q}) = \left\{\psi \in C^{\gamma}(\bar{Q})\colon \psi(\cdot,t) \in \mathcal{Z}(\bar{\Omega}) \ \ \forall t \in [0,T]\right\},$$

$$\mathcal{X}(\bar{Q}) = \left\{\psi \in C^{2,\alpha}(\bar{Q})\colon \widehat{L}\psi \in \mathcal{Z}(\bar{Q}), \ \ \psi|_{\partial\Omega\times[0,T]} = 0\right\},$$

$$\mathcal{M} = \mathcal{M}(Q) = \mathcal{Z}'_C(\bar{Q}) = \{v \in \mathcal{Z}'(\bar{Q})\colon (v(\cdot,t),\varphi(\cdot)) \in C[0,T] \quad \forall \varphi \in \mathcal{Z}(\bar{\Omega})\},$$

where

$$\widehat{L}\psi(x,t) = f_{-\alpha}(t)\hat{*}\psi(x,t) - \Delta\psi(x,t), \quad (x,t) \in Q,$$

$$h(t)\hat{*}\varphi(t) = \big(h(s),\varphi(s+t)\big), \quad t \in [0,T].$$

We consider inverse problem (3), (4) in the case $F_1 \in \mathcal{Z}'(\bar{\Omega})$, $F_2 \in C[0,T]$. Note that the solution u of the corresponding direct problem (3) may have the singularity at $t = 0$.

Definition 3. A pair $(u,g) \in \mathcal{M}(Q) \times C[0,T]$ is called a generalized solution of problem (3), (4) if the identity

$$\Big(u(x,t),\widehat{L}\psi(x,t)\Big) = \int_0^T g(t)\Big(F_0(\cdot,t,u(\cdot,t)),\psi(\cdot,t)\Big)dt + \\ +\Big(F_1(\cdot),\int_0^T f_{1-\alpha}(t)\psi(\cdot,t)dt\Big) \quad \forall \psi \in \mathcal{X}(\bar{Q}) \tag{16}$$

and condition (4) hold.

The necessary conditions

$$(F_1,\varphi_0) = F_2(0)$$

and

$$\big(F_0(x,\cdot,u(x,\cdot)),\psi(x,\cdot)\big) \in L_1(0,T) \ \ \forall\, \psi \in \mathcal{X}(\bar{Q}) \tag{17}$$

follow from the definition.

Assumption (A): $\varphi_0 \in \mathcal{Z}_0(\bar{\Omega})$, $\Delta\varphi_0 \in \mathcal{Z}(\bar{\Omega})$,

$$\big(F_0(x,\cdot,v(x,\cdot)),\varphi(x)\big) \in C[0,T] \quad \forall \varphi \in \mathcal{Z}(\bar{\Omega}), \ v \in \mathcal{M}(Q),$$

$$R_v(t) := \big(F_0(\cdot,t,v(\cdot,t)),\varphi_0(\cdot)\big) \neq 0 \quad \forall\, t \in [0,T], \ v \in \mathcal{M}(Q).$$

Let $F(x,t,z_1,z_2) = F_0(x,t,z_1) - F_0(x,t,z_2)$, $(x,t,z_1),(x,t,z_2) \in Q \times \mathbb{R}$.

It follows from (A) that

$$\big(F(x,\cdot,v_1,v_2),\varphi(x)\big) \in C[0,T] \ \ \forall\, v_1,v_2 \in \mathcal{M}(Q), \ \varphi \in \mathcal{Z}(\bar{\Omega}). \tag{18}$$

Take two solutions $(u_1,g_1),(u_2,g_2) \in \mathcal{M}(Q) \times C[0,T]$ of problem (3), (4). Denoting $u = u_1 - u_2$, $g = g_1 - g_2$, we obtain

$$\Big(u(x,t),\widehat{L}\psi(x,t)\Big) = \int_0^T g(t)\Big(F_0(\cdot,t,u_1(\cdot,t)),\psi(\cdot,t)\Big)dt + \\ + \int_0^T g_2(t)\Big(F(\cdot,t,u_1,u_2),\psi(\cdot,t)\Big)dt \quad \forall \psi \in \mathcal{X}(\bar{Q}), \tag{19}$$

and that

$$\big(u(\cdot,t),\varphi_0(\cdot)\big)=0,\quad t\in[0,T]. \tag{20}$$

As in [21] we prove that u satisfies identity (19) if and only if it satisfies the equation

$$\begin{gathered}(u,\varphi)=\int\limits_0^T g(\tau)\Big(F_0(\cdot,\tau,u_1(\cdot,\tau)),(\widehat{\mathcal{G}_0}\varphi)(\cdot,\tau)\Big)d\tau+\\+\int\limits_0^T g_2(\tau)\Big(F(\cdot,\tau,u_1,u_2),(\widehat{\mathcal{G}_0}\varphi)(\cdot,\tau)\Big)d\tau\quad\forall\varphi\in\mathcal{Z}(\bar Q).\end{gathered} \tag{21}$$

By the results in [21], $\widehat{\mathcal{G}_0}\varphi\in\mathcal{X}(\bar Q)$ for all $\varphi\in\mathcal{Z}(\bar Q)$. Therefore, taking (17) into account, we get that the right-hand side of (21) exists.

In particular, in the case $\varphi(x,t)=\widehat\varphi(x)\eta(t)$ with $\widehat\varphi\in\mathcal{Z}(\bar\Omega)$, $\eta\in C^{1,(0)}[0,T]$, formula (21) gets a form

$$\begin{gathered}\int\limits_0^T\big(u(\cdot,t),\widehat\varphi(\cdot)\big)\eta(t)dt=\int\limits_0^T g(\tau)\left(F_0(\cdot,\tau,u_1(\cdot,\tau)),\int\limits_\tau^T\eta(t)dt\int\limits_\Omega G_0(x,t,\cdot,\tau)\widehat\varphi(x)dx\right)d\tau+\\+\int\limits_0^T g(\tau)\left(F(\cdot,\tau,u_1,u_2),\int\limits_\tau^T\eta(t)dt\int\limits_\Omega G_0(x,t,\cdot,\tau)\widehat\varphi(x)dx\right)d\tau,\end{gathered}$$

that is

$$\begin{gathered}\int\limits_0^T\big(u(\cdot,t),\widehat\varphi(\cdot)\big)\eta(t)dt=\int\limits_0^T\eta(t)dt\int\limits_0^t g(\tau)\left(F_0(\cdot,\tau,u_1(\cdot,\tau)),\int\limits_\Omega G_0(x,t,\cdot,\tau)\widehat\varphi(x)dx\right)d\tau+\\+\int\limits_0^T\eta(t)dt\int\limits_0^t g_2(\tau)\left(F(\cdot,\tau,u_1,u_2),\int\limits_\Omega G_0(x,t,\cdot,\tau)\widehat\varphi(x)dx\right)d\tau.\end{gathered}$$

By arbitrariness of η and previous reasoning this equation may be written as follows

$$\begin{gathered}\big(u(\cdot,t),\varphi(\cdot)\big)=\int\limits_0^t g(\tau)\left(F_0(\cdot,\tau,u_1(\cdot,\tau)),\int\limits_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx\right)d\tau+\\+\int\limits_0^t g_2(\tau)\left(F(\cdot,\tau,u_1,u_2),\int\limits_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx\right)d\tau\ \ \forall t\in[0,T],\ \varphi\in\mathcal{Z}(\bar\Omega).\end{gathered} \tag{22}$$

As in [23] we prove that the solution of this equation satisfies identity (19) and

$$\big(D_t^\alpha u-\Delta u,\varphi\big)=\big(g_2(t)F(x,t,u_1,u_2)+g(t)F_0(x,t,u_1),\varphi(x)\big)\quad\forall\varphi\in\mathcal{Z}(\bar\Omega).$$

From this equation and over-determination condition (20) we get

$$\begin{gathered}\big(u(\cdot,t),\Delta\varphi_0(\cdot)\big)+g(t)\big(F_0(y,t,u_1(y,t)),\varphi_0(y)\big)+\\+g_2(t)\big(F(z,t,u_1,u_2),\varphi_0(z)\big)=0,\ \ t\in[0,T]\end{gathered}$$

and from here find

$$g(t) = -\frac{1}{R_{u_1}(t)}\Big[\big(u(\cdot,t),\Delta\varphi_0(\cdot)\big) + g_2(t)\big(F(z,t,u_1,u_2),\varphi_0(z)\big)\Big],\ \ t\in[0,T]. \tag{23}$$

Note that according to (A) and (18), $R_{u_1}\in C[0,T]$, $R_{u_1}(t)\neq 0$, $t\in[0,T]$, the function $\big(F(\cdot,t,u_1,u_2),\varphi_0(\cdot)\big)$ exists and belongs to $C[0,T]$.

Substituting the expression for g in (22) we obtain the equation

$$\begin{aligned}\big(u(\cdot,t),\varphi(\cdot)\big) = &-\int_0^t \frac{1}{R_{u_1}(\tau)}\Big[\big(u(\cdot,\tau),\Delta\varphi_0(\cdot)\big) + g_2(\tau)\big(F(\cdot,\tau,u_1,u_2),\varphi_0(\cdot)\big)\Big]\times\\ &\times\left(F_0(\cdot,\tau,u_1(\cdot,\tau)),\int_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx\right)d\tau + \\ &+\int_0^t g_2(\tau)\left(F(\cdot,\tau,u_1,u_2),\int_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx\right)d\tau\ \ \forall t\in[0,T],\ \varphi\in\mathcal{Z}(\bar\Omega)\end{aligned} \tag{24}$$

and that under the assumption (A) the pair $(u,g)\in\mathcal{M}(Q)\times C[0,T]$ is the solution of problem (19), (20) if and only if u satisfies equation (24), g is defined by (23).

From the results in [15, 21] we get that

$$\int_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx \in \mathcal{Z}_0(\bar\Omega)\cap C^{2+\gamma}(\bar\Omega)\quad \forall\varphi\in\mathcal{Z}(\bar\Omega),\ \ 0\le\tau<t\le T,$$

$$\Big|\int_\Omega G_0(x,t,y,\tau)\varphi(x)dx)\Big| \le \widehat{C}(t-\tau)^{\alpha-1}\,||\varphi||_{\mathcal{Z}(\bar\Omega)},\quad y\in\bar\Omega,\ 0\le\tau<t\le T. \tag{25}$$

Then, according to (A) and (18),

$$\left(F_0(\cdot,\tau,v(\cdot,\tau)),\int_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx\right)$$

and

$$\left(F(\cdot,\tau,v_1,v_2),\int_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx\right)$$

exist for all $\varphi\in\mathcal{Z}(\bar\Omega)$, $v,v_1,v_2\in\mathcal{M}(Q)$, $0\le\tau<t\le T$ and are integrable in t and τ. Therefore, under the assumption (A) the right-hand side of equation (24) exists and belongs to $C[0,T]$.

Assumption (B): $\varphi_0 \in \mathcal{Z}_0(\bar{\Omega})$, $\Delta\varphi_0 \in \mathcal{Z}(\bar{\Omega})$,

$$\exists \int_\Omega |F_0(x,\cdot,v(x,\cdot))|dx \in C[0,T] \quad \forall v \in \mathcal{M}(Q),$$

$$R_v(t) := \int_\Omega F_0(x,t,v(x,t))\varphi_0(x)dx \neq 0 \quad \forall\, t \in [0,T],\ v \in \mathcal{M}(Q),$$

$$\int_0^t (t-\tau)^{\alpha-1}d\tau \int_\Omega |F_0(y,\tau,v(y,\tau))|dy \text{ is monotonously increasing,}$$

$$\exists\, L = \text{const} > 0 \text{ such that}$$

$$|\big(F_0(\cdot,t,v_1(\cdot,t)) - F_0(\cdot,t,v_2(\cdot,t)),\varphi(\cdot)\big)| \le L|\big(v_1(\cdot,t) - v_2(\cdot,t),\varphi(\cdot)\big)|$$

$$\forall t \in (0,T],\quad v_1,v_2 \in \mathcal{M}(Q),\quad \varphi \in \mathcal{Z}_0(\bar{\Omega}).$$

Theorem 2. *Assume that* (B) *holds. Then there exists* $t_0 \in (0,T]$ *(*$Q_0 = \Omega \times (0,t_0]$*, respectively) such that the solution* $(u,g) \in \mathcal{M}(Q_0)\times C[0,t_0]$ *of problem* (3), (4) *is unique.*

Proof. Assumption (B) implies (A) with such F_0 and (17) for all $u \in \mathcal{M}(Q)$. Therefore, from the above, it is sufficient to prove that equation (24) has the unique solution $u = 0$ in $\mathcal{M}(Q_0)$ with some $t_0 \in (0,T]$. We denote

$$||v|| = ||v||_{\mathcal{M}} = \max_{t\in[0,T]} ||v(\cdot,t)||' < +\infty,$$

where $||v(\cdot,t)||' = \sup\limits_{\varphi\in\mathcal{Z}(\bar{\Omega})} \dfrac{\Big|\big(v(\cdot,t),\varphi(\cdot)\big)\Big|}{||\varphi||_{\mathcal{Z}(\bar{\Omega})}}$, and define the operator H on $\mathcal{M}(Q)$:

$$\big((Hv)(\cdot,t),\varphi(\cdot)\big) = -\int_0^t \frac{1}{R_{v_1}(\tau)}\Big[\big(v(\cdot,\tau),\Delta\varphi_0(\cdot)\big) + g_2(\tau)\big(F(z,\tau,v_1,v_2),\varphi_0(z)\big)\Big]\times$$
$$\times\left(F_0(\cdot,\tau,v_1(\cdot,\tau)),\int_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx\right)d\tau +$$
$$+\int_0^t g_2(\tau)\left(F(\cdot,\tau,v_1,v_2),\int_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx\right)d\tau$$
$$\forall t \in [0,T],\quad \varphi \in \mathcal{Z}(\bar{\Omega}),\quad v_1,v_2 \in \mathcal{M}(Q), \text{ and } v = v_1 - v_2.$$

By assumption (B) and properties of $G_0(x,t,y,\tau)$, in particular, (25), for all $\varphi \in \mathcal{Z}(\bar{\Omega})$ we get

$$\frac{\left|\left(F_0(\cdot,\tau,v_1(\cdot,\tau)),\int_\Omega G_0(x,t,\cdot,\tau)\varphi(x)dx\right)\right|}{||\varphi||_{\mathcal{Z}(\bar{\Omega})}} \le$$

$$\leq \frac{\int\limits_{\Omega} |F_0(y,\tau,v_1(y,\tau))| \Big[\int\limits_{\Omega} |G_0(x,t,y,\tau)\varphi(x)|dx\Big]dy}{||\varphi||_{\mathcal{Z}(\bar\Omega)}} \leq$$

$$\leq C_1(t-\tau)^{\alpha-1} \int\limits_{\Omega} |F_0(y,\tau,u_1(y,\tau))|dy,$$

$$\frac{\left|\left(F(z,\tau,v_1,v_2), \int\limits_{\Omega} G_0(x,t,y,\tau)\varphi(x)dx\right)\right|}{||\varphi||_{\mathcal{Z}(\bar\Omega)}} \leq$$

$$\leq \frac{L\left|\left(v(y,\tau), \int\limits_{\Omega} G_0(x,t,y,\tau)\varphi(x)dx\right)\right|}{||\varphi||_{\mathcal{Z}(\bar\Omega)}} \leq LC_1(t-\tau)^{\alpha-1}\frac{|\big(v(\cdot,\tau),\varphi(\cdot)\big)|}{||\varphi||_{\mathcal{Z}(\bar\Omega)}} \leq$$

$$\leq LC_1(t-\tau)^{\alpha-1}||v(\cdot,\tau)||', \quad 0 \leq \tau < t \leq T.$$

Then, denoting $a = \min\limits_{s\in[0,T]} R_{v_1}(s)$, we get

$$\frac{|\big((Hv)(\cdot,t),\varphi(\cdot)\big)|}{||\varphi||_{\mathcal{Z}(\bar\Omega)}} \leq \frac{C_1}{a||\varphi||_{\mathcal{Z}(\bar\Omega)}} \int\limits_0^t \Big[\frac{|\big(v(\cdot,\tau),\Delta\varphi_0(\cdot)\big)|}{||\Delta\varphi_0||_{\mathcal{Z}(\bar\Omega)}} ||\Delta\varphi_0||_{\mathcal{Z}(\bar\Omega)} +$$

$$+\frac{|g_2(\tau)|\,|\big(F(z,\tau,v_1,v_2),\varphi_0(z)\big)|}{||\varphi_0||_{\mathcal{Z}(\bar\Omega)}} ||\varphi_0||_{\mathcal{Z}(\bar\Omega)}\Big]\times$$

$$\times(t-\tau)^{\alpha-1}\int\limits_{\Omega} |F_0(y,\tau,v_1(y,\tau))|dy\,d\tau +$$

$$+L\int\limits_0^t |g_2(\tau)| \frac{|\big(v(\cdot,\tau), \int_{\Omega} G_0(x,t,\cdot,\tau)\varphi(x)dx\big)|}{||\varphi||_{\mathcal{Z}(\bar\Omega)}} d\tau \leq$$

$$\leq C_1 \int\limits_0^t (t-\tau)^{\alpha-1}\Big[\frac{||\Delta\varphi_0||_{\mathcal{Z}(\bar\Omega)} + ||\varphi_0||_{\mathcal{Z}(\bar\Omega)}}{a} \int\limits_{\Omega} |F_0(y,\tau,v_1(y,\tau))|dy +$$

$$+L \max_{\tau\in[0,T]} |g_2(\tau)|\Big] ||v(\cdot,\tau)||'\, d\tau \leq$$

$$\leq C_2 \int\limits_0^t (t-\tau)^{\alpha-1}\Big[1 + \int\limits_{\Omega} |F_0(y,\tau,v_1(y,\tau))|dy\Big]\, d\tau\; ||v||.$$

Under assumption (B) there exists $t_0 \in (0,T]$ such that

$$C_3 := C_2 \int\limits_0^t (t-\tau)^{\alpha-1}\Big[1 + \int\limits_{\Omega} |F_0(y,\tau,v_1(y,\tau))|dy\Big]\, d\tau < 1 \quad \forall t \in [0,t_0].$$

Therefore, we obtain

$$||Hv|| \leq C_3||v|| \quad \forall v \in \mathcal{M}(Q_0),$$

and similarly,

$$||Hv_1 - Hv_2|| \leq C_3||v_1 - v_2|| \quad \forall v_1, v_2 \in \mathcal{M}(Q_0)$$

with $C_3 < 1$. Therefore, $H : \mathcal{M}(Q_0) \to \mathcal{M}(Q_0)$ and is compressible. By the Banach fixed point theorem equation (24) has the unique solution $u = 0$ in $\mathcal{M}(Q_0)$. □

3. Conclusion

We found sufficient conditions of uniqueness of the inverse problem of the restoration a source term in a semilinear time fractional equation under integral type overdetermination condition. The cases of both regular and singular data in an initial condition was considered. The obtained result can be transferred to the case of more general equation with an elliptic second order differential expression having sufficiently regular, dependent on spatial variables coefficients.

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ОБЕРНЕНА ЗАДАЧА ДЛЯ НАПІВЛІНІЙНОГО РІВНЯННЯ З ДРОБОВОЮ ПОХІДНОЮ ЗА ЧАСОМ

Галина ЛОПУШАНСЬКА, Ольга М'ЯУС

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, 79000, Львів, Україна
e-mail: lhp@ukr.net
Національний університет "Львівська політехніка",
вул. Бандери, 12, 79013, Львів, Україна
e-mail: myausolya2016@gmail.com

Знаходимо достатні умови єдиності класичного та узагальненого розв'язку оберненої задачі відновлення залежного від часу неперервного множника у правій частині напівлінійного рівняння з дробовою похідною за часом. Використовуємо умову перевизначення інтегрального типу.

Ключові слова: похідна дробового порядку, обернена задача, умова перевизначення, вектор-функція Гріна, інтегральне рівняння.