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GROWTH ESTIMATES OF ANALYTIC FUNCTIONS IN $\mathbb{D}\times\mathbb{C}$ HAVING BOUNDED L-INDEX IN JOINT VARIABLES

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We prove some properties of power series expansion of analytic functions in $\mathbb{D}\times\mathbb{C}$ having bounded $\mathbf{L}$-index in joint variables, where $\mathbf{L}(z,w) = (l_1(z,w), l_2(z,w))$ with $l_j : \mathbb{D}\times\mathbb{C} \to \mathbb{R}_+$ $(j \in \{1,2\})$ are positive continuous functions and $l_1(z,w) > \beta/(1-|z|)$ for all $(z,w) \in \mathbb{D}\times\mathbb{C}$ and some $\beta > 1$. Moreover, we provide growth estimates of these function class. They describe the behavior of logarithm of maximum modulus of analytic function on a skeleton in a bidisc by behavior of the function $\mathbf{L}$. These estimates are sharp in a general case. The presented results are based on bidisc exhaustion of the Cartesian product of the unit disc and complex plane.

Key words: analytic function, bounded index in joint variables, unit disc, complex plane, Cartesian product, growth estimates.

1. Introduction

We need some standard notation. Denote $\mathbb{R}_+ = (0, +\infty)$, $\mathbf{0} = (0,0)$, $\mathbf{1} = (1,1)$, $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$. $R = (r_1, r_2) \in \mathbb{R}^2_+$, $(z,w) \in \mathbb{D}\times\mathbb{C}$. For $A = (a_1,a_2) \in \mathbb{R}^2$, $B = (b_1,b_2) \in \mathbb{R}^2$ we will use formal notations without violation of the existence of these expressions $AB = (a_1b_1, a_2b_2)$, $A/B = (a_1/b_1, a_2/b_2)$, $A^B = a_1^{b_1}a_2^{b_2}$. The notation $A < B$ means that $a_j < b_j$, $j \in \{1,2\}$; the relation $A \leqslant B$ is defined similarly. For $K = (k_1,k_2) \in \mathbb{Z}^2_+$ denote $\|K\| = k_1 + k_2$, $K! = k_1! \cdot k_2!$.

For $z \in \mathbb{C}^2$ and $w \in \mathbb{C}^2$ we define

$$\langle z, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2,$$

2010 Mathematics Subject Classification: 30D60, 32A10, 32A40 where $\overline{w}_k$ is the complex conjugate of w_k.

For $(z_0,w_0)\in\mathbb{D}\times\mathbb{C}$ we denote

$$\mathbb{D}^2((z_0,w_0),R):=\{(z,w)\in\mathbb{D}\times\mathbb{C}\colon\ |z-z_0|<r_1,\ |w-w_0|<r_2\}$$

the bidisc, its skeleton

$$\mathbb{T}^2((z_0,w_0),R):=\{(z,w)\in\mathbb{D}\times\mathbb{C}\colon\ |z-z_0|=r_1,\ |w-w_0|=r_2\}$$

and

$$\mathbb{D}^2[(z_0,w_0),R]:=\{(z,w)\in\mathbb{D}\times\mathbb{C}\colon\ |z-z_0|\le r_1,\ |w-w_0|\le r_2\}$$

the closed bidisc, $\mathbb{D}=\{z\in\mathbb{C}\colon\ |z|<1\}$. For $K=(k_1,k_2)\in\mathbb{Z}_+^2$, $(z,w)\in\mathbb{D}\times\mathbb{C}$ and the partial derivatives of function $F(z)=F(z,w)$ we use the notation

$$F^{(K)}(z,w)=\frac{\partial^{\|K\|}F(z,w)}{\partial z^{k_1}\partial w^{k_2}}=\frac{\partial^{k_1+k_2}F(z,w)}{\partial z^{k_1}\partial w^{k_2}}.$$

Let $\mathbf{L}(z,w)=(l_1(z,w),l_2(z,w))$, where $l_j(z)\colon\mathbb{D}\times\mathbb{C}\to\mathbb{R}_+$ are continuous functions ($j\in\{1,2\}$).

An analytic function $F\colon\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ is called ([1–3]) a function of *bounded* $\mathbf{L}$*-index (in joint variables),* if there exists $n_0\in\mathbb{Z}_+$ such that for all $(z,w)\in\mathbb{D}\times\mathbb{C}$ and for all $J\in\mathbb{Z}_+^2$

$$\frac{|F^{(J)}(z,w)|}{J!\mathbf{L}^J(z,w)}\leqslant\max_{\|K\|\leqslant n_0}\frac{|F^{(K)}(z,w)|}{K!\mathbf{L}^K(z,w)}. \tag{1}$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* F and is denoted by $N(F,\mathbf{L},\mathbb{D}\times\mathbb{C})=n_0$. It is an analog of the definition of an analytic function of bounded $\mathbf{L}$-index in joint variables (see definitions for various classes of analytic functions in [5, 7–9, 14, 15, 17]).

By $Q(\mathbb{D}\times\mathbb{C})$ we denote the class of functions $\mathbf{L}$ which satisfy the conditions

$$(\forall(z,w)\in\mathbb{D}\times\mathbb{C})\colon\ l_1(z,w)>\beta/(1-|z|), \tag{2}$$

$$\begin{gathered}(\forall r_1\in[0,\beta],\ \forall r_2\in(0,+\infty))\colon\\ 0<\lambda_{1,j}(R)\leqslant\lambda_{2,j}(R)<+\infty,\end{gathered} \tag{3}$$

where $\beta>1$ is some constant, and

$$\begin{aligned}\lambda_{1,j}((z_0,w_0),R)&=\inf_{(z,w)\in\mathbb{D}^2[(z_0,w_0),R/\mathbf{L}(z_0,w_0)]}l_j(z,w)/l_j(z_0,w_0)\\ \lambda_{2,j}((z_0,w_0),R)&=\sup_{(z,w)\in\mathbb{D}^2[(z_0,w_0),R/\mathbf{L}(z_0,w_0)]}l_j(z,w)/l_j(z_0,w_0),\\ \lambda_{1,j}(R)&=\inf_{(z_0,w_0)\in\mathbb{D}\times\mathbb{C}}\lambda_{1,j}((z_0,w_0),R),\\ \lambda_{2,j}(R)&=\sup_{(z_0,w_0)\in\mathbb{D}\times\mathbb{C}}\lambda_{2,j}((z_0,w_0),R),\ j\in\{1,2\}.\end{aligned}$$

A similar condition was used for other classes of analytic functions of bounded index as one so several variables [13, 18, 19].

For an analytic function $F(z)$ we put

$$M(R,z^0,F)=\max\{|F(z)|\colon z\in\mathbb{T}^2(z^0,R)\}.$$

The following theorems were obtained in [3].

Theorem 1 ([3]). *Let* $\mathbf{L} \in Q(\mathbb{D} \times \mathbb{C})$. *If an analytic function* $F : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ *has bounded* $\mathbf{L}$*-index in joint variables then for any* $R', R'' \in \mathcal{B}^2$, $R' < R''$, *there exists* $p_1 = p_1(R', R'') \geqslant 1$ *such that for each* $z^0 \in \mathbb{D} \times \mathbb{C}$

$$M\left(R''/\mathbf{L}(z^0), z^0, F\right) \leqslant p_1 M\left(R'/\mathbf{L}(z^0), z^0, F\right). \tag{4}$$

Theorem 2 ([3]). *Let* $\mathbf{L} \in Q(\mathbb{D} \times \mathbb{C})$. *An analytic function* $F : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ *has bounded* $\mathbf{L}$*-index in joint variables if and only if there exist* $p \in \mathbb{Z}_+$ *and* $c \in \mathbb{R}_+$ *such that for each* $z \in \mathbb{D} \times \mathbb{C}$ *the inequality*

$$\begin{aligned} &\max\left\{\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z) l_2^{j_2}(z)} : j_1 + j_2 = p+1\right\} \leq \\ &\leqslant c \max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z) l_2^{k_2}(z)} : k_1 + k_2 \leqslant p\right\} \end{aligned} \tag{5}$$

holds.

In view of Theorems 1 and 2 the authors wrote the following sentence: "Analogs of Theorems 1 and 2 were used [6, 16] to obtain growth estimates of analytic functions in the unit ball of bounded $\mathbf{L}$-index in joint variables and to deduce sufficient conditions of index boundedness for analytic solutions in the unit bidisc of some system of partial differential equations. It is natural to pose similar questions for functions analytic in the domain $\mathbb{D} \times \mathbb{C}$:"

Problem 1. *What are growth estimates analytic functions in* $\mathbb{D} \times \mathbb{C}$ *of bounded* $\mathbf{L}$*-index in joint variables?*

A complete answer to the question is our main goal in the paper. But in order to achieve this goal we will also study property of power series expansion of functions which are analytic in $\mathbb{D} \times \mathbb{C}$.

2. Properties of power series expansion of analytic functions in $\mathbb{D} \times \mathbb{C}$.

Let $(z_0, w_0) \in \mathbb{D} \times \mathbb{C}$. We develop an analytic function $F \colon \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ in the power series written in a diagonal form

$$F(z,w) = \sum_{k=0}^{\infty} p_k(z - z_0, w - w_0) = \sum_{k=0}^{\infty} \sum_{j_1+j_2=k} b_{j_1,j_2}(z - z^0)^{j_1}(w - w_0)^{j_2}, \tag{6}$$

where p_k are homogeneous polynomials of k-th degree, $b_J = b_{j_1,j_2} = \dfrac{F^{(J)}(z_0,w_0)}{J!}$. A polynomial p_{k_0}, $k_0 \in \mathbb{Z}_+$, is called a dominating polynomial in the power series expansion (6) on $\mathbb{T}^2((z_0, w_0), R)$ if for every $(z, w) \in \mathbb{T}^2((z_0, w_0), R)$ the inequality

$$\left|\sum_{k \neq k^0} p_k(z - z_0, w - w_0)\right| \leqslant \frac{1}{2} \max\{|b_{j_1,j_2}| r_1^{j_1} r_2^{j_2} : j_1 + j_2 = k^0\}$$

holds. Recently, there were obtained propositions on properties of the main polynomial for entire functions ([10]), for analytic function in a bidisc ([12]) and for analytic functions in the unit ball [6]. Below we will deduce analogs of these propositions for analytic functions in the Cartesian product of unit disc and complex plane.

Theorem 3. *Let $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$. If an analytic function $F\colon\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ has bounded $\mathbf{L}$-index in joint variables then there exists $p\in\mathbb{Z}_+$ that for all $d\in\Big(0;\frac{\beta}{\sqrt{2}}\Big]$ there exists $\eta(d)\in(0;d)$ such that for each $(z_0,w_0)\in\mathbb{D}\times\mathbb{C}$ and some $r=r(d,(z_0,w_0))\in(\eta(d),d)$, $k^0=k^0(d,(z_0,w_0))\leqslant p$ the polynomial p_{k^0} is a dominating polynomial in series* (6) *on $\mathbb{T}^2\Big((z_0,w_0),\frac{r\mathbf{1}}{\mathbf{L}(z_0,w_0)}\Big)$.*

Proof. Our proof is similar to the proof of the mentioned proposition from [6,10,12]. We repear considerations from those papers with some modifications. Let F be an analytic function of bounded $\mathbf{L}$-index in joint variables with $N=N(F,\mathbf{L},\mathbb{D}\times\mathbb{C})<+\infty$ and n_0 be the $\mathbf{L}$-index in joint variables at a point $(z_0,w_0)\in\mathbb{D}\times\mathbb{C}$, i.e. n_0 is the least number, for which inequality (1) holds at the point (z_0,w_0). Then for each $(z_0,w_0)\in\mathbb{D}\times\mathbb{C}$ $n_0\leqslant N$.

We put

$$a_J^*=\frac{|b_J|}{\mathbf{L}^J(z_0,w_0)}=\frac{|F^{(J)}(z_0,w_0)|}{J!\mathbf{L}^J(z^0,w_0)},$$
$$a_k=\max\{a_J^*\colon \|J\|=k\},\ c=2\{(N+3)!3!+(N+1)C_{N+1}^N\}.$$

Let $d\in\Big(0;\frac{\beta}{\sqrt{2}}\Big]$ be an arbitrary number. We also denote $r_m=\dfrac{d}{(d+1)c^m}$, $\mu_m=\max\{a_kr_m^k\colon k\in\mathbb{Z}_+\}$, $s_m=\min\{k\colon a_kr_m^k=\mu_m\}$ for $m\in\mathbb{Z}_+$.

Since $(z_0,w_0)\in\mathbb{D}\times\mathbb{C}$ is a fixed point the inequality $a_K^*\leqslant\max\{a_J^*\colon\ \|J\|\leqslant n_0\}$ is valid for all $K\in\mathbb{Z}_+^2$. Then $a_k\leqslant a_{n_0}$ for all $k\in\mathbb{Z}_+$. Hence, for all $k>n_0$, in view of $r_0<1$, we have $a_kr_0^k<a_{n_0}r_0^{n_0}$. This implies $s_0\leqslant n_0$. Since $cr_m=r_{m-1}$, we obtain that for each $k>s_{m-1}$ $(r_{m-1}<1)$

$$a_{s_{m-1}}r_m^{s_{m-1}}=a_{s_{m-1}}r_{m-1}^{s_{m-1}}c^{-s_{m-1}}\geqslant a_kr_{m-1}^kc^{-s_{m-1}}=a_kr_m^kc^{k-s_{m-1}}\geqslant ca_kr_m^k. \tag{7}$$

It yields that $s_m\leqslant s_{m-1}$ for all $m\in\mathbb{N}$. Thus, we can rewrite

$$\mu_0=\max\{a_kr_0^k\colon k\leqslant n_0\},\ \mu_m=\max\{a_kr_m^k\colon k\leqslant s_{m-1}\},\ m\in\mathbb{N}.$$

Let us introduce additional notations for $m\in\mathbb{N}$

$$\mu_0^*=\max\left\{a_kr_0^k\colon s_0\neq k\leqslant n_0\right\},\ s_0^*=\min\{k\colon k\neq s_0,a_kr_0^k=\mu_0^*\},$$
$$\mu_m^*=\max\{a_kr_m^k\colon s_m\neq k\leqslant s_{m-1}\},\ s_m^*=\min\{k\colon k\neq s_m,a_kr_m^k=\mu_m^*\}.$$

We will show that there exists $m_0\in\mathbb{Z}_+$ such that

$$\frac{\mu_{m_0}^*}{\mu_{m_0}}\leqslant\frac{1}{c}. \tag{8}$$

Suppose that for all $m\in\mathbb{Z}_+$ the inequality

$$\frac{\mu_m^*}{\mu_m}>\frac{1}{c} \tag{9}$$

holds. If $s^*_m < s_m$ ($s^*_m \neq s_m$ in view of definition) then we have

$$a_{s^*_m} r^{s^*_m}_{m+1} = \frac{a_{s^*_m} r^{s^*_m}_m}{c^{s^*_m}} = \frac{\mu^*_m}{c^{s^*_m}} > \frac{\mu_m}{c^{s^*_m+1}} = \frac{a_{s_m} r^{s_m}_m}{c^{s^*_m+1}} = \frac{a_{s_m} r^{s_m}_{m+1}}{c^{s^*_m+1-s_m}} \geqslant a_{s_m} r^{s_m}_{m+1}.$$

Besides, for every $k > s^*_m$, $k \neq s_m$, (i. e., $k-1 \geqslant s^*_m$) it can be deduced similarly that

$$a_{s^*_m} r^{s^*_m}_{m+1} = \frac{a_{s^*_m} r^{s^*_m}_m}{c^{s^*_m}} \geqslant \frac{a_k r^k_m}{c^{s^*_m}} \geqslant \frac{a_k r^k_m}{c^{k-1}} = c a_k r^k_{m+1}.$$

Hence, $a_{s^*_m} r^{s^*_m}_{m+1} > a_k r^k_{m+1}$ for all $k > s^*_m$. Then

$$s_{m+1} \leqslant s^*_m \leqslant s_m - 1. \tag{10}$$

On the contrary, if $s_m < s^*_m \leqslant s_{m-1}$, then the equality $s_{m+1} = s_m$ may hold. Indeed, by definition $s_{m+1} \leqslant s_m$. It means that the specified equality is possible. But if $s_{m+1} < s_m$ then $s_{m+1} \leqslant s_m - 1$ (they are natural numbers!). Hence, we obtain (10).

Thus, the inequalities $s^*_{m+1} \leqslant s_m$ and $s^*_m \neq s_{m+1}$ imply that $s^*_{m+1} < s_{m+1}$. As above instead of (10) we have

$$s_{m+2} \leqslant s^*_{m+1} \leqslant s_{m+1} - 1 = s_m - 1.$$

Therefore, if for all $m \in \mathbb{Z}_+$ (9) holds, then for every $m \in \mathbb{Z}_+$ either $s_{m+2} \leqslant s_{m+1} \leqslant s_m - 1$ or $s_{m+2} \leqslant s_m - 1$ holds, that is $s_{m+2} \leqslant s_m - 1$, because $s_{m+2} \leqslant s_{m+1}$. It follows that

$$s_m \leqslant s_{m-2} - 1 \leqslant \ldots \leqslant s_{m-2[m/2]} - [m/2] \leqslant s_0 - [m/2] \leqslant n_0 - [m/2] \leqslant N - [m/2].$$

In other words, $s_m < 0$ for $m > 2N+1$, which is impossible. Therefore, there exists $m_0 \leqslant 2N+1$ such that (8) holds. We put $r = r_{m_0}$, $\eta(d) = \frac{d}{(d+1)c^{2(N+1)}}$, $p = N$ and $k_0 = s_{m_0}$. Then for all $\|J\| \neq k_0 = s_{m_0}$ in $\mathbb{T}^2\Big((z_0, w_0), \frac{r_1}{\mathbf{L}(z_0,w_0)}\Big)$, in view (7) and (8) we obtain

$$|b_J||(z-z_0)^{j_1}(w-w_0)^{j_2}| = a^*_J r^{\|J\|} \leqslant a_{\|J\|} r^{\|J\|} \leqslant \frac{1}{c} a_{s_{m_0}} r^{s_{m_0}}_{m_0} = \frac{1}{c} a_{k_0} r^{k_0}.$$

Thus, for $(z,w) \in \mathbb{T}^2\Big((z_0, w_0), \frac{r\mathbf{1}}{\mathbf{L}(z_0,w_0)}\Big)$

$$\begin{aligned}\Bigg|\sum_{\|J\|\neq k_0} b_J (z-z_0)^{j_1}(w-w_0)^{j_2}\Bigg| &\leqslant \sum_{\|J\|\neq k_0} a^*_j r^{\|J\|} \leqslant \sum_{\substack{k=0,\\ k\neq k_0}}^{\infty} a_k C^k_{k+1} r^k = \\ &= \sum_{\substack{k=0,\\ k\neq s_{m_0}}}^{s_{m_0-1}} a_k C^k_{k+1} r^k + \sum_{k=s_{m_0-1}+1}^{\infty} a_k C^k_{k+1} r^k. \end{aligned} \tag{11}$$

We will estimate two sums in (11). From (8) it follows that $\mu^*_{m_0} \leqslant \frac{1}{c}\mu_{m_0}$ or

$$\max\{a_k r^k_{m_0} \colon k \neq s_{m_0},\ k \leq s_{m_0-1}\} \leqslant \frac{1}{c} \max\{a_k r^k_{m_0} \colon k \neq s_{m_0}, k \leqslant s_{m_0-1}\},$$

i. e. $a_kr^k \leqslant \frac{1}{c}a_{k_0}r^{k_0}$. Taking into account (10), it can be deduced that

$$\sum_{\substack{k=0,\\ k\neq s_{m_0}}}^{s_{m_0-1}} a_kC_{k+1}^k r^k \leqslant \frac{a_{k_0}r^{k_0}}{c}\sum_{k=0}^{N} C_{k+1}^k \leqslant \frac{a_{k_0}r^{k_0}}{c}(N+1)C_{N+1}^N. \tag{12}$$

For all $k \geqslant s_{m_0-1}+1$ $a_kr^k_{m_0-1} \leqslant \mu_{m_0-1}$ holds. Then $a_kr^k_{m_0} = \frac{a_kr^k_{m_0-1}}{c^k} \leqslant \frac{\mu_{m_0-1}}{c^k}$. In view of (8) we deduce

$$\sum_{k=s_{m_0-1}+1}^{\infty} a_kC_{k+1}^k r^k \leqslant \mu_{m_0-1}\sum_{k=s_{m_0-1}+1}^{\infty} C_{k+1}^k\frac{1}{c^k} \leqslant$$
$$\leqslant a_{s_{m_0-1}}r_{m_0}^{s_{m_0-1}}c^{s_{m_0-1}}\sum_{k=s_{m_0-1}+1}(k+1)(k+2)\frac{1}{c^k} \leqslant$$
$$\leqslant \frac{a_{s_{m_0}}r^{s_{m_0}}}{c}c^{s_{m_0-1}}\left(\sum_{k=s_{m_0-1}+1}^{\infty} x^{k+2}\right)^{(2)}\Bigg|_{x=\frac{1}{c}} = \frac{a_{k_0}r^{k_0}}{c}c^{s_{m_0-1}}\left\{\frac{x^{s_{m_0-1}+3}}{1-x}\right\}^{(2)}\Bigg|_{x=\frac{1}{c}} =$$
$$= \frac{a_{k_0}r^{k_0}}{c}c^{s_{m_0-1}}\sum_{j=0}^{n} C_2^j(2-j)!(s_{m_0-1}+3)\ldots(s_{m_0-1}-j+4)\times$$
$$\times \frac{x^{s_{m_0-1}+3-j}}{(1-x)^{3-j}}\Bigg|_{x=\frac{1}{c}} \leqslant \frac{a_{k_0}r^{k_0}}{c}c^{s_{m_0}-1}2!(N+3)!\sum_{j=0}^{2}\frac{(1/c)^{s_{m_0-1}+3-j}}{(1-1/c)^{3-j}} =$$
$$= 2!(N+3)!\frac{a_{k_0}r^{k_0}}{c}\sum_{j=0}^{2}\frac{1}{(c-1)^{3-j}} \leqslant 3!(N+3)!\frac{a_{k_0}r^{k_0}}{c}, \tag{13}$$

because $c \geqslant 2$. Hence, from (11)–(13) it follows that

$$\left|\sum_{\|J\|\neq k_0} b_J(z-z_0)^{j_1}(w-w_0)^{j_2}\right| \leqslant \frac{((N+1)C_{1+N}^N + 3!(N+3)!)a_{k_0}r^{k_0}}{c} \leqslant \frac{1}{2}a_{k_0}r^{k_0}.$$

It means that the polynomial P_{k_0} is a dominating polynomial in series (6) on the skeleton $\mathbb{T}^2\Big((z_0,w_0),\frac{r\mathbf{1}}{\mathbf{L}(z_0,w_0)}\Big)$. $\square$

Theorem 4. *Let $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$. If there exist $p\in\mathbb{Z}_+$, $d\in(0;1]$, $\eta\in(0;d)$ such that for each $(z_0,w_0)\in\mathbb{D}\times\mathbb{C}$ and some $R=(r_1,\ldots,r_n)$ with $r_j=r_j(d,(z_0,w_0))\in(\eta,d)$, $j\in\{1,2\}$, and certain $k^0=k^0(d,(z_0,w_0))\leqslant p$ the polynomial p_{k^0} is a dominating polynomial in series (6) on $\mathbb{T}^2((z_0,w_0),R/\mathbf{L}(z_0,w_0))$ then the analytic in $\mathbb{D}\times\mathbb{C}$ function F has bounded $\mathbf{L}$-index in joint variables.*

Proof. Suppose that there exist $p\in\mathbb{Z}_+$, $d\leq 1$ and $\eta\in(0;d)$ such that for each $(z_0,w_0)\in\mathbb{D}\times\mathbb{C}$ and some $R=(r_1,r_2)$ with $r_j=r_j(d,(z_0,w_0))\in(\eta,d)$, $j\in\{1,2\}$, and $k_0=k_0(1,(z_0,w_0))\leqslant p$ the polynomial P_{k_0} is a dominating polynomial in series (6) on $\mathbb{T}^2\Big((z_0,w_0),\frac{R}{\mathbf{L}(z_0,w_0)}\Big)$. Let us denote $r_0=\max\{r_1,r_2\}$. Then

$$\left|\sum_{j_1+j_2\neq k_0} b_{j_1,j_2}(z-z_0)^{j_1}(w-w_0)^{j_2}\right| = \left|F(z)-\sum_{j_1+j_2=k_0} b_{j_1,j_2}(z-z_0)^{j_1}(w-w_0)^{j_2}\right| \leqslant$$
$$\leq \frac{a_{k_0}r_0^{k_0}}{2}.$$

Using Cauchy's inequality we have

$$|b_{j_1,j_2}(z-z_0)^{j_1}(w-w_0)^{j_2}| = a_j^* R^J \leqslant \frac{a_{k_0}r_0^{k_0}}{2}$$

for all $(j_1,j_2)\in\mathbb{Z}_+^2$, $j_1+j_2\neq k_0$, that is for all $j_1+j_2=k\neq k_0$

$$a_k r_1^{j_1} r_2^{j_2} \leqslant \frac{a_{k_0}r_0^{k_0}}{2}. \tag{14}$$

Suppose that F is not a function of bounded $\mathbf{L}$-index in joint variables. Then in view of Theorem 2 for all $p_1\in\mathbb{Z}_+$ and $c\geqslant 1$ there exists $(z_0,w_0)\in\mathbb{D}\times\mathbb{C}$ such that the inequality

$$\max\left\{\frac{|F^{(j_1,j_2)}(z_0,w_0)|}{l_1^{j_1}(z_0,w_0)l_2^{j_2}(z_0,w_0)} : j_1+j_2=p_1+1\right\} >$$
$$> c\max\left\{\frac{|F^{(k_1,k_2)}(z_0,w_0)|}{l_1^{k_1}(z_0,w_0)l_2^{k_2}(z_0,w_0)} : k_1+k_2\leqslant p_1\right\}$$

holds. We put $p_1=p$ and $c=\left(\frac{(p+1)!}{\eta^{p+1}}\right)^2$. Then for these $z_0(p_1,c)$ and $w_0(p_1,c)$

$$\max\left\{\frac{|F^{(j_1,j_2)}(z_0,w_0)|}{j_1!j_2!l_1^{j_1}(z_0,w_0)l_2^{j_2}(z_0,w_0)} : j_1+j_2=p+1\right\} >$$
$$> \frac{1}{\eta^{p+1}}\max\left\{\frac{|F^{(k_1,k_2)}(z_0,w_0)|}{k_1!k_2!l_1^{k_1}(z_0,w_0)l_2^{k_2}(z_0,w_0)} : k_1+k_2\leqslant p\right\},$$

that is $a_{p+1}>\dfrac{a_{k_0}}{\eta^{p+1}}$. Hence, $a_{p+1}r_0^{p+1}>\dfrac{a_{k_0}r_0^{p+1}}{\eta^{p+1}}\geqslant a_{k_0}r^{k_0}$. The last inequality contradicts (14). Therefore, F is of bounded $\mathbf{L}$-index in joint variables. □

3. Estimates of growth of analytic functions in ball

At first we prove the following lemma.

Lemma 1. *If* $\mathbf{L}\in Q(\mathbb{B}\times\mathbb{C})$ *then for every fixed* $(z^*,w^*)\in\mathbb{D}\times\mathbb{C}$ *one has* $l_1(z^*+z,w^*)\to\infty$ *as* $|z^*+z|\to 1-0$ *and* $|w|l_2(z^*,w^*+w)\to\infty$ *as* $|w^*+w|\to\infty$.

Proof. In view of (2) we have $l_1(z^*+z,w^*)\geqslant\dfrac{\beta}{1-|z^*+z|}\to+\infty$ as $|z^*+z|\to 1-0$. The first part of the statement is proved.

Below we prove the second part, i.e. $|w|l_2(z^*,w^*+w)\to\infty$ as $|w^*+w|\to\infty$ On the contrary, if there exist a number $C>0$ and a sequence w_m such that $|w_m|l_2(z^*,w^*+w_m)=k_m\leqslant C$, i. e. $|w_m|=\dfrac{k_m}{l_2(z^*,w^*+w_m)}$. Then

$$\frac{1}{l_2(z^*,w^*+w_m)}l_2\left(z^*,w^*+w_m-\frac{k_m e^{i\,arg\,w_m}}{l_2(z^*,w^*+w_m)}\right)=\frac{|w_m|}{k_m}l_2(z^*,w^*)\to+\infty,$$

as $w_m\to\infty$, because k_m is bounded and $l_2(z^*,w^*)$ is finite. It means that $\lambda_{2,2}(0,C)=+\infty$ and $\mathbf{L}\notin Q(\mathbb{D}\times\mathbb{C})$. □

Denote

$$[0,2\pi]^2=[0,2\pi]\times[0,2\pi].$$

For $R=(r_1,r_2)\in\mathbb{R}^2_+$, $\Theta=(\theta_1,\theta_2)\in[0,2\pi]^2$, $A=(a_1,a_2)\in\mathbb{D}\times\mathbb{C}$ we write

$$Re^{i\Theta}=(r_1e^{i\theta_1},r_2e^{i\theta_2}),\ \arg A=(\arg a_1,\arg a_n).$$

By $K(\mathbb{D}\times\mathbb{C})$ we denote the class of positive continuous functions $\mathbf{L}=(l_1,l_2)$, where $l_j\colon\mathbb{D}\times\mathbb{C}\to\mathbb{R}_+$ satisfy (2) and there exists $c\geqslant 1$ such that for every $R\in\mathbb{R}^2_+$ with $|R|<1$ and $j\in\{1,2\}$ one has

$$\max_{\Theta_1,\Theta_2\in[0,2\pi]^2}\frac{l_j(Re^{i\Theta_2})}{l_j(Re^{i\Theta_1})}\leqslant c.$$

If $\mathbf{L}(z)=(l_1(|z_1|,|z_2|),l_2(|z_1|,|z_2|))$ then $\mathbf{L}\in K(\mathbb{D}\times\mathbb{C})$. It is easy to prove that $\dfrac{|e^z|+1}{1-|z|}\in Q(\mathbb{D})\setminus K(\mathbb{D})$, but $\dfrac{e^{e^{|z|}}}{1-|z|}\in K(\mathbb{D})\setminus Q(\mathbb{D})$. Similarly, the function $|e^z|+1\in Q(\mathbb{C})\setminus K(\mathbb{C})$, but $e^{e^{|z|}}\in K(\mathbb{C})\setminus Q(\mathbb{C})$. Therefore, the vector-function $\left(\frac{|e^z|+1}{1-|z|},|e^w|+1\right)$ belongs to the class $Q(\mathbb{D}\times\mathbb{C})$, but another vector-function $\left(\frac{e^{e^{|z|}}}{1-|z|},e^{e^{|w|}}\right)$ belongs to the class $K(\mathbb{D}\times\mathbb{C})\setminus Q(\mathbb{D}\times\mathbb{C})$.

Besides, if $\mathbf{L}_1,\mathbf{L}_2\in K(\mathbb{D}\times\mathbb{C})$ then $\mathbf{L}_1+\mathbf{L}_2\in K(\mathbb{D}\times\mathbb{C})$ and $\mathbf{L}_1\mathbf{L}_2\in K(\mathbb{D}\times\mathbb{C})$. For simplicity, let us write $M(F,R)=\max\{|F(z,w)|\colon(z,w)\in\mathbb{T}^2(\mathbf{0},R)\}$, where $r_1\in[0,1)$ and $r_2\geq 0$. Denote $\boldsymbol{\beta}=\left(\frac{\beta}{c},2\right)$.

Theorem 5. *Let $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})\cap K(\mathbb{D}\times\mathbb{C})$, $\beta>c$. If an analytic function $F:\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ has bounded $\mathbf{L}$-index in joint variables, then*

$$\ln\max\{|F(z,w)|\colon(z,w)\in\mathbb{T}^2(\mathbf{0},R)\}=$$
$$=O\Bigg(\min\Bigg\{\min_{\Theta\in[0,2\pi]^2}\left(\int_0^{r_1}l_1(te^{i\theta_1},r_2e^{i\theta_2})dt+\int_0^{r_2}l_2(r_1^0,t)dt\right);$$
$$\min_{\Theta\in[0,2\pi]^2}\left(\int_0^{r_1}l_1(te^{i\theta_1},r_2e^{i\theta_2})dt+\int_0^{r_2}l_2(r_1^0,t)dt\right)\Bigg\}\Bigg)\qquad \textit{as } r_1\to1-0,r_2\to+\infty,$$

where $R^0=(r_1^0,r_2^0)$ is a fixed radius, $r_1\in(0,1)$, $r_2^0>0$.

Proof. Let $R > \mathbf{0}$ with $r_1 \in (0,1)$, $\Theta \in [0,2\pi]^2$ and the point $(z^*,w^*) \in \mathbb{T}^2\big(\mathbf{0}, R + \frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\big)$ be such that

$$|F(z^*,w^*)| = \max\left\{|F(z,w)|\colon (z,w) \in \mathbb{T}^2\left(\mathbf{0}, R+\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}.$$

Denote $z_0 = \dfrac{z^* r_1}{r_1+\beta/(cl_1(Re^{i\Theta}))}$, $w_0 = \dfrac{w^* r_2}{r_2+2/l_2(Re^{i\Theta})}$. Then

$$|z_0 - z^*| = \left|\frac{z^* r_1}{r_1+\beta/(cl_1(Re^{i\Theta}))} - z^*\right| = \left|\frac{z^*\beta/(cl_1(Re^{i\Theta}))}{r_1+\frac{\beta}{cl_1(Re^{i\Theta})}}\right| = \frac{\beta}{cl_1(Re^{i\Theta})},$$

$$|w_0 - w^*| = \left|\frac{w^* r_2}{r_2+\frac{2}{l_2(Re^{i\Theta})}} - w^*\right| = \left|\frac{2w^*/(l_2(Re^{i\Theta}))}{r_2+\frac{2}{l_j(Re^{i\Theta})}}\right| = \frac{2}{l_j(Re^{i\Theta})},$$

$$\mathbf{L}(z_0,w_0)=\mathbf{L}\left(\frac{(z^*,w^*)R}{R+\boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})}\right)=\mathbf{L}\left(\frac{(R+\boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta}))e^{i\arg(z^*,w^*)}R}{R+\boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})}\right)=\mathbf{L}(Re^{i\arg z^*}).$$

Since $\mathbf{L} \in K(\mathbb{D}\times\mathbb{C})$, we have that

$$c\mathbf{L}(z_0,w_0) = c\mathbf{L}\big(Re^{i\arg(z^*,w^*)}\big) \geqslant \mathbf{L}\big(Re^{i\Theta}\big) \geqslant \frac{1}{c}\mathbf{L}(z^0).$$

We consider two skeletons $\mathbb{T}^2\big((z_0,w_0), \frac{\mathbf{1}}{\mathbf{L}(z_0,w_0)}\big)$ and $\mathbb{T}^2\big((z^0,w_0), \frac{\boldsymbol{\beta}}{\mathbf{L}(z_0,w_0)}\big)$. By Theorem 1 there exists $p_1 = p_1\big(\frac{\mathbf{1}}{c}, c\boldsymbol{\beta}\big) \geqslant 1$ such that (4) holds with $R' = \frac{\mathbf{1}}{c}$, $R'' = c\boldsymbol{\beta}$, i.e.

$$\begin{aligned}
\max&\left\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2\left(\mathbf{0}, R+\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\} = |F(z^*,w^*)| \leqslant\\
&\leqslant \max\left\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2\left((z_0,w_0), \frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\} \leq\\
&\leqslant \max\left\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2\left((z_0,w_0), \frac{c\boldsymbol{\beta}}{\mathbf{L}(z^0)}\right)\right\} \leqslant\\
&\leqslant p_1\max\left\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2\left((z_0,w_0), \frac{\mathbf{1}}{c\mathbf{L}(z^0)}\right)\right\} \leq\\
&\leqslant p_1\max\left\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2\left(\mathbf{0}, R+\frac{\mathbf{1}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}.
\end{aligned} \tag{15}$$

The function $\ln^+\max\{|F(z,w)|\colon z\in\mathbb{T}^2(\mathbf{0},R)\}$ is a convex function of the variables $\ln r_1$, $\ln r_2$ (see [20, p. 84]). Hence, the function admits a representation

$$\begin{aligned}
&\ln^+\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R)\}-\\
&-\ln^+\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},(r_1^0,r_2))\} = \int_{r_1^0}^{r_1}\frac{A_1(t,r_2)}{t}dt,
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\ln^+\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R)\}-\\
&-\ln^+\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},(r_1,r_2^0))\} = \int_{r_2^0}^{r_2}\frac{A_2(r_1,t)}{t}dt
\end{aligned} \tag{17}$$

for arbitrary $0 < r_j^0 \leqslant r_j$, where the functions $A_1(t, r_2)$, $A_2(r_1, t)$ are positive non-decreasing in variable t, $j \in \{1, 2\}$.

Using (15) we deduce

$$\begin{aligned}
\ln p_1 \geq \ln\max\left\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2\left(\mathbf{0}, R+\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}-\\
-\ln\max\left\{|F(z,w)|: (z,w)\in\mathbb{T}^2\left(\mathbf{0}, R+\frac{\mathbf{1}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}=\\
=\ln\max\left\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2\left(\mathbf{0}, R+\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}-\\
-\ln\max\left\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2\left(\mathbf{0}, R+\frac{(\beta/c,1)}{\mathbf{L}(Re^{i\Theta})}\right)\right\}+\\
+\ln\max\left\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2\left(\mathbf{0}, R+\frac{(\beta/c,1)}{\mathbf{L}(Re^{i\Theta})}\right)\right\}-\\
-\ln\max\left\{|F(z,w)|: (z,w)\in\mathbb{T}^2\left(\mathbf{0}, R+\frac{\mathbf{1}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}=\\
=\int\limits_{r_1+1/l_1(Re^{i\Theta})}^{r_1+\beta/(cl_1(Re^{i\Theta}))}\frac{1}{t}A_1\left(t, r_2+\frac{1}{l_2(Re^{i\Theta})}\right)dt+\\
+\int\limits_{r_2+1/l_2(Re^{i\Theta})}^{r_2+2/(l_2(Re^{i\Theta}))}\frac{1}{t}A_2\left(r_1+\frac{\beta/c}{l_1(Re^{i\Theta})}, t\right)dt\geqslant\\
\geqslant\ln\left(1+\frac{\beta/c-1}{r_1l_1(Re^{i\Theta})+1}\right)A_1\left(r_1, r_2+\frac{1}{l_2(Re^{i\Theta})}\right)+\\
+\ln\left(1+\frac{1}{r_2l_2(Re^{i\Theta})+1}\right)A_2\left(r_1+\frac{\beta/c}{l_1(Re^{i\Theta})}, r_2\right)
\end{aligned}\tag{18}$$

By Lemma 1 the function $r_1l_1(Re^{i\Theta}) \to +\infty$ as $r_1 \to 1-0$ and $r_2l_2(Re^{i\Theta}) \to +\infty$ as $r_2 \to \infty$. Hence, for $r_j \geqslant r_j^0$

$$\ln\left(1+\frac{\frac{\beta}{c}-1}{r_1l_1(Re^{i\Theta})+1}\right)\sim\frac{\frac{\beta}{c}-1}{r_1l_1(Re^{i\Theta})+1}\geqslant\frac{\frac{\beta}{c}-1}{2r_1l_1(Re^{i\Theta})},\quad r_1\to 1-0,$$

$$\ln\left(1+\frac{1}{r_2l_2(Re^{i\Theta})+1}\right)\sim\frac{1}{r_2l_2(Re^{i\Theta})+1}\geqslant\frac{1}{2r_2l_2(Re^{i\Theta})},\quad r_2\to\infty.$$

Thus, inequality (18) implies that

$$\begin{aligned}
A_1\left(r_1, r_2+\frac{1}{l_2(Re^{i\Theta})}\right)\leqslant\frac{2\ln p_1}{\beta/c-1}\,r_1l_1(Re^{i\Theta}),\\
A_2\left(r_1+\frac{\beta/c}{l_1(Re^{i\Theta})}, r_2\right)\leqslant 2\ln p_1 r_2l_2(Re^{i\Theta}).
\end{aligned}$$

Let $R^0 = (r_1^0, r_2^0)$, where every r_j^0 is chosen above. Applying (16) and (17) we obtain consequently with $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$

$$\begin{gathered}\ln\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R)\} = \\ = \ln\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R+(r_1^0-r_1)\mathbf{e}_1)\} + \int\limits_{r_1^0}^{r_1}\frac{A_1(t,r_2)}{t}dt = \\ = \ln\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R+(r_1^0-r_1)\mathbf{e}_1+(r_2^0-r_2)\mathbf{e}_2)\} + \\ + \int\limits_{r_1^0}^{r_1}\frac{A_1(t,r_2)}{t}dt + \int\limits_{r_2^0}^{r_2}\frac{A_2(r_1^0,t)}{t}dt = \\ = \ln\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R^0)\} + \int\limits_{r_1^0}^{r_1}\frac{A_1(t,r_2)}{t}dt + \int\limits_{r_2^0}^{r_2}\frac{A_2(r_1^0,t)}{t}dt \leqslant \\ \leqslant \ln\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R^0)\} + \\ + \frac{2\ln p_1}{\beta-1}\Big(\int\limits_{r_1^0}^{r_1} l_1(t,r_2)dt + \int\limits_{r_2^0}^{r_2} l_2(r_1^0,t)dt\Big) \leqslant \\ \leqslant \ln\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R^0)\} + \\ + \frac{2\ln p_1}{\beta/c-1}\left(\int\limits_{0}^{r_1} l_1(t,r_2)dt + 2\ln p_1\int\limits_{0}^{r_2} l_2(r_1^0,t)dt\right) \leqslant \\ \leqslant (1+o(1))\frac{2\ln p_1}{\beta-1}\left(\int\limits_{0}^{r_1} l_1(t,r_2)dt + \int\limits_{0}^{r_2} l_2(r_1^0,t)dt\right).\end{gathered}$$

The function $\ln\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R)\}$ is independent of Θ. Thus, the following estimate

$$\begin{gathered}\ln\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R)\} = \\ = O\left(\min_{\Theta\in[0,2\pi]^2}\Big(\int\limits_{0}^{r_1} l_1(te^{i\theta_1}, r_2e^{i\theta_2})dt + \int\limits_{0}^{r_2} l_2(r_1^0,t)dt\Big)\right)\end{gathered}$$

holds as $r_1\to 1-0$ and $r_2\to\infty$. Obviously, the similar equality can be proved for arbitrary permutation σ_2 of the set $\{1,2\}$. In particular,

$$\begin{gathered}\ln\max\{|F(z,w)|\colon (z,w)\in\mathbb{T}^2(\mathbf{0},R)\} = \\ = O\left(\min_{\Theta\in[0,2\pi]^2}\left(\int\limits_{0}^{r_2} l_1(r_1e^{i\theta_1}, te^{i\theta_2})dt + \int\limits_{0}^{r_2} l_2(te^{i\theta_1}, r_2^0e^{i\theta_2})dt\right)\right)\end{gathered}$$

Thus, estimate (5) holds. Theorem 5 is proved. □

Let us denote $a^+ = \max\{a, 0\}$, $u_j(t) = u_j(t, R, \Theta) = l_j(\frac{tR}{r^*}e^{i\Theta})$, where $a \in \mathbb{R}$, $t \in [0, r^*]$, $j \in \{1, 2\}$, $r^* = \max_{1\leqslant j\leqslant 2} r_j \neq 0$ and $R = (r_1, r_2)$ with $r_1 \in [0, 1]$, $r_2 > 0$. All the following results are analogs of results from [6, 11], obtained for entire functions for functions which are analytic in the unit ball. We repeat considerations from these papers with another conditions by R.

Theorem 6. *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable r_k, $k \in \{1, 2\}$, and $R = (r_1, r_2)$ with $r_1 \in [0, 1]$, $r_2 > 0$, $\Theta \in [0, 2\pi]^2$. If the function $\mathbf{L}$ satisfies (2) and an analytic function $F : \mathbb{D}\times\mathbb{C} \to \mathbb{C}$ has bounded $\mathbf{L}$-index $N = N(F, \mathbf{L})$ in joint variables then for every $\Theta \in [0, 2\pi]^2$ and for every $R = (r_1, r_2)$ with $r_1 \in [0, 1]$, $r_2 > 0$, and for each $S \in \mathbb{Z}^2_+$ one has*

$$\ln\max\left\{\frac{|F^{(S)}(Re^{i\Theta})|}{S!\mathbf{L}^S(Re^{i\Theta})}\colon\ \|S\|\leqslant N\right\} \leqslant \ln\max\left\{\frac{|F^{(S)}(\mathbf{0})|}{S!\mathbf{L}^S(\mathbf{0})}\colon\ \|S\|\leqslant N\right\} +$$

$$+\int_0^{r^*}\left(\max_{\|S\|\leqslant N}\left\{\sum_{j=1}^2\frac{r_j}{r^*}(k_j+1)l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)\right\} + \max_{\|S\|\leqslant N}\left\{\sum_{j=1}^2\frac{k_j(-u_j'(\tau))^+}{l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)}\right\}\right)d\tau. \tag{19}$$

Proof. Let $R \in \mathbb{R}^2\setminus\{\mathbf{0}\}$, $\Theta \in [0, 2\pi]^2$. Denote $\alpha_j = \frac{r_j}{r^*}$, $j \in \{1, 2\}$ and $A = (\alpha_1, \alpha_2)$. We consider the function

$$g(t) = \max\left\{\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\colon\ \|S\|\leqslant N\right\}, \tag{20}$$

where $At = (\alpha_1 t, \alpha_2 t)$, $Ate^{i\Theta} = (\alpha_1 te^{i\theta_1}, \alpha_2 te^{i\theta_2})$.

Since the function $\dfrac{|F^{(S)}(Ate^{i\Theta})|}{K!\mathbf{L}^K(Ate^{i\Theta})}$ is continuously differentiable by real $t \in [0, +\infty)$, outside the zero set of the function $|F^{(S)}(Ate^{i\Theta})|$, the function $g(t)$ is a continuously differentiable function on $[0, r^*)$, except, perhaps, for a countable set of points.

Therefore, using the inequality $\frac{d}{dr}|g(r)| \leqslant |g'(r)|$ which holds except for the points $r = t$ such that $g(t) = 0$, we deduce

$$\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right) = \frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\frac{d}{dt}|F^{(S)}(Ate^{i\Theta})| +$$

$$+|F^{(S)}(Ate^{i\Theta})|\frac{d}{dt}\frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})} \leqslant \frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\left|\sum_{j=1}^2 F^{(S+\mathbf{e}_j)}(Ate^{i\Theta})\alpha_j e^{i\theta_j}\right| -$$

$$-\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^2\frac{k_j u_j'(t)}{l_j(Ate^{i\Theta})} \leqslant \sum_{j=1}^2\frac{|F^{(S+\mathbf{e}_j)}(Ate^{i\Theta})|}{(S+\mathbf{1}_j)!\mathbf{L}^{S+\mathbf{e}_j}(Ate^{i\Theta})}\alpha_j(k_j+1)l_j(Ate^{i\Theta}) +$$

$$+\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^2\frac{k_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}. \tag{21}$$

For absolutely continuous functions h_1, h_2, $\ldots$, h_k and $h(x) := \max\{h_j(z)\colon 1 \leqslant j \leqslant k\}$, $h'(x) \leqslant \max\{h_j'(x)\colon 1 \leqslant j \leqslant k\}$, $x \in [a, b]$ (see [18, Lemma 4.1, p. 81]). The function g is absolutely continuous, therefore, from (21) it follows that

$$g'(t) \leqslant \max\left\{\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right): \|S\| \leqslant N\right\} \leqslant$$

$$\leqslant \max_{\|S\|\leqslant N}\left\{\sum_{j=1}^{2}\frac{\alpha_j(s_j+1)l_j(Ate^{i\Theta})|F^{(S+\mathbf{e}_j)}(Ate^{i\Theta})|}{(K+\mathbf{1}_j)!\mathbf{L}^{K+\mathbf{e}_j}(Ate^{i\Theta})}+\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^{2}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\} \leqslant$$

$$\leqslant g(t)\left(\max_{\|S\|\leqslant N}\left\{\sum_{j=1}^{2}\alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\}+\max_{\|S\|\leqslant N}\left\{\sum_{j=1}^{2}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}\right) =$$

$$= g(t)(\beta(t)+\gamma(t)),$$

where

$$\beta(t) = \max_{\|S\|\leqslant N}\left\{\sum_{j=1}^{2}\alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\}, \gamma(t) = \max_{\|S\|\leqslant N}\left\{\sum_{j=1}^{2}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}.$$

Thus, $\frac{d}{dt}\ln g(t) \leqslant \beta(t)+\gamma(t)$ and

$$g(t) \leqslant g(0)\exp\int_0^t(\beta(\tau)+\gamma(\tau))d\tau, \tag{22}$$

because $g(0) \neq 0$. But $r^*A = R$. Substituting $t = r^*$ in (22) and taking into account (20), we deduce

$$\ln\max\left\{\frac{|F^{(S)}(Re^{i\Theta})|}{S!\mathbf{L}^S(Re^{i\Theta})}: \ \|S\| \leqslant N\right\} \leqslant \ln\max\left\{\frac{|F^{(S)}(\mathbf{0})|}{S!\mathbf{L}^S(\mathbf{0})}: \ \|S\| \leqslant N\right\} +$$

$$+\int_0^{r^*}\left(\max_{\|S\|\leqslant N}\left\{\sum_{j=1}^{2}\alpha_j(s_j+1)l_j(A\tau e^{i\Theta})\right\}+\max_{\|S\|\leqslant N}\left\{\sum_{j=1}^{2}\frac{s_j(-u_j'(\tau))^+}{l_j(A\tau e^{i\Theta})}\right\}\right)d\tau,$$

i.e. (19) is proved. □

Theorem 7. *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable r_k, $k \in \{1,2\}$, and $R = (r_1, r_2)$ with $r_1 \in [0,1]$, $r_2 > 0$, $\Theta \in [0,2\pi]^2$. If the function $\mathbf{L}$ satisfies (2) and an analytic function F in $\mathbb{D}\times\mathbb{C}$ has bounded $\mathbf{L}$-index $N = N(F,\mathbf{L})$ in joint variables and there exists $C > 0$ such that the function $\mathbf{L}$ satisfies inequalities*

$$\sup_{|R|<1}\max_{t\in[0,r^*]}\max_{\Theta\in[0,2\pi]^2}\max_{1\leqslant j\leqslant 2}\frac{(-(u_j(t,R,\Theta))_t')^+}{\frac{r_j}{r^*}l_j^2(\frac{t}{r^*}Re^{i\Theta})} \leqslant C, \tag{23}$$

then

$$\varlimsup_{r_1\to 1-0, r_2\to\infty}\frac{\ln\max\{|F(z,w)|: \ (z,w)\in\mathbb{T}^2(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^w}\int_0^1\langle R, \mathbf{L}\left(\tau Re^{i\Theta}\right)\rangle d\tau} \leqslant (C+1)N+1. \tag{24}$$

Proof. By Lemma 1 if $\mathbf{L}$ satisfies (2) then

$$\max_{\Theta\in[0,2\pi]^2}\int_0^1\left\langle R,\mathbf{L}\left(\tau Re^{i\Theta}\right)\right\rangle d\tau\to+\infty \text{ as } r_1\to 1-0 \text{ and } r_2\to+\infty. \tag{25}$$

Denote $\widetilde{\beta}(t)=\sum_{j=1}^{2}\alpha_j l_j(Ate^{i\Theta})$. If, in addition, (23) holds then for some S^*, $\|S^*\|\leqslant N$ and $\widetilde{S}$, $\|\widetilde{S}\|\leqslant N$,

$$\frac{\gamma(t)}{\widetilde{\beta}(t)}=\frac{\sum_{j=1}^2\frac{s_j^*(-u_j'(t))^+}{l_j(Ate^{i\Theta})}}{\sum_{j=1}^2\alpha_j l_j(Ate^{i\Theta})}\leqslant\sum_{j=1}^{2}s_j^*\frac{(-u_j'(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})}\leqslant\sum_{j=1}^{2}s_j^*\cdot C\leqslant NC,$$

$$\frac{\beta(t)}{\widetilde{\beta}(t)}=\frac{\sum_{j=1}^2\alpha_j(\tilde{s}_j+1)l_j(Ate^{i\Theta})}{\sum_{j=1}^2\alpha_j l_j(Ate^{i\Theta})}=1+\frac{\sum_{j=1}^2\alpha_j\tilde{s}_j l_j(Ate^{i\Theta})}{\sum_{j=1}^2\alpha_j l_j(Ate^{i\Theta})}\leqslant$$
$$\leqslant 1+\sum_{j=1}^{2}\tilde{s}_j\leqslant 1+N.$$

But

$$|F(Ate^{i\Theta})|\leqslant g(t)\leqslant g(0)\exp\int_0^t(\beta(\tau)+\gamma(\tau))d\tau$$

and $r^*A=R$. Putting $t=r^*$ and taking into account (25), we obtain

$$\ln\max\{|F(z,w)|\colon\ (z,w)\in\mathbb{T}^2(\mathbf{0},R)\}=$$
$$=\ln\max_{\Theta\in[0,2\pi]^2}|F(Re^{i\Theta})|\leqslant\ln\max_{\Theta\in[0,2\pi]^2}g(r^*)\leqslant$$
$$\leqslant\ln g(0)+\max_{\Theta\in[0,2\pi]^2}\int_0^{r^*}(\beta(\tau)+\gamma(\tau))d\tau\leqslant$$
$$\leqslant\ln g(0)+(NC+N+1)\max_{\Theta\in[0,2\pi]^2}\int_0^{r^*}\widetilde{\beta}(\tau)d\tau=$$
$$=\ln g(0)+(NC+N+1)\max_{\Theta\in[0,2\pi]^2}\int_0^{r^*}\sum_{j=1}^{2}\alpha_j l_j(A\tau e^{i\Theta})d\tau=$$
$$=\ln g(0)+(NC+N+1)\max_{\Theta\in[0,2\pi]^2}\int_0^{r^*}\sum_{j=1}^{2}\frac{r_j}{r^*}l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)d\tau=$$
$$=\ln g(0)+(NC+N+1)\max_{\Theta\in[0,2\pi]^2}\int_0^{1}\sum_{j=1}^{2}r_j l_j(\tau Re^{i\Theta})d\tau.$$

Thus, we conclude that (24) holds. □

Theorem 8. *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable r_k, $k \in \{1,2\}$, and $R = (r_1, r_2)$ with $r_1 \in [0,1]$, $r_2 > 0$, $\Theta \in [0,2\pi]^2$. If the function $\mathbf{L}$ satisfies (2) and an analytic function $F : \mathbb{D} \times C \to \mathbb{C}$ has bounded $\mathbf{L}$-index $N = N(F, \mathbf{L})$ in joint variables and*

$$r^*(-(u_j(t,R,\Theta))'_{t=r^*})^+/(r_j l_j^2(Re^{i\Theta})) \to 0 \tag{26}$$

uniformly in all $\Theta \in [0,2\pi]^2$, $j \in \{1,2\}$, as $r_1 \to 1-0$, $r_2 \to +\infty$ then

$$\varlimsup_{|R|\to 1-0} \frac{\ln \max\{|F(z,w): \ (z,w) \in \mathbb{T}^2(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^2} \int\limits_0^1 \langle R, \mathbf{L}\,(\tau Re^{i\Theta})\rangle d\tau} \leqslant N+1. \tag{27}$$

Estimate (27) can be deduced by analogy to the proof of Theorem 7.

Our main result in this section is the following

Theorem 9. *Let $\mathbf{L}(R) = (l_1(R), l_2(R))$, $l_j(R)$ be a positive continuously differentiable non-decreasing function in each variable r_2, $k \in \{1,2\}$, and $R = (r_1, r_2)$ with $r_1 \in [0,1]$, $r_2 > 0$. If the function $\mathbf{L}$ satisfies (2) and an analytic function $F : \mathbb{D} \times C \to \mathbb{C}$ has bounded $\mathbf{L}$-index $N = N(F, \mathbf{L})$ in joint variables then*

$$\varlimsup_{r_1\to 1-0, r_2\to+\infty} \frac{\ln \max\{|F(z,w)|: \ (z,w) \in T^2(\mathbf{0},R)\}}{\int\limits_0^1 \langle R, \mathbf{L}\,(\tau R)\rangle d\tau} \leqslant N+1.$$

This statement is a direct consequence of Theorem 8, which is obtained for a more general function $\mathbf{L}$.

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ОЦІНКИ ЗРОСТАННЯ АНАЛІТИЧНИХ В $\mathbb{D}\times\mathbb{C}$ ФУНКЦІЙ ОБМЕЖЕНОГО L-ІНДЕКСУ ЗА СУКУПНІСТЮ ЗМІННИХ

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Доведено деякі властивості розвинення у степеневий ряд аналітичних в $\mathbb{D}\times\mathbb{C}$ функцій обмеженого $\mathbf{L}$-індексу за сукупністю змінних, де $\mathbf{L}(z,w) = (l_1(z,w), l_2(z,w))$ з $l_j : \mathbb{D}\times\mathbb{C} \to \mathbb{R}_+$ $(j \in \{1,2\})$ — додатні неперервні функції та $l_1(z,w) > \beta/(1-|z|)$ для всіх $(z,w) \in \mathbb{D}\times\mathbb{C}$ і деякого $\beta > 1$. Навіть більше, отримано оцінки зростання функцій з цього класу. Вони описують поводження логарифма максимуму модуля аналітичних функцій на кістяку в бікрузі через поводження функції $\mathbf{L}$. Ці оцінки точні в загальному випадку. Подані результати ґрунтуються на вичерпанні бікругами декартового добутку одиничного круга та комплексної площини.

Ключові слова: аналітична функція, обмежений індекс за сукупністю змінних, одиничний круг, комплексна площина, декартів добуток, оцінки зростання.