ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2019. Випуск 87. С. 37–45
Visnyk of the Lviv Univ. Series Mech. Math. 2019. Issue 87. P. 37–45
http://publications.lnu.edu.ua/bulletins/index.php/mmf
doi: http://dx.doi.org/10.30970/vmm.2019.87.037-045

УДК 512.536

ON THE DICHOTOMY OF A LOCALLY COMPACT SEMITOPOLOGICAL MONOID OF ORDER ISOMORPHISMS BETWEEN PRINCIPAL FILTERS OF $\mathbb{N}^n$ WITH ADJOINED ZERO

Taras MOKRYTSKYI

Ivan Franko National University of Lviv,
Universitetska Str., 1, 79000 Lviv, Ukraine
e-mail: tmokrytskyi@gmail.com

Let n be any positive integer and $\mathscr{IPF}(\mathbb{N}^n)$ be the semigroup of all order isomorphisms between principal filters of the n-th power of the set of positive integers $\mathbb{N}$ with the product order. We prove that a Hausdorff locally compact semitopological semigroup $\mathscr{IPF}(\mathbb{N}^n)$ with an adjoined zero is either compact or discrete.

Key words: Semigroup, inverse semigroup, bicyclic monoid, semitopological semigroup, topological semigroup, locally compact, compact, discrete.

Further we follow the terminology of [10, 11, 12, 20]. In this paper we denote the set of positive integers by $\mathbb{N}$, the set of non-negative integers by $\mathbb{N}_0$, a semigroup S with the an adjoined zero by S^0 (cf. [11]), the symmetric group of degree n by $\mathscr{S}_n$, i.e., $\mathscr{S}_n$ is the group of all permutations of an n-element set. All topological spaces, considered in this paper, are assumed to be Hausdorff.

A semigroup S is called *inverse* if for every $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. Later such an element y will be denoted by x^{-1} and will be called the *inverse* of x. A map $\operatorname{inv}\colon S \to S$ which assigns to every $s \in S$ its inverse is called the *inversion*.

If Y is a subspace of a topological space X and $A \subset Y$, then by $\operatorname{cl}_Y(A)$ we denote the topological closure of A in Y.

A *semitopological (topological) semigroup* is a topological space with separately continuous (jointly continuous) semigroup operation. An inverse topological semigroup with continuous inversion is called *a topological inverse semigroup*.

We recall that a topological space X is *locally compact* if every point x of X has an open neighbourhood $U(x)$ with the compact closure $\operatorname{cl}_X(U(x))$.

2010 Mathematics Subject Classification: 20M10, 22A15, 54D40, 54D45, 54H10

The *bicyclic semigroup* (or the *bicyclic monoid*) $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by elements p and q and the relation $pq = 1$.

The bicyclic semigroup plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For instance, a well-known Andersen's result [1] states that a (0–)simple semigroup with an idempotent is completely (0–)simple if and only if it does not contain an isomorphic copy of the bicyclic semigroup. The bicyclic monoid admits only the discrete semigroup topology. Bertman and West in [9] extended this result for the case of semitopological semigroups. No stable and Γ-compact topological semigroups contains the bicyclic monoid [2, 18]. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups was studied in [3, 4, 8, 17].

For an arbitrary positive integer n by $(\mathbb{N}^n, \leqslant)$ we denote the n-th power of the set of positive integers $\mathbb{N}$ with the product order:

$$(x_1, \ldots, x_n) \leqslant (y_1, \ldots, y_n) \qquad \text{if and only if} \qquad x_i \leqslant y_i \quad \text{for all} \quad i = 1, \ldots, n.$$

It is obvious that the set of all order isomorphisms between principal filters of the poset $(\mathbb{N}^n, \leqslant)$ with the operation of composition of partial maps form a semigroup. This semigroup will be denoted by $\mathscr{IPF}(\mathbb{N}^n)$. The semigroup $\mathscr{IPF}(\mathbb{N}^n)$ is a generalization of the bicyclic semigroup $\mathscr{C}(p,q)$. Hence it is natural to ask: *what algebraic and topological properties of the semigroup $\mathscr{IPF}(\mathbb{N}^n)$ are similar to those of the bicyclic monoid?* The structure of the semigroup $\mathscr{IPF}(\mathbb{N}^n)$ is studied in [16]. There was shown that $\mathscr{IPF}(\mathbb{N}^n)$ is a bisimple, E-unitary, F-inverse monoid, described Green's relations on $\mathscr{IPF}(\mathbb{N}^n)$ and its maximal subgroups. It was proved that $\mathscr{IPF}(\mathbb{N}^n)$ is isomorphic to the semidirect product of the direct n-th power of the bicyclic monoid $\mathscr{C}^n(p,q)$ by the permutation group $\mathscr{S}_n$, every non-identity congruence on $\mathscr{IPF}(\mathbb{N}^n)$ is group and the least group congruence on $\mathscr{IPF}(\mathbb{N}^n)$ was described. It was shown that every shift-continuous topology on $\mathscr{IPF}(\mathbb{N}^n)$ is discrete and embedding of the semigroup $\mathscr{IPF}(\mathbb{N}^n)$ into compact-like topological semigroups was discussed.

A dichotomy for the bicyclic monoid with an adjoined zero $\mathscr{C}^0 = \mathscr{C}(p,q) \sqcup \{0\}$ was proved in [13]: *every locally compact semitopological bicyclic monoid $\mathscr{C}^0$ with an adjoined zero is either compact or discrete.* The above dichotomy was extended by Bardyla in [5] to locally compact λ-polycyclic semitopological monoids, in [6] to locally compact semitopological graph inverse semigroups in [15] to locally compact semitopological interassociates of the bicyclic monoid with an adjoined zero, and were extended in [14] to locally compact semitopological 0-bisimple inverse ω semigroups with compact maximal subgroups. The lattice of all weak shift-continuous topologies on $\mathscr{C}^0$ is described in [7].

The main purpose of this paper is to obtain counterparts of the above results for locally compact semitopological monoid $\mathscr{IPF}(\mathbb{N}^n)$.

By $\mathscr{IPF}(\mathbb{N}^n)^0$ we denote the monoid $\mathscr{IPF}(\mathbb{N}^n)$ with an adjoined zero.

Lemma 1. *Let $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ be a locally compact non-discrete semitopological semigroup. Then:*

(1) *for every open neighbourhood $U(0)$ of the zero in $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ there exists an open compact neighbourhood $V(0)$ of the zero in $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ such that $V(0) \subset U(0)$;*

(2) *for every open neighbourhood $U(0)$ of the zero in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ and every open compact neighbourhood $V(0)$ of the zero in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ the set $V(0)\cap U(0)$ is compact and open, and the set $V(0)\setminus U(0)$ is finite.*

Proof. (1) Let $U(0)$ be an arbitrary open neighbourhood of the zero in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$. By Theorem 3.3.1 from [10] the space $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ is regular. Since it is locally compact, there exists an open neighbourhood $V(0)\subseteq U(0)$ of the zero in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ such that $cl_{\mathscr{IPF}(\mathbb{N}^n)^0}(V(0))\subseteq U(0)$. Since all non-zero elements of the semigroup $\mathscr{IPF}(\mathbb{N}^n)^0$ are isolated points in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$, $\operatorname{cl}_{\mathscr{IPF}(\mathbb{N}^n)^0}(V(0))=V(0)$, and hence our assertion holds.

(2) Let $V(0)$ be an arbitrary compact open neighbourhood of the zero in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$. Then for an arbitrary open neighbourhood $U(0)$ of the zero in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ the family

$$\mathscr{U}=\{U(0)\}\cup\{\{x\}\colon x\in V(0)\setminus U(0)\}$$

is an open cover of $V(0)$. Since the family $\mathscr{U}$ is disjoint, it is finite. So the set $V(0)\setminus U(0)$ is finite and hence the set $V(0)\cap U(0)$ is compact. □

Remark 1. On the bicyclic semigroup $\mathscr{C}(p,q)$ the semigroup operation is determined in the following way:

$$p^iq^j\cdot p^kq^l=\begin{cases} p^iq^{j-k+l}, & \text{if } j>k;\\ p^iq^l, & \text{if } j=k;\\ p^{i-j+k}q^l, & \text{if } j<k,\end{cases}$$

which is equivalent to the following multiplication:

$$p^iq^j\cdot p^kq^l=p^{i+\max\{j,k\}-j}q^{l+\max\{j,k\}-k}.$$

The above implies that the bicyclic semigroup $\mathscr{C}(p,q)$ is isomorphic to the semigroup $(\mathbb{N}_0\times\mathbb{N}_0,*)$ which is defined on the square $\mathbb{N}_0\times\mathbb{N}_0$ of the set of non-negative integers with the following multiplication:

(1) $$(i,j)*(k,l)=(i+\max\{j,k\}-j,l+\max\{j,k\}-k).$$

We note that the semigroup $(\mathbb{N}_0\times\mathbb{N}_0,*)$ is isomorphic to the semigroup $(\mathbb{N}\times\mathbb{N},*)$ which is defined on the square $\mathbb{N}\times\mathbb{N}$ of the set of all positive integers with the same operation $*$. It is obvious that the map $f\colon\mathbb{N}_0\times\mathbb{N}_0\to\mathbb{N}\times\mathbb{N}$, $(i,j)\mapsto(i+1,j+1)$ is an isomorphism between semigroups $(\mathbb{N}_0\times\mathbb{N}_0,*)$ and $(\mathbb{N}\times\mathbb{N},*)$.

In this paper we will use the semigroup $(\mathbb{N}\times\mathbb{N},*)$ as a representation of the bicyclic semigroup $\mathscr{C}(p,q)$.

For an arbitrary positive integer n by $\mathscr{C}(p,q)^n$ we shall denote the n-th direct power of $(\mathbb{N}\times\mathbb{N},*)$, i.e., $\mathscr{C}(p,q)^n$ is the n-th power of $\mathbb{N}\times\mathbb{N}$ with the point-wise semigroup operation defined by (1). Also, by $[\mathbf{x},\mathbf{y}]$ we denote the ordered collection $((x_1,y_1),\ldots,(x_n,y_n))$ of $\mathscr{C}(p,q)^n$, where $\mathbf{x}=(x_1,\ldots,x_n)$ and $\mathbf{y}=(y_1,\ldots,y_n)$, and for arbitrary permutation $\sigma\colon\{1,\ldots,n\}\to\{1,\ldots,n\}$ we put

$$(\mathbf{x})\sigma=\left(x_{(1)\sigma^{-1}},\ldots,x_{(n)\sigma^{-1}}\right).$$

We recall (cf. [16]) that the semigroup $\mathscr{IPF}(\mathbb{N}^n)$ is isomorphic to the semidirect product $\mathscr{S}_n \ltimes \mathscr{C}(p,q)^n$ and hence according the above arguments we can consider the semigroup $\mathscr{IPF}(\mathbb{N}^n)$ as the set $\mathscr{S}_n \times (\mathbb{N}\times\mathbb{N})^n$ with the following semigroup operation

$$(\alpha,[\mathbf{x},\mathbf{y}])\cdot(\beta,[\mathbf{u},\mathbf{v}]) = (\alpha\circ\beta,[(\mathbf{x})\beta,(\mathbf{y})\beta]*[\mathbf{u},\mathbf{v}]) =$$
$$= (\alpha\circ\beta,[(\mathbf{x})\beta+\max\{(\mathbf{y})\beta,\mathbf{u}\}-(\mathbf{y})\beta,\mathbf{v}+\max\{(\mathbf{y})\beta,\mathbf{u}\}-\mathbf{u}])$$

For any permutation $\sigma\in\mathscr{S}_n$ of an n-element set and for any ordered tuple $\mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{N}^n$ we put

$$L_\sigma^{\mathbf{a}} = \{(\sigma,[\mathbf{a},\mathbf{x}])\in\mathscr{IPF}(\mathbb{N}^n)\colon \mathbf{x}\in\mathbb{N}^n\}.$$

For any integer $i\in\{1,\ldots,n\}$ define an element $\mathbf{2}_i$ as an element of $\mathbb{N}^n$ with the property that only i-th coordinate of $\mathbf{2}_i$ is equal to 2 and all other coordinate are equal to 1, i.e. $\mathbf{2}_i=(1,\ldots,\underbrace{2}_{i},\ldots,1)$.

Lemma 2. *Let $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ be a locally compact non-discrete semitopological semigroup. Then for any neighborhood $U(0)$ of the zero 0 and for any permutation $\sigma\in\mathscr{S}_n$ there exists $\mathbf{a}\in\mathbb{N}^n$ such that the set $L_\sigma^{\mathbf{a}}\cap U(0)$ is infinite.*

Proof. Suppose to the contrary that there exists neighborhood $U(0)$ of the zero 0 and permutation $\sigma\in\mathscr{S}_n$ such that for any $\mathbf{a}\in\mathbb{N}^n$ the set $L_\sigma^{\mathbf{a}}\cap U(0)$ is finite. Then Lemma 1(1) and the separate continuity of the semigroup operation in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ imply that there exists an open compact neighbourhood $V(0)$ of the zero 0 in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ such that $V(0)\cdot(1,[\mathbf{1},\mathbf{2}_1])\subset U(0)$.

Since for any fixed element $\mathbf{a}\in\mathbb{N}^n$ the set $L_\sigma^{\mathbf{a}}\cap U(0)$ is finite, there exists an element

$$m_{\mathbf{a}}=(\sigma,[\mathbf{a},(x_1,\ldots,x_n)])\in L_\sigma^{\mathbf{a}}\cap U(0)$$

with property that

$$U(0)\not\ni(\sigma,[\mathbf{a},(x_1+1,\ldots,x_n)])=m_{\mathbf{a}}\cdot(1,[\mathbf{1},\mathbf{2}_1]). \tag{2}$$

Consider the set $M=\{m_{\mathbf{a}}\colon\mathbf{a}\in\mathbb{N}^n\}$. Then property (2) implies that $M\cap V(0)=\varnothing$. Thus $U(0)\setminus V(0)\supset M$ which contradicts Lemma 1(2) because the set M is infinite. ☐

Lemma 3. *Let n be a positive integer, A and B be infinite subsets of $\mathbb{N}^n$ such that $A\sqcup B=\mathbb{N}^n$ and $A\cap B=\emptyset$. Then there exist an infinite subset $C\subset A$ and a positive integer $k\in\{1,\ldots,n\}$ such that at least one of the sets $(C)g_k$ and $(C)g_k^{-1}$ is a subset of B, where g_k is the map from $\mathbb{N}^n$ to $\mathbb{N}^n$ is defined in the following way: $(x_1,\ldots,x_n)g_k=(x_1,\ldots,x_k+1,\ldots,x_n)$.*

Proof. If $n=1$ consider the the set $C=\{a\in A\colon a+1\in B\}$, C is infinite and $(C)g_1\subset B$.

Let $n\geqslant 2$. An ordered tuple $p=(\mathbf{p}^1,\mathbf{p}^2,\ldots,\mathbf{p}^{r-1},\mathbf{p}^r)\in(\mathbb{N}^n)^r$ of elements of $\mathbb{N}^n$ is called a *path from point $\mathbf{a}$ to point $\mathbf{b}$* if $\mathbf{p}^1=\mathbf{a}$, $\mathbf{p}^k=\mathbf{b}$ and for any index $i\in\{2,\ldots,k\}$ there exists some $m_i\in\{1,\ldots,n\}$ such that $(\mathbf{p}^{i-1})g_{m_i}=\mathbf{p}^i$ or $(\mathbf{p}^{i-1})g_{m_i}^{-1}=\mathbf{p}^i$.

For any $X\subset\mathbb{N}^n$ we denote

$$\downarrow X=\{\mathbf{a}\in\mathbb{N}^n\colon\text{there exists }\mathbf{x}\in X\text{ such that }\mathbf{a}\leqslant\mathbf{x}\}.$$

Put $A_0=B_0=\varnothing$. For any $i\geqslant 1$ choose elements $\mathbf{a}^i\in A\setminus\downarrow(A_{i-1}\cup B_{i-1})$ and $\mathbf{b}^i\in B\setminus\downarrow(A_{i-1}\cup B_{i-1})$ and choose a path $p_i=(\mathbf{p}^1,\ldots,\mathbf{p}^k)$ from $\mathbf{a}^i$ to $\mathbf{b}^i$ with the property that all $\mathbf{p}^j \notin A_{i-1} \cup B_{i-1}$. By choosing the path p_i, there exists point $\mathbf{p}^j$ of this path such that $\mathbf{p}^j \in A$ and $\mathbf{p}^{j+1} \in B$, so define the sets $A_i = A_{i-1} \cup \{\mathbf{p}^j\}$ and $B_i = B_{i-1} \cup \{\mathbf{p}^{j+1}\}$.

Next, we define $\tilde{C} = \bigcup\limits_{i=1}^{\infty} A_i$. We remark that for any $\mathbf{a} \in \tilde{C}$ there exist $k \in \{1, \ldots, n\}$ and $s \in \{1, -1\}$ such that $(\mathbf{a})g_k^s \in \bigcup\limits_{i=1}^{\infty} B_i \subset B$, denote these numbers by $k_{\mathbf{a}}$ and $s_{\mathbf{a}}$, respectively. Since the set $\tilde{C}$ is infinite there exists an infinite subset $C \subset \tilde{C}$ such that for any $\mathbf{c} \in \tilde{C}$ the numbers $k_{\mathbf{c}}$ and $s_{\mathbf{c}}$ coincide. □

Lemma 4. *Let $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ be a locally compact non-discrete semitopological semigroup. Then for any neighborhood $U(0)$ of the zero 0 and for any permutation $\sigma \in \mathscr{S}_n$ there exists $\boldsymbol{a} \in \mathbb{N}^n$ such that the set $L_\sigma^{\mathbf{a}} \setminus U(0)$ is finite.*

Proof. Fix any neighborhood $U(0)$ of the zero 0 and any permutation $\sigma \in \mathscr{S}_n$. Lemma 2 implies that there exists $\mathbf{a} \in \mathbb{N}^n$ such that the set $L_\sigma^{\mathbf{a}} \cap U(0)$ is infinite.

The set $L_\sigma^{\mathbf{a}}$ is a disjoint union: $L_\sigma^{\mathbf{a}} = (L_\sigma^{\mathbf{a}} \cap U(0)) \sqcup (L_\sigma^{\mathbf{a}} \setminus U(0))$. The statements of the lemma would be proved when the set $L_\sigma^{\mathbf{a}} \setminus U(0)$ is finite, and hence we assume the opposite, i.e. that the set $L_\sigma^{\mathbf{a}} \setminus U(0)$ is infinite.

We consider the bijection $f_\sigma^a \colon L_\sigma^{\mathbf{a}} \to \mathbb{N}^n$ defined by the formula

$$(\sigma, [\mathbf{a}, \mathbf{x}])f_\sigma^a = \mathbf{x}.$$

Lemma 3 implies that in the set $L_\sigma^{\mathbf{a}} \cap U(0)$ there exist an infinite subset C and an integer number $k \in \{1, \ldots, n\}$ such that at least one of two sets $(C)(f_\sigma^a \circ g_k)$ and $(C)(f_\sigma^a \circ g_k^{-1})$ is a subset of $(L_\sigma^{\mathbf{a}} \setminus U(0))f_\sigma^a$.

We remark that the composition $f_\sigma^a \circ g_k \circ (f_\sigma^a)^{-1}$ coincides with the restriction of right translation $\rho_{(1,[\mathbf{0},\mathbf{1}_k])}$ to the set $L_\sigma^{\mathbf{a}}$, i.e.,

$$f_\sigma^a \circ g_k \circ (f_\sigma^a)^{-1} = \rho_{(1,[\mathbf{1},\mathbf{2}_k])}\big|_{L_\sigma^{\mathbf{a}}}$$

and similarly

$$f_\sigma^a \circ g_k^{-1} \circ (f_\sigma^a)^{-1} = \rho_{(1,[\mathbf{2}_k,\mathbf{1}])}\big|_{L_\sigma^{\mathbf{a}} \setminus \{(\sigma,[\mathbf{a},\mathbf{x}]) \colon \mathbf{x} \in \mathbb{N}^n,\ x_k = 2\}}$$

Lemma 1 and the separate continuity of the semigroup operation in $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ imply that there exists an open compact neighbourhood $V(0)$ of the zero 0 in $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ such that $V \cdot (1, [\mathbf{1}, \mathbf{2}_k]) \subset U(0)$ and $V \cdot (1, [\mathbf{2}_k, \mathbf{1}]) \subset U(0)$.

In any case we have that the set C is a subset of $U(0) \setminus V(0)$. Indeed:

(i) if $(C)(f_\sigma^a \circ g_k)$ is a subset of $(L_\sigma^{\mathbf{a}} \setminus U(0))f_\sigma^a$ then we have that

$$\begin{aligned} C \cdot (1, [\mathbf{1}, \mathbf{2}_k]) &= (C)\rho_{(1,[\mathbf{1},\mathbf{2}_k])} = \\ &= (C)\rho_{(1,[\mathbf{1},\mathbf{2}_k])}\big|_{L_\sigma^{\mathbf{a}}} = \\ &= (C)(f_\sigma^a \circ g_k \circ (f_\sigma^a)^{-1}) \subset \\ &\subset L_\sigma^{\mathbf{a}} \setminus U(0); \end{aligned}$$

(ii) if $(C)(f_\sigma^a \circ g_k^{-1})$ is a subset of $(L_\sigma^{\mathbf{a}} \setminus U(0))f$ then we have that

$$\begin{aligned}C \cdot (1,[\mathbf{2}_k,\mathbf{1}]) &= (C)\rho_{(1,[\mathbf{2}_k,\mathbf{1}])} = \\ &= (C)\rho_{(1,[\mathbf{2}_k,\mathbf{1}])}\big|_{L^{\mathbf{a}}_\sigma\setminus\{(\sigma,[\mathbf{a},\mathbf{x}]) \,\mid\, \mathbf{x}\in\mathbb{N}^n,\ x_k=2\}} = \\ &= (C)(f^a_\sigma\circ g_k^{-1}\circ(f^a_\sigma)^{-1}) \subset \\ &\subset L^{\mathbf{a}}_\sigma\setminus U(0);\end{aligned}$$

and since C is an infinite set, this contradicts Lemma 1(2). □

Lemma 5. *Let $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ be a locally compact non-discrete semitopological semigroup. Then for any neighborhood $U(0)$ of the zero 0, any permutation $\sigma\in\mathscr{S}_n$ and any element $\boldsymbol{a}\in\mathbb{N}^n$ the set $L^{\boldsymbol{a}}_\sigma\setminus U(0)$ is finite.*

Proof. Consider any neighborhood $U(0)$ of the zero 0 and any permutation $\sigma\in\mathscr{S}_n$. Lemma 4 implies that there exists $\mathbf{b}\in\mathbb{N}^n$ such that the set $L^{\mathbf{b}}_\sigma\setminus U(0)$ is finite. Fix any $\mathbf{a}\in\mathbb{N}^n\setminus\{\mathbf{b}\}$. Define elements $\mathbf{q},\mathbf{p}\in\mathbb{N}^n$ in the following way: for any $i\in\{1,.,n\}$ put

$$q_i=1,\quad p_i=b_i-a_i,\qquad \text{if}\quad b_i\geqslant a_i;$$

$$p_i=1,\quad q_i=a_i-b_i,\qquad \text{if}\quad b_i<a_i.$$

We remark that $\mathbf{q}-\mathbf{p}=\mathbf{a}-\mathbf{b}$ and $\max\{\mathbf{p},\mathbf{b}\}=\mathbf{b}$. Then, the restriction of the left translation $\lambda_{(1,[(\mathbf{q})\sigma^{-1},(\mathbf{p})\sigma^{-1}])}$ on the set $L^{\mathbf{b}}_\sigma$ is a bijection between $L^{\mathbf{b}}_\sigma$ and $L^{\mathbf{a}}_\sigma$: for any $(\sigma,[\mathbf{b},\mathbf{x}])\in L^{\mathbf{b}}_\sigma$ we have that

$$\begin{aligned}(\sigma,[\mathbf{b},\mathbf{x}])\lambda_{(1,[(\mathbf{q})\sigma^{-1},(\mathbf{p})\sigma^{-1}])} &= (1,[(\mathbf{q})\sigma^{-1},(\mathbf{p})\sigma^{-1}])\cdot(\sigma,[\mathbf{b},\mathbf{x}]) = \\ &= (\sigma,[\mathbf{q},\mathbf{p}]*[\mathbf{b},\mathbf{x}]) = \\ &= (\sigma,[\max\{\mathbf{p},\mathbf{b}\}-\mathbf{p}+\mathbf{q},\max\{\mathbf{p},\mathbf{b}\}-\mathbf{b}+\mathbf{x}]) = \\ &= (\sigma,[\mathbf{b}-\mathbf{p}+\mathbf{q},\mathbf{x}]) = (\sigma,[\mathbf{a},\mathbf{x}]).\end{aligned}$$

Lemma 1 and the separate continuity of the semigroup operation in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ imply that there exists an open compact neighbourhood $V(0)$ of the zero 0 in $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ such that $(1,[(\mathbf{q})\sigma^{-1},(\mathbf{p})\sigma^{-1}])\cdot V(0)\subset U(0)$. Since

$$\begin{aligned}L^{\mathbf{a}}_\sigma\setminus U(0) &\subseteq L^{\mathbf{a}}_\sigma\setminus(1,[(\mathbf{q})\sigma^{-1},(\mathbf{p})\sigma^{-1}])\cdot V(0) = \\ &= L^{\mathbf{a}}_\sigma\setminus(V(0))\lambda_{(1,[(\mathbf{q})\sigma^{-1},(\mathbf{p})\sigma^{-1}])} = \\ &= (L^{\mathbf{b}}_\sigma\setminus V(0))\lambda_{(1,[(\mathbf{q})\sigma^{-1},(\mathbf{p})\sigma^{-1}])},\end{aligned}$$

$L^{\mathbf{a}}_\sigma\setminus U(0)$ is finite, as the set $L^{\mathbf{b}}_\sigma\setminus V(0)$ is finite. □

Lemma 6. *Let $(\mathscr{IPF}(\mathbb{N}^n)^0,\tau)$ be a locally compact non-discrete semitopological semigroup. Then for any neighborhood $U(0)$ of the zero 0 and for any permutation $\sigma\in\mathscr{S}_n$ there exist only finite number of elements $\boldsymbol{a}\in\mathbb{N}^n$ such that the set $L^{\boldsymbol{a}}_\sigma\setminus U(0)$ is non empty, i.e. the set $\{\boldsymbol{a}\in\mathbb{N}^n\colon L^{\boldsymbol{a}}_\sigma\setminus U(0)\neq\varnothing\}$ is finite.*

Proof. Suppose to the contrary that there exist neighborhood $U(0)$ of the zero 0 and a permutation $\sigma\in\mathscr{S}_n$ such that the set $M=\left\{\mathbf{b}\in\mathbb{N}^n\colon L^{\mathbf{b}}_\sigma\setminus U(0)\neq\varnothing\right\}$ is infinite. Since for any $\mathbf{b}\in M$, by Lemma 5, the set $L^{\mathbf{b}}_\sigma\setminus U(0)$ is finite, there exist an element

$\mathbf{x_b} \in \mathbb{N}^n$ and a positive integer $k_\mathbf{b} \in \{1, \ldots, n\}$ such that $(\sigma, [\mathbf{b}, \mathbf{x_b}]) \notin L_\sigma^\mathbf{b} \setminus U(0)$ and $(\sigma, [\mathbf{b}, \mathbf{x_b} - (0, \ldots, \underbrace{1}_{k_\mathbf{b}}, \ldots, 0)]) \in L_\sigma^\mathbf{b} \setminus U(0)$. This defines the maps:

$$\gamma\colon M \to \mathbb{N}^n, \ \mathbf{b} \mapsto \mathbf{x_b},$$

$$\phi\colon M \to \{1, \ldots, n\}, \ \mathbf{b} \mapsto k_\mathbf{b}.$$

Since M is infinite and $(M)\phi$ finite, there exist an infinite subset $M' \subset M$ and positive integer $k_{M'} \in \{1, \ldots, n\}$ such that for any two elements $\mathbf{u}, \mathbf{v} \in M'$ the equality

$$(\mathbf{u})\phi = (\mathbf{v})\phi = k_{M'}$$

holds.

Lemma 1 and the separate continuity of the semigroup operation in $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ imply that there exists an open compact neighbourhood $V(0)$ of the zero 0 in $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ such that $V(0) \cdot (1, [\mathbf{2}_{k_{M'}}, 1]) \subset U(0)$.

Put $P = \{(\sigma, [\mathbf{b}, (\mathbf{b})\gamma]) \colon \mathbf{b} \in M'\}$. Then the choice of M' implies $P \subset U(0) \setminus V(0)$, which contradicts Lemma 1(2), because the set M' is infinite. □

Corollary 1. *Let $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ be a locally compact non-discrete semitopological semigroup. Then for any neighborhood $U(0)$ of the zero 0 the set $\mathscr{IPF}(\mathbb{N}^n)^0 \setminus U(0)$ is finite.*

Proof. Since

$$\mathscr{IPF}(\mathbb{N}^n) = \bigsqcup_{\sigma \in \mathscr{S}_n} \{\sigma\} \times (\bigsqcup_{\mathbf{a} \in \mathbb{N}^n} L_\sigma^\mathbf{a}),$$

Lemma 6 implies that the set

$$\mathscr{IPF}(\mathbb{N}^n) \setminus U(0) = \bigsqcup_{\sigma \in \mathscr{S}_n} \{\sigma\} \times \bigsqcup_{\mathbf{a} \in \mathbb{N}^n} L_\sigma^\mathbf{a} \setminus U(0) = \bigsqcup_{\sigma \in \mathscr{S}_n} \{\sigma\} \times \bigsqcup_{\mathbf{a} \in \mathbb{N}^n} L_\sigma^\mathbf{a} \setminus U(0)$$

is finite. □

Example 1. We define a topology τ_{Ac} on the semigroup $\mathscr{IPF}(\mathbb{N}^n)^0$ in the followinh way:

- (i) every element of the semigroup $\mathscr{IPF}(\mathbb{N}^n)$ is an isolated point in the space $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau_{Ac})$;
- (ii) the family $\mathcal{B}_{Ac}(0) = \left\{U \subset \mathscr{IPF}(\mathbb{N}^n)^0 : U \ni 0 \text{ and } \mathscr{IPF}(\mathbb{N}^n) \setminus U \text{ is finite}\right\}$ determines a base of the topology τ_{Ac} at zero $0 \in \mathscr{IPF}(\mathbb{N}^n)^0$

i.e., τ_{Ac} is the topology of the Alexandroff one-point compactification of the discrete space $\mathscr{IPF}(\mathbb{N}^n)$ with the remainder 0. The semigroup operation in $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ is separately continuous, because all elements of the semigroup $\mathscr{IPF}(\mathbb{N}^n)$ are isolated points in the space $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau)$ and any first order equation in $\mathscr{IPF}(\mathbb{N}^n$ has finitely many solutions (see Proposition 2.26 in [16]).

Remark 2. In [16] it was showed that the discrete topology τ_d is a unique shift continuous Hausdorff topology on the semigroup $\mathscr{IPF}(\mathbb{N}^n)$. Therefore, τ_{Ac} is the unique compact topology on this semigroup such that $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau_{Ac})$ is a compact semitopological semigroup.

Lemma 1 and Remark 2 imply the following dichotomy for a locally compact semitopological semigroup $\mathscr{IPF}(\mathbb{N}^n)^0$.

Theorem 1. *If $\mathscr{IPF}(\mathbb{N}^n)^0$ is a Hausdorff locally compact semitopological semigroup, then either $\mathscr{IPF}(\mathbb{N}^n)^0$ is discrete or $\mathscr{IPF}(\mathbb{N}^n)^0$ is topologically isomorphic to $(\mathscr{IPF}(\mathbb{N}^n)^0, \tau_{Ac})$.*

By Corollary 3.3 of [16] the semigroup $\mathscr{IPF}(\mathbb{N}^n)$ does not embed into a compact Hausdorff topological semigroup. Hence Theorem 1 implies the following corollary:

Corollary 2. *If $\mathscr{IPF}(\mathbb{N}^n)^0$ is a Hausdorff locally compact topological semigroup then the space $\mathscr{IPF}(\mathbb{N}^n)^0$ is discrete.*

Acknowledgements

The author is grateful to the referee and his PhD advisor Oleg Gutik for several useful comments and suggestions.

References

1. O. Andersen, *Ein Bericht über die Struktur abstrakter Halbgruppen*, PhD Thesis, Hamburg, 1952.
2. L. W. Anderson, R. P. Hunter, and R. J. Koch, *Some results on stability in semigroups*. Trans. Amer. Math. Soc. **117** (1965), 521–529. DOI: 10.2307/1994222
3. T. Banakh, S. Dimitrova, and O. Gutik, *The Rees-Suschkiewitsch Theorem for simple topological semigroups*, Mat. Stud. **31** (2009), no. 2, 211–218.
4. T. Banakh, S. Dimitrova, and O. Gutik, *Embedding the bicyclic semigroup into countably compact topological semigroups*, Topology Appl. **157**:18 (2010), 2803–2814. DOI: 10.1016/j.topol.2010.08.020
5. S. Bardyla, *Classifying locally compact semitopological polycyclic monoids*, Math. Bull. Shevchenko Sci. Soc. **13** (2016), 21–28.
6. S. Bardyla, *On locally compact semitopological graph inverse semigroups*, Mat. Stud. **49** (2018), no. 1, 19–28. DOI: 10.15330/ms.49.1.19-28
7. S. Bardyla and O. Gutik, *On the lattice of weak topologies on the bicyclic monoid with adjoined zero*, Preprint (arXiv:1908.04566).
8. S. Bardyla and A. Ravsky, *Closed subsets of compact-like topological spaces*, Preprint (arXiv:1907.12129).
9. M. O. Bertman and T. T. West, *Conditionally compact bicyclic semitopological semigroups*, Proc. Roy. Irish Acad. Sec. A **76** (1976), no. 21–23, 219–226.
10. J. H. Carruth, J. A. Hildebrant, and R. J. Koch, *The theory of topological semigroups*, Vol. I, Marcel Dekker, Inc., New York and Basel, 1983; Vol. II, Marcel Dekker, Inc., New York and Basel, 1986.
11. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vols. I and II, Amer. Math. Soc. Surveys **7**, Providence, R.I., 1961 and 1967.
12. R. Engelking, *General Topology*, 2nd ed., Heldermann, Berlin, 1989.
13. O. Gutik, *On the dichotomy of a locally compact semitopological bicyclic monoid with adjoined zero*, Visn. L'viv. Univ., Ser. Mekh.-Mat. **80** (2015), 33–41.
14. O. Gutik, *On locally compact semitopological 0-bisimple inverse ω-semigroups*, Topol. Algebra Appl. **6** (2018), 77–101. DOI: 10.1515/taa-2018-0008
15. O. Gutik and K. Maksymyk, *On semitopological interassociates of the bicyclic monoid*, Visn. L'viv. Univ., Ser. Mekh.-Mat. **82** (2016), 98–108.

16. O. Gutik and T. Mokrytskyi, *The monoid of order isomorphisms between principal filters of* $\mathbb{N}^n$, Eur. J. Math. (to appear). DOI: 10.1007/s40879-019-00328-5
17. O. Gutik and D. Repovš, *On countably compact* 0*-simple topological inverse semigroups*, Semigroup Forum **75** (2007), no. 2, 464–469. DOI: 10.1007/s00233-007-0706-x
18. J. A. Hildebrant and R. J. Koch, *Swelling actions of* Γ*-compact semigroups*, Semigroup Forum **33** (1986), no. 1, 65–85. DOI: 10.1007/BF02573183
19. M. Petrich, *Inverse semigroups*, John Wiley & Sons, New York, 1984.
20. W. Ruppert, *Compact semitopological semigroups: An intrinsic theory*, Lect. Notes Math., **1079**, Springer, Berlin, 1984.

Стаття: надійшла до редколегії 22.04.2019
доопрацьована 06.08.2019
прийнята до друку 13.11.2019

ПРО ДИХОТОМІЮ ЛОКАЛЬНО КОМПАКНОГО НАПІВТОПОЛОГІЧНОГО МОНОЇДА ПОРЯДКОВИХ ІЗОМОРФІЗМІВ МІЖ ГОЛОВНИМИ ФІЛЬТРАМИ МНОЖИНИ $\mathbb{N}^n$ З ПРИЄДНАНИМ НУЛЕМ

Тарас МОКРИЦЬКИЙ

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, 79000, м. Львів
e-mail: tmokrytskyi@gmail.com

Нехай n — довільне натуральне число і нехай $\mathscr{IPF}(\mathbb{N}^n)$ — напівгрупа всіх порядкових ізоморфізмів між головними фільтрами n-го степеня натуральних чисел $\mathbb{N}$ з порядком добутку. Доведено, що гаусдорфова локально компактна напівтопологічна напівгрупа $\mathscr{IPF}(\mathbb{N}^n)$ з приєднаним нулем є або компактною або дискретною.

Ключові слова: напівгрупа, інверсна напівгрупа, біциклічний моноїд, напівтопологічна напівгрупа, топологічна напівгрупа, локально компактний, компактний, дискретний.