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EXTENSION OF SEMIGROUPS BY SYMMETRIC INVERSE SEMIGROUPS OF A BOUNDED FINITE RANK

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We study the semigroup extension $\mathscr{I}_\lambda^n(S)$ of a semigroup S by symmetric inverse semigroup of a bounded finite rank n. We describe idempotents and regular elements of the semigroup $\mathscr{I}_\lambda^n(S)$ and show that the semigroup $\mathscr{I}_\lambda^n(S)$ is regular, orthodox, inverse or stable if and only if so is S. Green's relations are described on the semigroup $\mathscr{I}_\lambda^n(S)$ for an arbitrary monoid S. We introduce the conception of a semigroup with strongly tight ideal series, and prove that for any infinite cardinal λ and any positive integer n the semigroup $\mathscr{I}_\lambda^n(S)$ has a strongly tight ideal series provided so has S. Finally, we show that for every compact Hausdorff semitopological monoid (S,τ_S) there exists its unique compact topological extension $(\mathscr{I}_\lambda^n(S),\tau_{\mathscr{I}}^{\mathsf{c}})$ in the class of Hausdorff semitopological semigroups.

Key words: inverse semigroup, symmetric inverse semigroup of finite transformations, Green's relations, semigroup has a tight ideal series, semitopologica; semigroup, compact semigroup.

1. Introduction, motivation and main definitions

In this paper we follow the terminology of [11, 31].

If S is a semigroup, then by $E(S)$ we denote the subset of all idempotents of S. On the set of idempotents $E(S)$ there exists the natural partial order: $e \leqslant f$ *if and only if* $ef = fe = e$.

A semigroup S is called:

- *regular*, if for every $a \in S$ there exists an element b in S such that $a = aba$;
- *orthodox*, if S is regular and $E(S)$ is a subsemigroup of S;

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- *inverse* if every a in S possesses a unique inverse, i.e. if there exists a unique element a^{-1} in S such that

$$aa^{-1}a = a \qquad \text{and} \qquad a^{-1}aa^{-1} = a^{-1}.$$

It is obvious that every inverse semigroup is orthodox and every orthodox semigroup is regular. A map which associates to any element of an inverse semigroup its inverse is called the *inversion*.

Let λ be an arbitrary non-zero cardinal. A map α from a subset D of λ into λ is called a *partial transformation* of X. In this case the set D is called the *domain* of α and is denoted by $\operatorname{dom}\alpha$. Also, the set $\{x \in \lambda \colon y\alpha = x \text{ for some } y \in \lambda\}$ is called the *range* of α and is denoted by $\operatorname{ran}\alpha$. The cardinality of $\operatorname{ran}\alpha$ is called the *rank* of α and denoted by $\operatorname{rank}\alpha$. For convenience we denote by $\varnothing$ the empty transformation, that is a partial mapping with $\operatorname{dom}\varnothing = \operatorname{ran}\varnothing = \varnothing$.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \colon y\alpha \in \operatorname{dom}\beta\}, \qquad \text{for} \quad \alpha, \beta \in \mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the cardinal λ (see [11]). The symmetric inverse semigroup was introduced by V. V. Wagner [33] and it plays a major role in the theory of semigroups.

Put

$$\mathscr{I}_\lambda^\infty = \{\alpha \in \mathscr{I}_\lambda \colon \operatorname{rank}\alpha \text{ is finite}\} \quad \text{and} \quad \mathscr{I}_\lambda^n = \{\alpha \in \mathscr{I}_\lambda \colon \operatorname{rank}\alpha \leqslant n\},$$

for $n = 1, 2, 3, \ldots$. Obviously, $\mathscr{I}_\lambda^\infty$ and $\mathscr{I}_\lambda^n$ $(n = 1, 2, 3, \ldots)$ are inverse semigroups, $\mathscr{I}_\lambda^\infty$ is an ideal of $\mathscr{I}_\lambda$, and $\mathscr{I}_\lambda^n$ is an ideal of $\mathscr{I}_\lambda^\infty$, for each $n = 1, 2, 3, \ldots$. Further, we shall call the semigroup $\mathscr{I}_\lambda^\infty$ the *symmetric inverse semigroup of finite transformations* and $\mathscr{I}_\lambda^n$ the *symmetric inverse semigroup of finite transformations of the rank* $\leqslant n$. The elements of semigroups $\mathscr{I}_\lambda^\infty$ and $\mathscr{I}_\lambda^n$ are called *finite one-to-one transformations* (*partial bijections*) of the cardinal λ. By

$$\left(\begin{smallmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{smallmatrix}\right)$$

we denote a partial one-to-one transformation which maps x_1 onto y_1, ..., x_n onto y_n, and by 0 the empty transformation. Obviously, in such case we have $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$ $(i, j = 1, \ldots, n)$. The empty partial map $\varnothing \colon \lambda \rightharpoonup \lambda$ is denoted by 0. It is obvious that 0 is zero of the semigroup $\mathscr{I}_\lambda^n$.

Let λ be a non-zero cardinal. On the set $B_\lambda = (\lambda \times \lambda) \cup \{0\}$, where $0 \notin \lambda \times \lambda$, we define the semigroup operation " $\cdot$ " as follows

$$(a, b) \cdot (c, d) = \begin{cases} (a, d), & \text{if } b = c; \\ 0, & \text{if } b \neq c, \end{cases}$$

and $(a, b) \cdot 0 = 0 \cdot (a, b) = 0 \cdot 0 = 0$ for $a, b, c, d \in \lambda$. The semigroup B_λ is called the *semigroup of* $\lambda \times \lambda$*-matrix units* (see [11]). Obviously, for any cardinal $\lambda > 0$, the semigroup of $\lambda \times \lambda$-matrix units B_λ is isomorphic to $\mathscr{I}_\lambda^1$.

Let S be a semigroup with zero and λ be a non-zero cardinal. We define the semigroup operation on the set $B_\lambda(S) = (\lambda \times S \times \lambda) \cup \{0\}$ as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $a, b \in S$. If $S = S^1$ then the semigroup $B_\lambda(S)$ is called the *Brandt λ-extension of the semigroup* S [15, 19]. Obviously, if S has zero then $\mathcal{J} = \{0\} \cup \{(\alpha, 0_S, \beta) \colon 0_S \text{ is the zero of } S\}$ is an ideal of $B_\lambda(S)$. We put $B^0_\lambda(S) = B_\lambda(S)/\mathcal{J}$ and the semigroup $B^0_\lambda(S)$ is called the *Brandt λ^0-extension of the semigroup S with zero* [22].

A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation.

The Brandt λ-extension $B_\lambda(S)$ (or the Brandt λ^0-extension $B^0_\lambda(S)$) of a semigroup S can be considered as some semigroup extension of the semigroup S by the semigroup of $\lambda \times \lambda$-matrix units B_λ. An analogue of such extension gives the following construction.

2. The construction of of the semigroup extension $\mathscr{I}^n_\lambda(S)$

In this paper using the semigroup $\mathscr{I}^n_\lambda$ we propose the following semigroup extension.

Construction 1. Let S be a semigroup, λ be a non-zero cardinal, n and k be a positive integers such that $k \leqslant n \leqslant \lambda$. We identify every element $\alpha \in \mathscr{I}^n_\lambda$ with its graph $\mathsf{Gr}(\alpha) \subset \lambda \times \lambda$ and put

$$\mathscr{I}^n_\lambda(S) = \{\alpha_S \colon \mathsf{Gr}(\alpha) \to S \colon \alpha \in \mathscr{I}^n_\lambda\}$$

and every map from the empty map 0 into S is identified with the empty map $\varnothing \colon \lambda \times \lambda \rightharpoonup S$ and denote it by 0. An arbitrary element $0 \neq \operatorname{rank} \alpha \leqslant n$ is denoted by

$$\begin{pmatrix} x_1 & \cdots & x_k \\ s_1 & \cdots & s_k \\ y_1 & \cdots & y_k \end{pmatrix},$$

where $\alpha = \begin{pmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{pmatrix}$, and $((x_1, y_1))\,\alpha = s_1, \ \ldots, \ ((x_k, y_k))\,\alpha = s_k$. Also for $\alpha_S \in \mathscr{I}^n_\lambda(S)$ such that

$$\alpha_S = \begin{pmatrix} x_1 & \cdots & x_k \\ s_1 & \cdots & s_k \\ y_1 & \cdots & y_k \end{pmatrix}$$

we denote $\mathbf{d}(\alpha_S) = \{x_1, \ldots, x_k\}$ and $\mathbf{r}(\alpha_S) = \{y_1, \ldots, y_k\}$.

Now, we define a binary operation "$\cdot$" on the set $\mathscr{I}^n_\lambda(S)$ in the following way:

(*i*) $\alpha_S \cdot 0 = 0 \cdot \alpha_S = 0 \cdot 0 = 0$ for every $\alpha_S \in \mathscr{I}^n_\lambda(S)$;

(*ii*) if $\alpha \cdot \beta = 0$ in $\mathscr{I}^n_\lambda$ then $\alpha_S \cdot \beta_S = 0$ for any $\alpha_S, \beta_S \in \mathscr{I}^n_\lambda(S)$;

(*iii*) if $\alpha_S = \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix}$, $\beta_S = \begin{pmatrix} c_1 & \cdots & c_j \\ t_1 & \cdots & t_j \\ d_1 & \cdots & d_j \end{pmatrix}$ and

$$\alpha \cdot \beta = \begin{pmatrix} a_1 & \cdots & a_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \begin{pmatrix} c_1 & \cdots & c_j \\ d_1 & \cdots & d_j \end{pmatrix} = \begin{pmatrix} a_{i_1} & \cdots & a_{i_m} \\ d_{j_1} & \cdots & d_{j_m} \end{pmatrix} \neq 0 \quad \text{in } \mathscr{I}^n_\lambda,$$

then $\alpha_S \cdot \beta_S = \begin{pmatrix} a_{i_1} & \cdots & a_{i_m} \\ s_{i_1} t_{j_1} & \cdots & s_{i_m} t_{j_m} \\ d_{j_1} & \cdots & d_{j_m} \end{pmatrix}$.

Simple verifications show that the defined binary operation on $\mathscr{I}^n_\lambda(S)$ is associative and hence $\mathscr{I}^n_\lambda(S)$ is a semigroup. It is obvious that $\mathscr{I}^1_\lambda(S)$ is isomorphic to the Brandt λ-extension $B_\lambda(S)$ of the semigroup S.

We remark that if the semigroup S contains zero 0_S then

$$\mathscr{J}_0=\{0\}\cup\left\{\alpha_S=\begin{pmatrix} a_1 & \cdots & a_i\\ 0_S & \cdots & 0_S\\ b_1 & \cdots & b_i\end{pmatrix}\colon 0_S \text{ is the zero of } S\right\}$$

is an ideal of $\mathscr{I}_\lambda^n(S)$.

Also, we define a binary relation $\equiv_0$ on the semigroup $\mathscr{I}_\lambda^n(S)$ in the following way. For $\alpha_S,\beta_S\in\mathscr{I}_\lambda^n(S)$ we put $\alpha_S\equiv_0\beta_S$ if and only if at least one of the following conditions holds:

(1) $\alpha_S=\beta_S$;
(2) $\alpha_S,\beta_S\in\mathscr{J}_0$;
(3) $\alpha_S,\beta_S\notin\mathscr{J}_0$ and each of the conditions
  (*i*) $(x,y)\alpha_S$ is determined and $(x,y)\alpha_S\neq 0_S$; and
  (*ii*) $(x,y)\beta_S$ is determined and $(x,y)\beta_S\neq 0_S$
  implies the equality $(x,y)\alpha_S=(x,y)\beta_S$.

It is obvious that $\equiv_0$ is an equivalence relation on the semigroup $\mathscr{I}_\lambda^n(S)$.

The following proposition can be proved by immediate verifications.

Proposition 1. *The relation $\equiv_0$ is a congruence on the semigroup $\mathscr{I}_\lambda^n(S)$.*

We define $\overline{\mathscr{I}_\lambda^n}(S)=\mathscr{I}_\lambda^n(S)/_{\equiv_0}$.

In this paper we study algebraic properties of the semigroups $\mathscr{I}_\lambda^n(S)$ and $\overline{\mathscr{I}_\lambda^n}(S)$. We describe idempotents and regular elements of the semigroups $\mathscr{I}_\lambda^n(S)$ and $\overline{\mathscr{I}_\lambda^n}(S)$, show that the semigroup $\mathscr{I}_\lambda^n(S)$ $(\overline{\mathscr{I}_\lambda^n}(S))$ is regular, orthodox, inverse or stable if and only if so is S. Green's relations are described in the semigroup $\mathscr{I}_\lambda^n(S)$ for an arbitrary monoid S. We introduce the conception of a semigroup with strongly tight ideal series, and proved that for any infinite cardinal λ and any positive integer n the semigroup $\mathscr{I}_\lambda^n(S)$ has a strongly tight ideal series provides so has S. Finally, we show that for every compact Hausdorff semitopological monoid (S,τ_S) there exists its unique compact topological extension $(\mathscr{I}_\lambda^n(S),\tau_{\mathscr{I}}^{\mathsf{c}})$ in the class of Haudorff semitopological semigroups.

3. Algebraic properties of the semigroup extensions $\mathscr{I}_\lambda^n(S)$ and $\overline{\mathscr{I}_\lambda^n}(S)$

The following proposition describes the subset of idempotents of the semigroup $\mathscr{I}_\lambda^n(S)$.

Proposition 2. *For every positive integer $i\leqslant n$ a non-zero element $\alpha_S=\begin{pmatrix} a_1 & \cdots & a_i\\ s_1 & \cdots & s_i\\ b_1 & \cdots & b_i\end{pmatrix}$ of the semigroup $\mathscr{I}_\lambda^n(S)$ is an idempotent if and only if $a_1=b_1,\ldots,a_i=b_i$ and $s_1,\ldots,s_i\in E(S)$.*

Proof. The implication $(\Leftarrow)$ is trivial.

$(\Rightarrow)$ Suppose that $\alpha_S\cdot\alpha_S=\alpha_S$. Then the definition of the semigroup $\mathscr{I}_\lambda^n(S)$ implies that the symbols $a_1,\ldots,a_i$ are distinct. Similarly we obtain that the symbols $b_1,\ldots,b_i$ are distinct, too. The above arguments and the equality $\alpha_S\cdot\alpha_S=\alpha_S$ imply that $\{a_1,\ldots,a_i\}=\{b_1,\ldots,b_i\}$. Assume that $a_k\neq b_k=a_l$ for some integers $k,l\in\{1,\ldots,i\}$,

$k \neq l$. Then we have that $a_l \neq b_l \neq b_k$, which contradicts the equality $\alpha_S \cdot \alpha_S = \alpha_S$. The obtained contradiction implies the equalities $a_1 = b_1, \ldots, a_i = b_i$. Now, we get that

$$\alpha_S \cdot \alpha_S = \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ a_1 & \cdots & a_i \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ a_1 & \cdots & a_i \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 s_1 & \cdots & s_i s_i \\ a_1 & \cdots & a_i \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ a_1 & \cdots & a_i \end{pmatrix} = \alpha_S,$$

and hence $s_1 s_1 = s_1, \ldots, s_i s_i = s_i$. This completes the proof of the proposition. □

Proposition 3. *For every positive integer $i \leqslant n$ a non-zero element $\alpha_S = \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix}$ of the semigroup $\mathscr{I}_\lambda^n(S)$ is regular if and only if so are $s_1, \ldots, s_i$ in S.*

Proof. The implication ($\Leftarrow$) is trivial. Indeed, $\alpha_S = \alpha_S \beta_S \alpha_S$ for $\beta_S = \begin{pmatrix} b_1 & \cdots & b_i \\ t_1 & \cdots & t_i \\ a_1 & \cdots & a_i \end{pmatrix}$, where elements $t_1, \ldots, t_i$ of the semigroup S are such that $s_1 = s_1 t_1 s_1, \ldots, s_i = s_i t_i s_i$.

($\Rightarrow$) Suppose that α_S is a regular element of the semigroup $\mathscr{I}_\lambda^n(S)$. Then there exists an element $\beta_S = \begin{pmatrix} c_1 & \cdots & c_k \\ t_1 & \cdots & t_k \\ d_1 & \cdots & d_k \end{pmatrix}$ of the semigroup $\mathscr{I}_\lambda^n(S)$, $0 < k \leqslant n$, such that $\alpha_S = \alpha_S \cdot \beta_S \cdot \alpha_S$. Now, this implies that $\{b_1, \ldots, b_i\} \subseteq \{c_1, \ldots, c_k\}$ and hence $k \geqslant i$. Without loss of generality we may assume that $b_1 = c_1, \ldots, b_i = c_i$. Then the equality $\alpha_S = \alpha_S \cdot \beta_S \cdot \alpha_S$ and the semigroup operation of $\mathscr{I}_\lambda^n(S)$ imply that $d_1 = a_1, \ldots, d_i = a_i$ and hence we have that

$$\begin{aligned} \alpha_S = \alpha_S \cdot \beta_S \cdot \alpha_S &= \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \begin{pmatrix} c_1 & \cdots & c_k \\ t_1 & \cdots & t_k \\ d_1 & \cdots & d_k \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} = \\ &= \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \begin{pmatrix} b_1 & \cdots & b_i & c_{i+1} & \cdots & c_k \\ t_1 & \cdots & t_i & t_{i+1} & \cdots & t_k \\ a_1 & \cdots & a_i & d_{i+1} & \cdots & d_k \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} = \\ &= \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 t_1 s_1 & \cdots & s_i t_i s_i \\ b_1 & \cdots & b_i \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix}. \end{aligned}$$

This implies that the equalities $s_1 = s_1 t_1 s_1, \ldots, s_i = s_i t_i s_i$ hold in S, which completes the proof of our proposition. □

Two elements a and b of a semigroup S are said to be *inverses* of each other if

$$aba = a \qquad \text{and} \qquad bab = b.$$

The definition of the semigroup operation in $\mathscr{I}_\lambda^n(S)$ implies the following proposition.

Proposition 4. *Let λ be a non-zero cardinal, n and i be any positive integers such that $i \leqslant n \leqslant \lambda$. Let S be a semigroup and $a_1, \ldots, a_i, b_1, \ldots, b_i \in \lambda$. If the elements s_1 and t_1, $\ldots$, s_i and t_i are pairwise inverses of each other in S then the elements*

$$\begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} b_1 & \cdots & b_i \\ t_1 & \cdots & t_i \\ a_1 & \cdots & a_i \end{pmatrix}$$

are pairwise inverses of each other in the semigroup $\mathscr{I}_\lambda^n(S)$.

For arbitrary semigroup S, every positive integer $i \leqslant n$, any collection non-empty subsets $A_1, \ldots, A_i$ of S, and any two collections of distinct elements $\{a_1, \ldots, a_i\}$ and $\{b_1, \ldots, b_i\}$ of the cardinal λ we define a subset

$$[A_1, \ldots, A_i]_{(b_1, \ldots, b_i)}^{(a_1, \ldots, a_i)} = \left\{ \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} : s_1 \in A_1, \ldots, s_i \in A_i \right\}$$

of $\mathscr{I}_\lambda^n(S)$. I the case when $A_1 = \ldots = A_i = A$ in S we put

$$[A]_{(b_1,\ldots,b_i)}^{(a_1,\ldots,a_i)} = [A_1,\ldots,A_i]_{(b_1,\ldots,b_i)}^{(a_1,\ldots,a_i)}.$$

It is obvious that for every subset A of the semigroup S and any permutation $\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ we have that

$$[A]_{(b_{(1)\sigma},\ldots,b_{(i)\sigma})}^{(a_{(1)\sigma},\ldots,a_{(i)\sigma})} = [A]_{(b_1,\ldots,b_i)}^{(a_1,\ldots,a_i)}.$$

Proposition 5. *Let λ be a non-zero cardinal and n be any positive integer $\leqslant \lambda$. Then for arbitrary semigroup S, every positive integer $i \leqslant n$ and any collection of distinct elements $\{a_1,\ldots,a_i\}$ of λ the direct power S^i is isomorphic to a subsemigroup $S_{(a_1,\ldots,a_i)}^{(a_1,\ldots,a_i)}$ of $\mathscr{I}_\lambda^n(S)$.*

Proof. The semigroup operation of $\mathscr{I}_\lambda^n(S)$ implies that $S_{a_1,\ldots,a_i}^{a_1,\cdots,a_i}$ is a subsemigroup of $\mathscr{I}_\lambda^n(S)$ for any collection of distinct elements $\{a_1,\ldots,a_i\}$ of λ. We define an isomorphism $\mathfrak{h}\colon S^i \to S_{(a_1,\ldots,a_i)}^{(a_1,\ldots,a_i)}$ by the formula $(s_1,\ldots,s_i)\mathfrak{h} = \left(\begin{smallmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ a_1 & \cdots & a_i \end{smallmatrix}\right)$. □

Proposition 6. *For every semigroup S, any non-zero cardinal λ and any positive integer $n \leqslant \lambda$ the following statements hold:*

(i) $\mathscr{I}_\lambda^n(S)$ is regular if and only if so is S;
(ii) $\mathscr{I}_\lambda^n(S)$ is orthodox if and only if so is S;
(iii) $\mathscr{I}_\lambda^n(S)$ is inverse if and only if so is S.

Proof. Statement (i) follows from Proposition 3.

(ii) $(\Leftarrow)$ Suppose that S is an orthodox semigroup. Then statement (i) implies that the semigroup $\mathscr{I}_\lambda^n(S)$ is regular. By Proposition 2 every non-zero idempotent of the semigroup $\mathscr{I}_\lambda^n(S)$ has the form $\left(\begin{smallmatrix} a_1 & \cdots & a_i \\ e_1 & \cdots & e_i \\ a_1 & \cdots & a_i \end{smallmatrix}\right)$, where $0 < i \leqslant n$ and $e_1,\ldots,e_i$ are idempotents of S. This implies that the product of two idempotents of $\mathscr{I}_\lambda^n(S)$ is again an idempotent, and hence the semigroup $\mathscr{I}_\lambda^n(S)$ is orthodox.

$(\Rightarrow)$ Suppose that $\mathscr{I}_\lambda^n(S)$ is an orthodox semigroup. By Proposition 5, $S_{(a)}^{(a)}$ is a subsemigroup of $\mathscr{I}_\lambda^n(S)$ for every $a \in \lambda$ and hence $S_{(a)}^{(a)}$ is orthodox. Then Proposition 5 implies the semigroup S is orthodox, too.

(iii) $(\Leftarrow)$ Suppose that S is an inverse semigroup. By statement (i) the semigroup $\mathscr{I}_\lambda^n(S)$ is regular. Then using Proposition 2 we get that idempotents commute in $\mathscr{I}_\lambda^n(S)$ and hence by Theorem 1.17 of [11], $\mathscr{I}_\lambda^n(S)$ is an inverse semigroup.

$(\Rightarrow)$ Suppose that $\mathscr{I}_\lambda^n(S)$ is an inverse semigroup. By Proposition 5, $S_{(a)}^{(a)}$ is a subsemigroup of $\mathscr{I}_\lambda^n(S)$ for every $a \in \lambda$, and by Proposition 4 it is an inverse subsemigroup. Hence by Proposition 5, S is an inverse semigroup. □

Since any homomorphic image of a regular (resp., orthodox, inverse) semigroup is a regular (resp., orthodox, inverse) semigroup (see [11, Section 7.4] and [29, Lemma 2.2]), Proposition 6 implies the following corollary.

Corollary 1. *For every semigroup S, any non-zero cardinal λ and any positive integer $n \leqslant \lambda$ the following statements hold:*

(i) $\overline{\mathscr{I}_\lambda^n}(S)$ is regular if and only if so is S;

(*ii*) $\overline{\mathscr{I}_\lambda^n}(S)$ *is orthodox if and only if so is* S*;*

(*iii*) $\overline{\mathscr{I}_\lambda^n}(S)$ *is inverse if and only if so is* S*.*

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [13] or [11, Section 2.1]):

$$\begin{array}{ccc} a\mathscr{R}b & \text{if and only if} & aS^1 = bS^1; \\ a\mathscr{L}b & \text{if and only if} & S^1a = S^1b; \\ a\mathscr{J}b & \text{if and only if} & S^1aS^1 = S^1bS^1; \\ & \mathscr{D} = \mathscr{L}\circ\mathscr{R} = \mathscr{R}\circ\mathscr{L}; & \\ & \mathscr{H} = \mathscr{L}\cap\mathscr{R}. & \end{array}$$

Remark 1. It is obvious that for non-zero elements $\alpha_S = \left(\begin{smallmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{smallmatrix}\right)$ and $\beta_S = \left(\begin{smallmatrix} c_1 & \cdots & c_k \\ t_1 & \cdots & t_k \\ d_1 & \cdots & d_k \end{smallmatrix}\right)$ of the semigroup $\mathscr{I}_\lambda^n(S)$ any of conditions $\alpha_S\mathscr{R}\beta_S$, $\alpha_S\mathscr{L}\beta_S$, $\alpha_S\mathscr{D}\beta_S$, $\alpha_S\mathscr{J}\beta_S$, or $\alpha_S\mathscr{H}\beta_S$ implies the equality $i = k$.

The following proposition describes the Green relations on the semigroup $\mathscr{I}_\lambda^n(S)$.

Proposition 7. *Let* S *be a monoid,* λ *be any non-zero cardinal and* $n \leqslant \lambda$*. Let* $\alpha_S = \left(\begin{smallmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{smallmatrix}\right)$ *and* $\beta_S = \left(\begin{smallmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{smallmatrix}\right)$ *be non-zero elements of the semigroup* $\mathscr{I}_\lambda^n(S)$*. Then the following conditions hold:*

(*i*) $\alpha_S\mathscr{R}\beta_S$ *in* $\mathscr{I}_\lambda^n(S)$ *if and only if there exists a permutation* $\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ *such that* $a_1 = c_{(1)\sigma}, \ldots, a_i = c_{(i)\sigma}$ *and* $s_1\mathscr{R}t_{(1)\sigma}, \ldots, s_i\mathscr{R}t_{(i)\sigma}$ *in* S*;*

(*ii*) $\alpha_S\mathscr{L}\beta_S$ *in* $\mathscr{I}_\lambda^n(S)$ *if and only if there exists a permutation* $\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ *such that* $b_1 = d_{(1)\sigma}, \ldots, b_i = d_{(i)\sigma}$ *and* $s_1\mathscr{L}t_{(1)\sigma}, \ldots, s_i\mathscr{L}t_{(i)\sigma}$ *in* S*;*

(*iii*) $\alpha_S\mathscr{D}\beta_S$ *in* $\mathscr{I}_\lambda^n(S)$ *if and only if there exists a permutation* $\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ *such that* $s_1\mathscr{D}t_{(1)\sigma}, \ldots, s_i\mathscr{D}t_{(i)\sigma}$ *in* S*;*

(*iv*) $\alpha_S\mathscr{H}\beta_S$ *in* $\mathscr{I}_\lambda^n(S)$ *if and only if there exist permutations* $\sigma,\rho\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ *such that* $s_1\mathscr{R}t_{(1)\sigma}, \ldots, s_i\mathscr{R}t_{(i)\sigma}$ *and* $s_1\mathscr{L}t_{(1)\rho}, \ldots, s_i\mathscr{L}t_{(i)\rho}$ *in* S*;*

(*v*) $\alpha_S\mathscr{J}\beta_S$ *in* $\mathscr{I}_\lambda^n(S)$ *if and only if there exists a permutation* $\pi\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ *such that* $s_1\mathscr{J}t_{(1)\pi}, \ldots, s_i\mathscr{J}t_{(i)\pi}$ *in* S*.*

Proof. (*i*) ($\Rightarrow$) Suppose that $\alpha_S\mathscr{R}\beta_S$ in $\mathscr{I}_\lambda^n(S)$. Then there exist non-zero elements $\gamma_S = \left(\begin{smallmatrix} e_1 & \cdots & e_k \\ u_1 & \cdots & u_k \\ f_1 & \cdots & f_k \end{smallmatrix}\right)$ and $\delta_S = \left(\begin{smallmatrix} g_1 & \cdots & g_j \\ v_1 & \cdots & v_j \\ h_1 & \cdots & h_j \end{smallmatrix}\right)$ of the semigroup $\mathscr{I}_\lambda^n(S)$ such that $\alpha_S = \beta_S\gamma_S$, $\beta_S = \alpha_S\delta_S$, $i \leqslant j \leqslant n$ and $i \leqslant k \leqslant n$. Also, the definition of the semigroup operation of $\mathscr{I}_\lambda^n(S)$ implies that without loss of generality we may assume that $j = k = i$. Then the equalities $\alpha_S = \beta_S\gamma_S$ and $\beta_S = \alpha_S\delta_S$ imply that $\{a_1,\ldots,a_i\} = \{c_1,\ldots,c_i\}$, $\{b_1,\ldots,b_i\} = \{g_1,\ldots,g_i\}$ and $\{d_1,\ldots,d_i\} = \{e_1,\ldots,e_i\}$. Now, the semigroup operation of $\mathscr{I}_\lambda^n(S)$ implies that there exist permutations $\sigma,\rho,\zeta\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ such that $a_1 = c_{(1)\sigma}, \ldots, a_i = c_{(i)\sigma}$, $d_1 = e_{(1)\rho}, \ldots, d_i = e_{(i)\rho}$, and $b_1 = g_{(1)\zeta}, \ldots, b_i = g_{(i)\zeta}$, and hence we have that

$$\begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{pmatrix} \cdot \begin{pmatrix} e_1 & \cdots & e_i \\ u_1 & \cdots & u_i \\ f_1 & \cdots & f_i \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{pmatrix} \cdot \begin{pmatrix} d_1 & \cdots & d_i \\ u_{(1)\rho} & \cdots & u_{(i)\rho} \\ f_{(1)\rho} & \cdots & f_{(i)\rho} \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_i \\ t_1u_{(1)\rho} & \cdots & t_iu_{(i)\rho} \\ f_{(1)\rho} & \cdots & f_{(i)\rho} \end{pmatrix} =$$

$$= \begin{pmatrix} a_1 & \cdots & a_i \\ t_{(1)\sigma}u_{((1)\rho)\sigma} & \cdots & t_{(i)\sigma}u_{((i)\rho)\sigma} \\ f_{((1)\rho)\sigma} & \cdots & f_{((i)\rho)\sigma} \end{pmatrix}$$

and

$$\left(\begin{smallmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} g_1 & \cdots & g_i \\ v_1 & \cdots & v_i \\ h_1 & \cdots & h_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} b_1 & \cdots & b_i \\ v_{(1)\zeta} & \cdots & v_{(i)\zeta} \\ h_{(1)\zeta} & \cdots & h_{(i)\zeta} \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & \cdots & a_i \\ s_1 v_{(1)\zeta} & \cdots & s_i v_{(i)\zeta} \\ h_{(1)\zeta} & \cdots & h_{(i)\zeta} \end{smallmatrix}\right) =$$

$$= \left(\begin{smallmatrix} c_1 & \cdots & c_i \\ s_{(1)\sigma^{-1}} v_{((1)\zeta)\sigma^{-1}} & \cdots & s_{(i)\sigma^{-1}} v_{((i)\zeta)\sigma^{-1}} \\ h_{((1)\zeta)\sigma^{-1}} & \cdots & h_{((i)\zeta)\sigma^{-1}} \end{smallmatrix}\right).$$

Therefore we get that

$$\begin{aligned} & s_1 = t_{(1)\sigma} u_{((1)\rho)\sigma}, \quad \ldots, \quad s_i = t_{(i)\sigma} u_{((i)\rho)\sigma}, \\ & \text{and} \qquad t_1 = s_{(1)\sigma^{-1}} v_{((1)\zeta)\sigma^{-1}}, \quad \ldots, \quad t_i = s_{(i)\sigma^{-1}} v_{((i)\zeta)\sigma^{-1}}. \end{aligned} \tag{1}$$

Since $\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ is a permutation, conditions (1) imply that $s_1 \mathscr{R} t_{(1)\sigma}$, $\ldots$, $s_i \mathscr{R} t_{(i)\sigma}$ in S.

($\Leftarrow$) Suppose that for $\alpha_S, \beta_S \in \mathscr{I}_\lambda^n(S)$ there exists a permutation $\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ such that $a_1 = c_{(1)\sigma}, \ldots, a_i = c_{(i)\sigma}$ and $s_1 \mathscr{R} t_{(1)\sigma}, \ldots, s_i \mathscr{R} t_{(i)\sigma}$ in S. Then there exist $u_1, \ldots, u_i, v_1, \ldots, v_i \in S^1$ such that

$$s_1 = t_{(1)\sigma} u_1, \quad \ldots, \quad s_i = t_{(i)\sigma} u_i, \quad t_1 = s_{(1)\sigma^{-1}} v_1, \quad \ldots, \quad t_i = s_{(i)\sigma^{-1}} v_i.$$

Thus we get that

$$\left(\begin{smallmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} c_{(1)\sigma} & \cdots & c_{(i)\sigma} \\ t_{(1)\sigma} u_1 & \cdots & t_{(i)\sigma} u_i \\ b_1 & \cdots & b_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} c_1 & \cdots & c_i \\ t_1 u_{(1)\sigma^{-1}} & \cdots & t_i u_{(i)\sigma^{-1}} \\ b_{(1)\sigma^{-1}} & \cdots & b_{(i)\sigma^{-1}} \end{smallmatrix}\right) = \left(\begin{smallmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} d_1 & \cdots & d_i \\ u_{(1)\sigma^{-1}} & \cdots & u_{(i)\sigma^{-1}} \\ b_{(1)\sigma^{-1}} & \cdots & b_{(i)\sigma^{-1}} \end{smallmatrix}\right)$$

and

$$\left(\begin{smallmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_{(1)\sigma^{-1}} & \cdots & a_{(i)\sigma^{-1}} \\ s_{(1)\sigma^{-1}} v_1 & \cdots & s_{(i)\sigma^{-1}} v_i \\ d_1 & \cdots & d_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & \cdots & a_i \\ s_1 v_{(1)\sigma} & \cdots & s_i v_{(i)\sigma} \\ d_{(1)\sigma} & \cdots & d_{(i)\sigma} \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} b_1 & \cdots & b_i \\ v_{(1)\sigma} & \cdots & v_{(i)\sigma} \\ d_{(1)\sigma} & \cdots & d_{(i)\sigma} \end{smallmatrix}\right),$$

and hence $\alpha_S \mathscr{R} \beta_S$ in $\mathscr{I}_\lambda^n(S)$.

The proof of statement (ii) is similar to the proof of (i).

(iii) ($\Rightarrow$) Suppose that $\alpha_S \mathscr{D} \beta_S$ in $\mathscr{I}_\lambda^n(S)$. Then there exists a non-zero element $\gamma_S = \left(\begin{smallmatrix} e_1 & \cdots & e_i \\ u_1 & \cdots & u_i \\ f_1 & \cdots & f_i \end{smallmatrix}\right)$ of the semigroup $\mathscr{I}_\lambda^n(S)$ such that $\alpha_S \mathscr{R} \gamma_S$ and $\gamma_S \mathscr{L} \beta_S$ in $\mathscr{I}_\lambda^n(S)$. By statement (i) there exists a permutation $\zeta\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ such that $e_1 = a_{(1)\zeta}$, $\ldots, e_i = a_{(i)\zeta}$ and $u_1 \mathscr{R} s_{(1)\zeta}$, $\ldots, u_i \mathscr{R} s_{(i)\zeta}$ in S and by statement (ii) there exists a permutation $\varsigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ such that $f_1 = d_{(1)\varsigma}, \ldots, f_i = d_{(i)\varsigma}$ and $u_1 \mathscr{L} s_{(1)\varsigma}$, $\ldots, u_i \mathscr{L} s_{(i)\varsigma}$ in S. This implies that $s_1 \mathscr{D} t_{(1)\sigma}$, $\ldots, s_i \mathscr{D} t_{(i)\sigma}$ in S for the permutation $\sigma = \zeta \circ \varsigma^{-1}$ of $\{1,\ldots,i\}$.

($\Leftarrow$) Suppose that there exists a permutation $\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ such that $s_1 \mathscr{D} t_{(1)\sigma}, \ldots, s_i \mathscr{D} t_{(i)\sigma}$ in S. Then the definition of the relation $\mathscr{D}$ implies that there exist $u_1, \ldots, u_i \in S$ such that $s_1 \mathscr{R} u_1$, $\ldots$, $s_i \mathscr{R} u_i$ and $u_1 \mathscr{L} t_{(1)\sigma}$, $\ldots$, $u_i \mathscr{L} t_{(i)\sigma}$ in S. Now, for the element $\gamma_S = \left(\begin{smallmatrix} a_1 & \cdots & a_i \\ u_1 & \cdots & u_i \\ d_{(1)\sigma} & \cdots & d_{(i)\sigma} \end{smallmatrix}\right)$ of the semigroup $\mathscr{I}_\lambda^n(S)$ by statements (i) and (ii) we have that $\alpha_S \mathscr{R} \gamma_S$ and $\gamma_S \mathscr{L} \beta_S$ in $\mathscr{I}_\lambda^n(S)$.

(iv) follows from statements (i) and (ii).

(v) ($\Rightarrow$) Suppose that $\alpha_S \mathscr{J} \beta_S$ in $\mathscr{I}_\lambda^n(S)$. Then there exist non-zero elements $\gamma_S^l = \left(\begin{smallmatrix} e_1^l & \cdots & e_{k_l}^l \\ u_1^l & \cdots & u_{k_l}^l \\ f_1^l & \cdots & f_{k_l}^l \end{smallmatrix}\right)$, $\gamma_S^r = \left(\begin{smallmatrix} e_1^r & \cdots & e_{k_r}^r \\ u_1^r & \cdots & u_{k_r}^r \\ f_1^r & \cdots & f_{k_r}^r \end{smallmatrix}\right)$, $\delta_S^l = \left(\begin{smallmatrix} g_1^l & \cdots & g_{j_l}^l \\ v_1^l & \cdots & v_{j_l}^l \\ h_1^l & \cdots & h_{j_l}^l \end{smallmatrix}\right)$ and $\delta_S^r = \left(\begin{smallmatrix} g_1^r & \cdots & g_{j_r}^r \\ v_1^r & \cdots & v_{j_r}^r \\ h_1^r & \cdots & h_{j_r}^r \end{smallmatrix}\right)$ of the semigroup $\mathscr{I}_\lambda^n(S)$ such that $\alpha_S = \gamma_S^l \beta_S \gamma_S^r$, $\beta_S = \delta_S^l \alpha_S \delta_S^r$ and $i \leqslant k_l, k_r, j_l, j_r \leqslant n$ (see [13] or

[14, Section II.1]). Also, the definition of the semigroup operation of $\mathscr{I}_\lambda^n(S)$ implies that without loss of generality we may assume that $k_l = k_r = j_l = j_r = i$. Then the equalities $\alpha_S = \gamma_S^l \beta_S \gamma_S^r$ and $\beta_S = \delta_S^l \alpha_S \delta_S^r$ imply that

$$\{a_1, \ldots, a_i\} = \{g_1^l, \ldots, g_i^l\} = \{h_1^l, \ldots, h_i^l\},$$

$$\{b_1, \ldots, b_i\} = \{f_1^r, \ldots, f_i^r\} = \{g_1^r, \ldots, g_i^r\},$$

$$\{c_1, \ldots, c_i\} = \{g_1^l, \ldots, g_i^l\} = \{f_1^l, \ldots, f_i^l\}$$

and

$$\{d_1, \ldots, d_i\} = \{e_1^r, \ldots, e_i^r\} = \{h_1^r, \ldots, h_i^r\}.$$

Now, the semigroup operation of $\mathscr{I}_\lambda^n(S)$ implies that there exist permutations

$$\sigma, \rho, \zeta, \varsigma, \nu, \kappa \colon \{1, \ldots, i\} \to \{1, \ldots, i\}$$

such that $a_1 = e^l_{(1)\sigma}, \ldots, a_i = e^l_{(i)\sigma}$, $c_1 = f^l_{(1)\rho}, \ldots, c_i = f^l_{(i)\rho}$, $d_1 = e^r_{(1)\zeta}, \ldots, d_i = e^r_{(i)\zeta}$, $c_1 = g^l_{(1)\varsigma}, \ldots, c_i = g^l_{(i)\varsigma}$, $a_1 = h^l_{(1)\nu}, \ldots, a_i = h^l_{(i)\nu}$ and $b_1 = g^r_{(1)\kappa}, \ldots, b_i = g^r_{(i)\kappa}$, and hence we have that

$$\begin{aligned}
\begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} &= \begin{pmatrix} e_1^l & \cdots & e_{k_l}^l \\ u_1^l & \cdots & u_{k_l}^l \\ f_1^l & \cdots & f_{k_l}^l \end{pmatrix} \cdot \begin{pmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{pmatrix} \cdot \begin{pmatrix} e_1^r & \cdots & e_{k_r}^r \\ u_1^r & \cdots & u_{k_r}^r \\ f_1^r & \cdots & f_{k_r}^r \end{pmatrix} = \\
&= \begin{pmatrix} e^l_{(1)\rho} & \cdots & e^l_{(i)\rho} \\ u^l_{(1)\rho} & \cdots & u^l_{(i)\rho} \\ c_1 & \cdots & c_i \end{pmatrix} \cdot \begin{pmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{pmatrix} \cdot \begin{pmatrix} d_1 & \cdots & d_i \\ u^r_{(1)\zeta} & \cdots & u^r_{(i)\zeta} \\ f^r_{(1)\zeta} & \cdots & f^r_{(i)\zeta} \end{pmatrix} = \begin{pmatrix} e^l_{(1)\rho} & \cdots & e^l_{(i)\rho} \\ u^l_{(1)\rho} t_1 u^r_{(1)\zeta} & \cdots & u^l_{(i)\rho} t_i u^r_{(i)\zeta} \\ f^r_{(1)\zeta} & \cdots & f^r_{(i)\zeta} \end{pmatrix} = \\
&= \begin{pmatrix} e_1^l & \cdots & e_i^l \\ u_1^l t_{(1)\rho^{-1}} u^r_{((1)\zeta)\rho^{-1}} & \cdots & u_1^l t_{(i)\rho^{-1}} u^r_{((i)\zeta)\rho^{-1}} \\ f^r_{((1)\zeta)\rho^{-1}} & \cdots & f^r_{((i)\zeta)\rho^{-1}} \end{pmatrix} = \\
&= \begin{pmatrix} a_1 & \cdots & a_i \\ u^l_{(1)\sigma} t_{((1)\rho^{-1})\sigma} u^r_{(((1)\zeta)\rho^{-1})\sigma} & \cdots & u^l_{(i)\sigma} t_{((i)\rho^{-1})\sigma} u^r_{(((i)\zeta)\rho^{-1})\sigma} \\ f^r_{(((1)\zeta)\rho^{-1})\sigma} & \cdots & f^r_{(((i)\zeta)\rho^{-1})\sigma} \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{pmatrix} &= \begin{pmatrix} g_1^l & \cdots & g_i^l \\ v_1^l & \cdots & v_i^l \\ h_1^l & \cdots & h_i^l \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \begin{pmatrix} g_1^r & \cdots & g_i^r \\ v_1^r & \cdots & v_i^r \\ h_1^r & \cdots & h_i^r \end{pmatrix} = \\
&= \begin{pmatrix} g_1^l & \cdots & g_i^l \\ v_1^l & \cdots & v_i^l \\ h_1^l & \cdots & h_i^l \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \begin{pmatrix} b_1 & \cdots & b_i \\ v^r_{(1)\kappa} & \cdots & v^r_{(i)\kappa} \\ h^r_{(1)\kappa} & \cdots & h^r_{(i)\kappa} \end{pmatrix} = \begin{pmatrix} g^l_{(1)\nu} & \cdots & g^l_{(i)\nu} \\ v^l_{(1)\nu} s_1 v^r_{(1)\kappa} & \cdots & v^l_{(i)\nu} s_i v^r_{(i)\kappa} \\ h^r_{(1)\kappa} & \cdots & h^r_{(i)\kappa} \end{pmatrix} = \\
&= \begin{pmatrix} g_1^l & \cdots & g_i^l \\ v_1^l s_{(1)\nu^{-1}} v^r_{((1)\kappa)\nu^{-1}} & \cdots & v_i^l s_{(i)\nu^{-1}} v^r_{((i)\kappa)\nu^{-1}} \\ h^r_{((1)\kappa)\nu^{-1}} & \cdots & h^r_{((i)\kappa)\nu^{-1}} \end{pmatrix} = \\
&= \begin{pmatrix} c_1 & \cdots & c_i \\ v^l_{(1)\varsigma} s_{((1)\nu^{-1})\varsigma} v^r_{(((1)\kappa)\nu^{-1})\varsigma} & \cdots & v^l_{(i)\varsigma} s_{((1)\nu^{-1})\varsigma} v^r_{(((i)\kappa)\nu^{-1})\varsigma} \\ h^r_{(((1)\kappa)\nu^{-1})\varsigma} & \cdots & h^r_{(((i)\kappa)\nu^{-1})\varsigma} \end{pmatrix}.
\end{aligned}$$

Then the definition of the semigroup $\mathscr{I}_\lambda^n(S)$ implies the equalities

$$d_1 = h^r_{(((1)\kappa)\nu^{-1})\varsigma}, \qquad \ldots, \qquad d_i = h^r_{(((i)\kappa)\nu^{-1})\varsigma}.$$

Now, by the equality $\alpha_S = \gamma^l_S \beta_S \gamma^r_S$ we get that

$$\begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} = \begin{pmatrix} e^l_1 & \cdots & e^l_{k_l} \\ u^l_1 & \cdots & u^l_{k_l} \\ f^l_1 & \cdots & f^l_{k_l} \end{pmatrix} \cdot \begin{pmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{pmatrix} \cdot \begin{pmatrix} e^r_1 & \cdots & e^r_{k_r} \\ u^r_1 & \cdots & u^r_{k_r} \\ f^r_1 & \cdots & f^r_{k_r} \end{pmatrix} =$$

$$= \begin{pmatrix} e^l_1 & \cdots & e^l_{k_l} \\ u^l_1 & \cdots & u^l_{k_l} \\ f^l_1 & \cdots & f^l_{k_l} \end{pmatrix} \cdot \begin{pmatrix} c_1 & \cdots & c_i \\ v^l_{(1)\zeta} s_{((1)\nu-1)\varsigma} v^r_{(((1)\kappa)\nu-1)\varsigma} & \cdots & v^l_{(i)\zeta} s_{((1)\nu-1)\varsigma} v^r_{(((i)\kappa)\nu-1)\varsigma} \\ d_1 & \cdots & d_i \end{pmatrix} \cdot \begin{pmatrix} e^r_1 & \cdots & e^r_{k_r} \\ u^r_1 & \cdots & u^r_{k_r} \\ f^r_1 & \cdots & f^r_{k_r} \end{pmatrix} =$$

$$= \begin{pmatrix} e^l_{(1)\rho} & \cdots & e^l_{(i)\rho} \\ u^l_{(1)\rho} & \cdots & u^l_{(i)\rho} \\ c_1 & \cdots & c_i \end{pmatrix} \cdot \begin{pmatrix} c_1 & \cdots & c_i \\ v^l_{(1)\zeta} s_{((1)\nu-1)\varsigma} v^r_{(((1)\kappa)\nu-1)\varsigma} & \cdots & v^l_{(i)\zeta} s_{((1)\nu-1)\varsigma} v^r_{(((i)\kappa)\nu-1)\varsigma} \\ d_1 & \cdots & d_i \end{pmatrix} \cdot \begin{pmatrix} d_1 & \cdots & d_i \\ u^r_{(1)\zeta} & \cdots & u^r_{(i)\zeta} \\ f^r_{(1)\zeta} & \cdots & f^r_{(i)\zeta} \end{pmatrix} =$$

$$= \begin{pmatrix} e^l_{(1)\rho} & \cdots & e^l_{(i)\rho} \\ u^l_{(1)\rho} v^l_{(1)\zeta} s_{((1)\nu-1)\varsigma} v^r_{(((1)\kappa)\nu-1)\varsigma} u^r_{(1)\zeta} & \cdots & u^l_{(i)\rho} v^l_{(i)\zeta} s_{((1)\nu-1)\varsigma} v^r_{(((i)\kappa)\nu-1)\varsigma} u^r_{(1)\zeta} \\ f^r_{(1)\zeta} & \cdots & f^r_{(i)\zeta} \end{pmatrix}$$

which implies the equalities

$$s_1 = u^l_{(1)\sigma} v^l_{(((1)\zeta)\rho^{-1})\sigma} s_{((((1)\nu^{-1})\varsigma)\rho^{-1})\sigma} v^r_{(((((1)\kappa)\nu^{-1})\varsigma)\rho^{-1})\sigma} u^r_{(((1)\zeta)\rho^{-1})\sigma}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$s_i = u^l_{(i)\sigma} v^l_{(((i)\zeta)\rho^{-1})\sigma} s_{((((i)\nu^{-1})\varsigma)\rho^{-1})\sigma} v^r_{(((((i)\kappa)\nu^{-1})\varsigma)\rho^{-1})\sigma} u^r_{(((i)\zeta)\rho^{-1})\sigma}.$$

Hence for the permutation $\pi = \nu^{-1}\varsigma\rho^{-1}\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ we have that $s_1 \mathscr{J} t_{(1)\pi}$, $\ldots, s_i \mathscr{J} t_{(i)\pi}$ in S.

($\Leftarrow$) Suppose that for elements $\alpha_S, \beta_S \in \mathscr{I}^n_\lambda(S)$ there exists a permutation $\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ such that $s_1 \mathscr{J} t_{(1)\sigma}, \ldots, s_i \mathscr{J} t_{(i)\sigma}$ in S. Then there exist $u_1,\ldots,u_i, v_1,\ldots,v_i, x_1,\ldots,x_i, y_1,\ldots,y_i \in S^1$ such that

$$s_1 = x_1 t_{(1)\sigma} u_1, \quad \ldots, \quad s_i = x_i t_{(i)\sigma} u_i, \quad t_1 = y_1 s_{(1)\sigma^{-1}} v_1, \quad \ldots, \quad t_i = y_i s_{(i)\sigma^{-1}} v_i.$$

Thus, we have that

$$\begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} = \begin{pmatrix} c_{(1)\sigma} & \cdots & c_{(i)\sigma} \\ x_1 t_{(1)\sigma} u_1 & \cdots & x_i t_{(i)\sigma} u_i \\ b_{(1)\sigma} & \cdots & b_{(i)\sigma} \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_i \\ x_{(1)\sigma^{-1}} t_1 u_{(1)\sigma^{-1}} & \cdots & x_{(i)\sigma^{-1}} t_i u_{(i)\sigma^{-1}} \\ b_1 & \cdots & b_i \end{pmatrix} =$$
$$= \begin{pmatrix} c_1 & \cdots & c_i \\ x_{(1)\sigma^{-1}} & \cdots & x_{(i)\sigma^{-1}} \\ c_1 & \cdots & c_i \end{pmatrix} \cdot \begin{pmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \begin{pmatrix} b_1 & \cdots & b_i \\ u_{(1)\sigma^{-1}} & \cdots & u_{(i)\sigma^{-1}} \\ b_1 & \cdots & b_i \end{pmatrix}$$

and

$$\begin{pmatrix} c_1 & \cdots & c_i \\ t_1 & \cdots & t_i \\ d_1 & \cdots & d_i \end{pmatrix} = \begin{pmatrix} a_{(1)\sigma^{-1}} & \cdots & a_{(i)\sigma^{-1}} \\ y_1 s_{(1)\sigma^{-1}} v_1 & \cdots & y_i s_{(i)\sigma^{-1}} v_i \\ d_{(1)\sigma^{-1}} & \cdots & d_{(i)\sigma^{-1}} \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_i \\ y_{(1)\sigma} s_1 v_{(1)\sigma} & \cdots & y_{(i)\sigma} s_i v_{(i)\sigma} \\ d_1 & \cdots & d_i \end{pmatrix} =$$
$$= \begin{pmatrix} a_1 & \cdots & a_i \\ y_{(1)\sigma} & \cdots & y_{(i)\sigma} \\ a_1 & \cdots & a_i \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ d_1 & \cdots & d_i \end{pmatrix} \cdot \begin{pmatrix} d_1 & \cdots & d_i \\ v_{(1)\sigma} & \cdots & v_{(i)\sigma} \\ d_1 & \cdots & d_i \end{pmatrix},$$

and hence we get that $\alpha_S \mathscr{J} \beta_S$ in $\mathscr{I}^n_\lambda(S)$. □

Remark 2. Proposition 7(*iv*) implies that if there exists a permutation $\sigma\colon \{1,\ldots,i\} \to \{1,\ldots,i\}$ such that $s_1 \mathscr{H} t_{(1)\sigma}, \ldots, s_i \mathscr{H} t_{(i)\sigma}$ in S then $\alpha_S \mathscr{H} \beta_S$ in $\mathscr{I}^n_\lambda(S)$. But Example 1 implies that the converse statement is not true.

Example 1. Let λ be any cardinal $\geqslant 2$ and $\mathscr{C}(p,q)$ be the bicyclic monoid. The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The distinct elements of $\mathscr{C}(p,q)$ are exhibited in the following useful array

$$\begin{array}{ccccc} 1 & p & p^2 & p^3 & \cdots \\ q & qp & qp^2 & qp^3 & \cdots \\ q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\ q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

and the semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

We fix arbitrary distinct elements a_1 and a_1 of λ and put

$$\alpha=\begin{pmatrix} a_1 & a_1 \\ qp & q^2p^2 \\ a_1 & a_1 \end{pmatrix} \qquad \text{and} \qquad \beta=\begin{pmatrix} a_1 & a_2 \\ qp^2 & q^2p \\ a_2 & a_1 \end{pmatrix}.$$

Then we have that

$$\alpha=\begin{pmatrix} a_1 & a_2 \\ qp^2 & q^2p \\ a_2 & a_1 \end{pmatrix}\cdot\begin{pmatrix} a_1 & a_2 \\ p & q \\ a_2 & a_1 \end{pmatrix} \qquad \text{and} \qquad \beta=\begin{pmatrix} a_1 & a_1 \\ qp & q^2p^2 \\ a_1 & a_1 \end{pmatrix}\cdot\begin{pmatrix} a_1 & a_2 \\ p & q \\ a_2 & a_1 \end{pmatrix}$$

and hence $\alpha\mathscr{R}\beta$ in $\mathscr{I}_\lambda^n(S)$, and similarly we have that

$$\alpha=\begin{pmatrix} a_1 & a_2 \\ p & q \\ a_2 & a_1 \end{pmatrix}\cdot\begin{pmatrix} a_1 & a_2 \\ qp^2 & q^2p \\ a_2 & a_1 \end{pmatrix} \qquad \text{and} \qquad \beta=\begin{pmatrix} a_1 & a_2 \\ p & q \\ a_2 & a_1 \end{pmatrix}\cdot\begin{pmatrix} a_1 & a_1 \\ qp & q^2p^2 \\ a_1 & a_1 \end{pmatrix}$$

and hence $\alpha\mathscr{L}\beta$ in $\mathscr{I}_\lambda^n(S)$. Thus $\alpha\mathscr{H}\beta$ in $\mathscr{I}_\lambda^n(S)$, but the elements qp and q^2p^2 are not pairwise $\mathscr{H}$-equivalent to qp^2 and q^2p for any permutation $\sigma\colon\{1,2\}\to\{1,2\}$.

Recall [28], a semigroup S is said to be:

(a) *left stable* if for $a,b\in S$, $Sa\subseteq Sab$ implies $Sa=Sab$;
(b) *right stable* if for $c,d\in S$, $cS\subseteq dcS$ implies $cS=dcS$;
(b) *stable* if it is both left and right stable.

We observe that in the book [11] an other definition of a stable semigroup is given, and these two notion are distinct. A semigroup stable in the sense of Koch and Wallace is always stable in the sense of the book [11], but not conversely (see: [30]). For the semigroups with an identity element and for regular semigroups these two definitions of stability coincide.

The following proposition states that the construction of the semigroup $\mathscr{I}_\lambda^n(S)$ preserves left an right stabilities.

Proposition 8. *For every semigroup S, any non-zero cardinal λ and any positive integer $n\leqslant\lambda$ the following statements hold:*

(*i*) *$\mathscr{I}_\lambda^n(S)$ is right stable if and only if so is S;*
(*ii*) *$\mathscr{I}_\lambda^n(S)$ is left stable if and only if so is S;*
(*iii*) *$\mathscr{I}_\lambda^n(S)$ is stable if and only if so is S.*

Proof. (*i*) ($\Leftarrow$) Suppose that the semigroup S is right stable and assume that $\alpha_S=\begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix}$ and $\beta_S=\begin{pmatrix} c_1 & \cdots & c_k \\ t_1 & \cdots & t_k \\ d_1 & \cdots & d_k \end{pmatrix}$ are elements of the semigroup $\mathscr{I}_\lambda^n(S)$ such that

$\alpha_S \mathscr{I}_\lambda^n(S) \subseteq \beta_S \alpha_S \mathscr{I}_\lambda^n(S)$. Then the above inclusion and the definition of the semigroup operation on $\mathscr{I}_\lambda^n(S)$ imply that $i \leqslant k$ and the inclusion

$$\{a_1, \ldots, a_i\} \subseteq \{c_1, \ldots, c_k\} \cap \{d_1, \ldots, d_k\}$$

holds. Without loss of generality we may assume that $d_1 = a_1, \ldots, d_i = a_i$. Then the inclusion $\alpha_S \mathscr{I}_\lambda^n(S) \subseteq \beta_S \alpha_S \mathscr{I}_\lambda^n(S)$ implies that there exists a permutation $\sigma\colon \{1, \ldots, i\} \to \{1, \ldots, i\}$ such that $c_1 = a_{(1)\sigma}, \ldots, c_i = a_{(i)\sigma}$. Hence by the definition of the semigroup operation of $\mathscr{I}_\lambda^n(S)$ we get that

$$\begin{aligned}
\beta_S \alpha_S \mathscr{I}_\lambda^n(S) &= \begin{pmatrix} c_1 & \cdots & c_k \\ t_1 & \cdots & t_k \\ d_1 & \cdots & d_k \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \mathscr{I}_\lambda^n(S) = \begin{pmatrix} c_1 & \cdots & c_i & c_{i+1} & \cdots & c_k \\ t_1 & \cdots & t_i & t_{i+1} & \cdots & t_k \\ d_1 & \cdots & d_i & d_{i+1} & \cdots & d_k \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \mathscr{I}_\lambda^n(S) = \\
&= \begin{pmatrix} a_{(1)\sigma} & \cdots & a_{(i)\sigma} & c_{i+1} & \cdots & c_k \\ t_1 & \cdots & t_i & t_{i+1} & \cdots & t_k \\ a_1 & \cdots & a_i & d_{i+1} & \cdots & d_k \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \mathscr{I}_\lambda^n(S) = \begin{pmatrix} a_{(1)\sigma} & \cdots & a_{(i)\sigma} \\ t_1 & \cdots & t_i \\ a_1 & \cdots & a_i \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \mathscr{I}_\lambda^n(S) = \\
&= \begin{pmatrix} a_{(1)\sigma} & \cdots & a_{(i)\sigma} \\ t_1 s_1 & \cdots & t_i s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \mathscr{I}_\lambda^n(S) = \begin{pmatrix} a_1 & \cdots & a_i \\ t_{(1)\sigma^{-1}} s_{(1)\sigma^{-1}} & \cdots & t_{(i)\sigma^{-1}} s_{(i)\sigma^{-1}} \\ b_{(1)\sigma^{-1}} & \cdots & b_{(i)\sigma^{-1}} \end{pmatrix} \cdot \mathscr{I}_\lambda^n(S) = \\
&= \{0\} \cup \bigcup \Big\{ [t_{(1)\sigma^{-1}} s_{(1)\sigma^{-1}} S, \ldots, t_{(i)\sigma^{-1}} s_{(i)\sigma^{-1}} S]_{(p_1,\ldots,p_i)}^{(a_1,\ldots,a_i)} \colon p_1, \ldots, p_i \in \lambda \Big\} \cup \\
&\quad \cup \bigcup \Big\{ [t_{(l_1)\sigma^{-1}} s_{(l_1)\sigma^{-1}} S, \ldots, t_{(l_{i-1})\sigma^{-1}} s_{(l_{i-1})\sigma^{-1}} S]_{(p_1,\ldots,p_{i-1})}^{(l_1,\ldots,l_{i-1})} \colon l_1, \ldots, l_{i-1} \text{ are} \\
&\qquad \text{distinct elements of } \{1, \ldots, i\} \text{ and } p_1, \ldots, p_{i-1} \in \lambda \Big\} \cup \cdots \cup \\
&\quad \cup \bigcup \Big\{ [t_{(l)\sigma^{-1}} s_{(l)\sigma^{-1}} S]_{(p)}^{(l)} \colon l \in \{1, \ldots, i\} \text{ and } p \in \lambda \Big\}
\end{aligned}$$

and

$$\begin{aligned}
\alpha_S \mathscr{I}_\lambda^n(S) &= \begin{pmatrix} a_1 & \cdots & a_i \\ s_1 & \cdots & s_i \\ b_1 & \cdots & b_i \end{pmatrix} \cdot \mathscr{I}_\lambda^n(S) = \{0\} \cup \bigcup \Big\{ [s_1 S, \ldots, s_i S]_{(p_1,\ldots,p_i)}^{(a_1,\ldots,a_i)} \colon p_1, \ldots, p_i \in \lambda \Big\} \cup \\
&\cup \bigcup \Big\{ [s_{l_1} S, \ldots, s_{l_{i-1}} S]_{(p_1,\ldots,p_{i-1})}^{(l_1,\ldots,l_{i-1})} \colon l_1, \ldots, l_{i-1} \text{ are distinct elements of } \{1, \ldots, i\} \\
&\qquad \text{and } p_1, \ldots, p_{i-1} \in \lambda \Big\} \cup \cdots \cup \\
&\cup \bigcup \Big\{ [s_l S]_{(p)}^{(l)} \colon l \in \{1, \ldots, i\} \text{ and } p \in \lambda \Big\}.
\end{aligned}$$

Hence, the inclusion $\alpha_S \mathscr{I}_\lambda^n(S) \subseteq \beta_S \alpha_S \mathscr{I}_\lambda^n(S)$ and semigroup operations of the semigroups $\mathscr{I}_\lambda^n(S)$ and S imply that $s_l S \subseteq t_{(l)\sigma^{-1}} s_{(l)\sigma^{-1}} S$, for every $l \in \{1, \ldots, i\}$. Since the semigroup of all permutations of a finite set is finite, we conclude that there exists a positive integer j such that $\sigma^j \colon \{1, \ldots, i\} \to \{1, \ldots, i\}$ is the identity map and therefore we get that $\sigma^{j-1} = \sigma$. This implies that for every $l \in \{1, \ldots, i\}$ we have that

$$\begin{aligned}
s_l S \subseteq t_{(l)\sigma^{-1}} s_{(l)\sigma^{-1}} S &\subseteq t_{(l)\sigma^{-1}} t_{(l)\sigma^{-2}} s_{(l)\sigma^{-2}} S \subseteq \\
&\subseteq \cdots \subseteq \\
&\subseteq t_{(l)\sigma^{-1}} t_{(l)\sigma^{-2}} \cdots t_{(l)\sigma^{-j+1}} s_{(l)\sigma^{-j+1}} S = \\
&= t_{(l)\sigma^{-1}} t_{(l)\sigma^{-2}} \cdots t_l s_l S.
\end{aligned}$$

Then the right stability of the semigroup S implies the equality

$$s_l S = t_{(l)\sigma^{-1}} t_{(l)\sigma^{-2}} \cdots t_l s_l S$$

and hence we have that $s_l S = t_{(l)\sigma^{-1}} s_{(l)\sigma^{-1}} S$, for every $l \in \{1, \ldots, i\}$. Then the inclusion $\alpha_S \mathscr{I}_\lambda^n(S) \subseteq \beta_S \alpha_S \mathscr{I}_\lambda^n(S)$ and above formulae imply the equality $\alpha_S \mathscr{I}_\lambda^n(S) = \beta_S \alpha_S \mathscr{I}_\lambda^n(S)$, and hence the semigroup $\mathscr{I}_\lambda^n(S)$ is right stable.

($\Rightarrow$) Suppose that the semigroup $\mathscr{I}_\lambda^n(S)$ is right stable and $sS \subseteq tsS$ for $s, t \in S$. We fix an arbitrary $a \in \lambda$ and put $\alpha_S = \left(\begin{smallmatrix} a \\ s \\ a \end{smallmatrix}\right)$ and $\beta_S = \left(\begin{smallmatrix} a \\ t \\ a \end{smallmatrix}\right)$. Hence by the definition of the semigroup operation of $\mathscr{I}_\lambda^n(S)$ we get that

$$\alpha_S \mathscr{I}_\lambda^n(S) = \left(\begin{smallmatrix} a \\ s \\ a \end{smallmatrix}\right) \mathscr{I}_\lambda^n(S) = \{0\} \cup \bigcup \left\{ [sS]_{(p)}^{(a)} : p \in \lambda \right\}$$

and

$$\beta_S \alpha_S \mathscr{I}_\lambda^n(S) = \left(\begin{smallmatrix} a \\ t \\ a \end{smallmatrix}\right) \left(\begin{smallmatrix} a \\ s \\ a \end{smallmatrix}\right) \mathscr{I}_\lambda^n(S) = \left(\begin{smallmatrix} a \\ ts \\ a \end{smallmatrix}\right) \mathscr{I}_\lambda^n(S) = \{0\} \cup \bigcup \left\{ [tsS]_{(p)}^{(a)} : p \in \lambda \right\},$$

and hence by the inclusion $sS \subseteq tsS$ we have that $\alpha_S \mathscr{I}_\lambda^n(S) \subseteq \beta_S \alpha_S \mathscr{I}_\lambda^n(S)$. Now the right stability of $\mathscr{I}_\lambda^n(S)$ implies the equality $\alpha_S \mathscr{I}_\lambda^n(S) = \beta_S \alpha_S \mathscr{I}_\lambda^n(S)$. This implies $[sS]_{(p)}^{(a)} = [tsS]_{(p)}^{(a)}$ in $\mathscr{I}_\lambda^n(S)$ for every $p \in \lambda$, and hence $sS = tsS$.

The proof of statement (ii) is dual to that of statement (i).

(iii) follows from statements (i) and (ii). □

4. On semigroups with a tight ideal series

Fix an arbitrary positive integer m and any $p \in \{0, \ldots, m\}$. Let A be a non-empty set and let B be a non-empty proper subset of A. By $[B \subset A]_p^m$ we denote all elements $(x_1, \ldots, x_m)$ of the power A^m which satisfy the following property: *at most p coordinates of $(x_1, \ldots, x_m)$ belong to $A \setminus B$.* It is obvious that $[B \subset A]_m^m = A^m$ for any positive integer m, any non-empty set A and any non-empty proper subset B of A.

The above definition implies the following two lemmas.

Lemma 1. *Let m be an arbitrary positive integer and $p \in \{1, \ldots, m\}$. Let A be a non-empty set and let B be a non-empty proper subset of A. Then the set $[B \subset A]_p^m \setminus [B \subset A]_{p-1}^m$ consists of all elements $(x_1, \ldots, x_m)$ of the power A^m such that exactly p coordinates of $(x_1, \ldots, x_m)$ belong to $A \setminus B$.*

Lemma 2. *Let m be an arbitrary positive integer and $p \in \{0, 1, \ldots, m\}$. Let S be a semigroup, A and B be ideals in S such that $B \subset A$. Then $[B \subset A]_p^m$ is an ideal of the direct power S^m.*

An subset D of a semigroup S is said to be ω-unstable if D is infinite and $aB \cup Ba \nsubseteq D$ for any $a \in D$ and any infinite subset $B \subseteq D$.

Definition 1 ([18]). An *ideal series* (see, for example, [11]) for a semigroup S is a chain of ideals

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = S.$$

We call the ideal series *tight* if I_0 is a finite set and $D_k = I_k \setminus I_{k-1}$ is an ω-unstable subset for each $k = 1, \ldots, n$.

It is obvious that for every infinite cardinal λ and any positive integer n the semigroup $\mathscr{I}_\lambda^n$ has a tight ideal series. A finite direct product of semigroups with tight ideal series is a semigroup with a tight ideal series and a homomorphic image of a semigroup with a tight ideal series with finite preimages is a semigroup with a tight ideal series too [18].

A subset D of a semigroup S is said to be *strongly ω-unstable* if D is infinite and $aB\cup Bb \nsubseteq D$ for any $a,b\in D$ and any infinite subset $B\subseteq D$. It is obvious that a subset D of a semigroup S is strongly ω-unstable then D is ω-unstable.

Definition 2. We call the ideal series $I_0\subseteq I_1\subseteq I_2\subseteq\cdots\subseteq I_n=S$ *strongly tight* if I_0 is a finite set and $D_k=I_k\setminus I_{k-1}$ is a strongly ω-unstable subset for each $k=1,\ldots,n$.

An example of a semigroup with a strongly tight ideal series gives the following proposition.

Proposition 9. *Let λ be any infinite cardinal and n be any positive integer. Then*

$$I_0=\{0\}\subseteq I_1=\mathscr{I}_\lambda^1\subseteq I_2=\mathscr{I}_\lambda^2\subseteq\cdots\subseteq I_n=\mathscr{I}_\lambda^n,$$

is the strongly tight ideal series in the semigroup $\mathscr{I}_\lambda^n$.

Proof. The definition of the semigroup $\mathscr{I}_\lambda^n$ implies that $I_0\subseteq I_1\subseteq I_2\subseteq\cdots\subseteq I_n$ is an ideal series in $\mathscr{I}_\lambda^n$.

Fix an arbitrary integer $i=1,\ldots,n$. For any infinite subset B of $\mathscr{I}_\lambda^i\setminus\mathscr{I}_\lambda^{i-1}$ at least one of the following families of sets

$$\mathfrak{d}(B)=\{\operatorname{dom}\gamma\colon\gamma\in B\}\qquad\text{or}\qquad\mathfrak{r}(B)=\{\operatorname{ran}\gamma\colon\gamma\in B\}$$

is infinite. Then the definition of the semigroup operation in $\mathscr{I}_\lambda^n$ implies that $\alpha B\nsubseteq\mathscr{I}_\lambda^i\setminus\mathscr{I}_\lambda^{i-1}$ in the case when the set $\mathfrak{d}(B)$ is infinite, and $B\beta\nsubseteq\mathscr{I}_\lambda^i\setminus\mathscr{I}_\lambda^{i-1}$ in the case when the set $\mathfrak{r}(B)$ is infinite, for any $\alpha,\beta\in\mathscr{I}_\lambda^i\setminus\mathscr{I}_\lambda^{i-1}$. ☐

Later for an arbitrary non-empty set A, any positive integer n and any $i\in\{1,\ldots,n\}$ by $\pi_i\colon A^n\to A$, $(a_1,\ldots,a_n)\mapsto a_i$ we shall denote the projection on the i-th factor of the power A^n.

Proposition 10. *Let n be a positive integer $\geqslant 2$ and let $I_0\subseteq I_1\subseteq I_2\subseteq\cdots\subseteq I_m=S$ be the strongly tight ideal series for a semigroup S. Then the series*

$$\begin{aligned}
&I_0^n\subseteq[I_0\subset I_1]_1^n\subseteq[I_0\subset I_1]_2^n\subseteq\cdots\subseteq[I_0\subset I_1]_{n-1}^n\subseteq[I_0\subset I_1]_n^n=I_1^n\subseteq\\
&\subseteq[I_1\subset I_2]_1^n\subseteq[I_1\subset I_2]_2^n\subseteq\cdots\subseteq[I_1\subset I_2]_{n-1}^n\subseteq[I_1\subset I_2]_n^n=I_2^n\subseteq\quad\cdots\quad\subseteq\\
&\subseteq[I_{m-1}\subset I_m]_1^n\subseteq[I_{m-1}\subset I_m]_2^n\subseteq\cdots\subseteq[I_{m-1}\subset I_m]_{n-1}^n\subseteq[I_{m-1}\subset I_m]_n^n=I_m^n=S^n
\end{aligned}\tag{2}$$

is a strongly tight ideal series for the direct power S^n.

Proof. It is obvious that I_0^n is a finite ideal of S^n. Also by Lemma 2 all subsets in series (2) are ideals in S^n.

Fix any $k\in\{1,\ldots,m\}$ and any $p\in\{1,\ldots,n\}$. We claim that the sets

$$[I_{k-1}\subset I_k]_p^n\setminus[I_{k-1}\subset I_k]_{p-1}^n\qquad\text{and}\qquad[I_{k-1}\subset I_k]_1^n\setminus I_{k-1}^n$$

are strongly ω-unstable in S^n. Indeed, fix an arbitrary infinite subset

$$B\subseteq[I_{k-1}\subset I_k]_p^n\setminus[I_{k-1}\subset I_k]_{p-1}^n$$

and any points

$$a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in [I_{k-1} \subset I_k]^n_p \setminus [I_{k-1} \subset I_k]^n_{p-1}.$$

Then there exists a coordinate $i \in \{1, \dots, n\}$ such that the set $\pi_i(B) \subseteq I_k \setminus I_{k-1}$ is infinite. If $a_i \notin I_k \setminus I_{k-1}$ or $b_i \notin I_k \setminus I_{k-1}$ then

$$(a_i \cdot \pi_i(B) \cup \pi_i(B) \cdot b_i) \cap I_k \setminus I_{k-1} = \varnothing,$$

and hence

$$aB \cup Bb \nsubseteq [I_{k-1} \subset I_k]^n_p \setminus [I_{k-1} \subset I_k]^n_{p-1}.$$

If $a_i, b_i \in I_k \setminus I_{k-1}$ then $(a_i \cdot \pi_i(B) \cup \pi_i(B) \cdot b_i) \nsubseteq I_k \setminus I_{k-1}$, because the set $I_k \setminus I_{k-1}$ is strongly ω-unstable in S, and hence $aB \cup Bb \nsubseteq [I_{k-1} \subset I_k]^n_p \setminus [I_{k-1} \subset I_k]^n_{p-1}$. The proof of the statement that the set $[I_{k-1} \subset I_k]^n_1 \setminus I^n_{k-1}$ is strongly ω-unstable in S^n is similar. □

Later we fix an arbitrary positive integer n. Then for any semigroup S and any positive integer $k \leqslant n$, since $\mathscr{I}^k_\lambda(S)$ is a subsemigroup of $\mathscr{I}^n_\lambda(S)$, by $\iota\colon \mathscr{I}^k_\lambda(S) \to \mathscr{I}^n_\lambda(S)$ we denote this natural embedding. Similar arguments imply that, without loss of generality, for any subsemigroup T of S and any positive integer $k \leqslant n$ since $\mathscr{I}^k_\lambda(T)$ is a subsemigroup of $\mathscr{I}^n_\lambda(S)$ by $\iota\colon \mathscr{I}^k_\lambda(T) \to \mathscr{I}^n_\lambda(S)$, we denote this natural embedding.

Let $A \neq \varnothing$ and k be any positive integer. A subset $B \subseteq A^k$ is said to be *k-symmetric* if the following condition holds: $(b_1, \dots, b_k) \in B$ implies $\big(b_{(1)\sigma}, \dots, b_{(k)\sigma}\big) \in B$ for every permutation $\sigma\colon \{1, \dots, k\} \to \{1, \dots, k\}$.

Remark 3. We observe that every element of the tight ideal series (2) is m-symmetric in S^n, and moreover the sets

$$[I_{k-1} \subset I_k]^n_p \setminus [I_{k-1} \subset I_k]^n_{p-1} \qquad \text{and} \qquad [I_{k-1} \subset I_k]^n_1 \setminus I^n_{k-1}$$

are m-symmetric in S^n, too, for all $k \in \{1, \dots, m\}$ and $p \in \{1, \dots, n\}$.

We need the following construction.

Construction 2. Let λ be a cardinal $\geqslant 1$, n be any positive integer, k be any positive integer $\leqslant \min\{n, \lambda\}$, and S be a semigroup. For any ordered collections of k distinct elements $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ of λ^k, we define a map

$$\mathfrak{f}^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)}\colon S^k \to S^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)}$$

by the formula

$$(s_1, \dots, s_k)\mathfrak{f}^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)} = \begin{pmatrix} a_1 & \cdots & a_k \\ s_1 & \cdots & s_k \\ b_1 & \cdots & b_k \end{pmatrix}.$$

For any non-empty subset $A \subseteq S^k$ and any positive integer $k \leqslant n$ we define the following subsets

$$[A]^{(*)_k}_{\mathscr{I}^n_\lambda(S)} = \bigcup \Big\{ (A)\mathfrak{f}^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)} : (a_1, \dots, a_k) \text{ and } (b_1, \dots, b_k) \text{ are ordered collections of } k \text{ distinct elements of } \lambda^k \Big\}$$

and

$$\overline{[A]}^{(*)_k}_{\mathscr{I}^n_\lambda(S)} = \begin{cases} [A]^{(*)_k}_{\mathscr{I}^n_\lambda(S)} \cup \mathscr{I}^{k-1}_\lambda(S), & \text{if } k \geqslant 1; \\ [A]^{(*)_1}_{\mathscr{I}^n_\lambda(S)} \cup \{0\}, & \text{if } k = 1, \end{cases}$$

of the semigroup $\mathscr{I}^n_\lambda(S)$.

The following lemma can be immediately derived from the definition of k-symmetric sets.

Lemma 3. *Let λ be a cardinal $\geqslant 1$, k be any positive integer $\leqslant \lambda$ and S be a semigroup. Let $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ be arbitrary ordered collections of k distinct elements of λ^k. If $A \neq \varnothing$ is a k-symmetric subset of S^k, then*

$$(A)\mathfrak{f}^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)} = (A)\mathfrak{f}^{(a_{(1)\sigma},\ldots,a_{(k)\sigma})}_{(b_{(1)\sigma},\ldots,b_{(k)\sigma})}$$

for every permutation $\sigma\colon \{1, \ldots, k\} \to \{1, \ldots, k\}$.

Theorem 1. *Let λ be an infinite cardinal and n be a positive integer. If S is a finite semigroup, then*

$$I_0 = \{0\} \subseteq I_1 = \mathscr{I}^1_\lambda(S) \subseteq I_2 = \mathscr{I}^2_\lambda(S) \subseteq \cdots \subseteq I_n = \mathscr{I}^n_\lambda(S)$$

is a strongly tight ideal series for the semigroup $\mathscr{I}^n_\lambda(S)$.

Proof. It is obvious that for every $i = 0, 1, \ldots, n$ the set I_i is an ideal in $\mathscr{I}^n_\lambda(S)$ and moreover the set I_0 is finite.

Fix an arbitrary $i = 1, \ldots, n$ and any infinite subset $B \subseteq I_i \setminus I_{i-1}$. Since the semigroup S is finite, every infinite subset X of $I_i \setminus I_{i-1}$ intersects infinitely many sets of the form $S^{(a_1,\ldots,a_i)}_{(b_1,\ldots,b_i)}$. Then the definition of the semigroup $\mathscr{I}^n_\lambda(S)$ implies that at least one of the families of sets

$$\mathfrak{d}(B) = \{\mathbf{d}\,\gamma\colon \gamma \in B\} \qquad \text{or} \qquad \mathfrak{r}(B) = \{\mathbf{r}\,\gamma\colon \gamma \in B\}$$

is infinite. Then the definition of the semigroup operation in $\mathscr{I}^n_\lambda(S)$ implies that $\alpha B \nsubseteq I_i \setminus I_{i-1}$ in the case when the set $\mathfrak{d}(B)$ is infinite, and $B\beta \nsubseteq I_i \setminus I_{i-1}$ in the case when the set $\mathfrak{r}(B)$ is infinite, for any $\alpha, \beta \in I_i \setminus I_{i-1}$. □

Theorem 2. *Let λ be an infinite cardinal, n be a positive integer and let*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m = S$$

be a strongly tight ideal series for a semigroup S. Then the series

$$
\begin{aligned}
&J_0=\{0\}\subseteq J_{1,0}=\overline{[I_0]}^{(*)_1}_{\mathscr{I}^n_\lambda(S)}\subseteq\\
&\subseteq J_{1,1}=\overline{[I_1]}^{(*)_1}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{1,2}=\overline{[I_2]}^{(*)_1}_{\mathscr{I}^n_\lambda(S)}\subseteq\cdots\subseteq J_{1,m}=\overline{[I_m]}^{(*)_1}_{\mathscr{I}^n_\lambda(S)}=\mathscr{I}^1_\lambda(S)\subseteq\\
&\subseteq J_{2,0}=\overline{[I_0^2]}^{(*)_2}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{2,1}=\overline{[[I_0\subset I_1]_1^2]}^{(*)_2}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{2,2}=\overline{[I_1^2]}^{(*)_2}_{\mathscr{I}^n_\lambda(S)}\subseteq\\
&\subseteq J_{2,3}=\overline{[[I_1\subset I_2]_1^2]}^{(*)_2}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{2,4}=\overline{[I_2^2]}^{(*)_2}_{\mathscr{I}^n_\lambda(S)}\subseteq\ \cdots\ \subseteq\\
&\subseteq J_{2,2m-1}=\overline{[[I_{m-1}\subset I_m]_1^2]}^{(*)_2}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{2,2m}=\overline{[[I_m]_2^2]}^{(*)_2}_{\mathscr{I}^n_\lambda(S)}=\mathscr{I}^2_\lambda(S)\subseteq\quad\cdots\quad\subseteq\\
&\subseteq J_{n,0}=\overline{[I_0^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,1}=\overline{[[I_0\subset I_1]_1^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,2}=\overline{[[I_0\subset I_1]_2^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq\\
&\subseteq J_{n,3}=\overline{[[I_0\subset I_1]_3^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,4}=\overline{[[I_0\subset I_1]_4^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq\cdots\subseteq\\
&\subseteq J_{n,n-1}=\overline{[[I_0\subset I_1]_{n-1}^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,n}=\overline{[I_1^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq\\
&\subseteq J_{n,n+1}=\overline{[[I_1\subset I_2]_1^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,n+2}=\overline{[[I_1\subset I_2]_2^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq\\
&\subseteq J_{n,n+3}=\overline{[[I_1\subset I_2]_3^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,n+4}=\overline{[[I_1\subset I_2]_4^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq\cdots\subseteq\\
&\subseteq J_{n,2n-1}=\overline{[[I_1\subset I_2]_{n-1}^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,2n}=\overline{[I_2^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq\quad\cdots\quad\subseteq\\
&\subseteq J_{n,(m-1)n+1}=\overline{[[I_{m-1}\subset I_m]_1^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,(m-1)n+2}=\overline{[[I_{m-1}\subset I_m]_2^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq\\
&\subseteq J_{n,(m-1)n+3}=\overline{[[I_{m-1}\subset I_m]_3^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,(m-1)n+4}=\overline{[[I_{m-1}\subset I_m]_4^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq\\
&\subseteq\cdots\subseteq J_{n,mn-1}=\overline{[[I_{m-1}\subset I_m]_{n-1}^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\subseteq J_{n,mn}=\overline{[I_m^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}=\mathscr{I}^n_\lambda(S)
\end{aligned}
\tag{3}
$$

is a strongly tight ideal series for the semigroup $\mathscr{I}^n_\lambda(S)$.

Proof. The definition of the semigroup $\mathscr{I}^n_\lambda(S)$ and Lemma 2 imply that all subsets in series (3) are ideals in $\mathscr{I}^n_\lambda(S)$.

Since I_0 is a finite ideal in S, the equalities

$$
\begin{gathered}
J_{1,0}\setminus J_0=\overline{[I_0]}^{(*)_1}_{\mathscr{I}^n_\lambda(S)}\setminus\{0\}=[I_0]^{(*)_1}_{\mathscr{I}^n_\lambda(S)}\\
J_{2,0}\setminus J_{1,m}=\overline{[I_0^2]}^{(*)_1}_{\mathscr{I}^n_\lambda(S)}\setminus\mathscr{I}^1_\lambda(S)=[I_0^2]^{(*)_1}_{\mathscr{I}^n_\lambda(S)}\\
\cdots\qquad\cdots\qquad\cdots\\
J_{n,0}\setminus J_{n-1,m(n-1)}=\overline{[I_0^n]}^{(*)_n}_{\mathscr{I}^n_\lambda(S)}\setminus\mathscr{I}^{n-1}_\lambda(S)=[I_0^n]^{(*)_n}_{\mathscr{I}^n_\lambda(S)}
\end{gathered}
$$

and the semigroup operation of $\mathscr{I}^n_\lambda(S)$ imply that

$$J_{1,0}\setminus J_0,\quad J_{2,0}\setminus J_{1,m},\quad\ldots,\quad J_{n,0}\setminus J_{n-1,m(n-1)}$$

are strongly ω-unstable subsets in $\mathscr{I}^n_\lambda(S)$.

Next we shall show that the set $J_{k,p}\setminus J_{k,p-1}$ is strongly ω-unstable in $\mathscr{I}^n_\lambda(S)$ for all $k=1,\ldots,n$ and $p=1,\ldots,km$.

Fix any infinite subset B of $J_{k,p} \setminus J_{k,p-1}$ and any $\alpha, \beta \in J_{k,p} \setminus J_{k,p-1}$. If $\mathbf{d}(B) \neq \mathbf{r}(\alpha)$ then the semigroup operation of $\mathscr{I}_\lambda^n(S)$ implies that $\alpha B \nsubseteq J_{k,p} \setminus J_{k,p-1}$. Similarly, if $\mathbf{d}(\beta) \neq \mathbf{r}(B)$ then $B\beta \nsubseteq J_{k,p} \setminus J_{k,p-1}$.

Next we suppose that $\mathbf{d}(B) = \mathbf{r}(\alpha)$, $\mathbf{d}(\beta) = \mathbf{r}(B)$,

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_k \\ s_1 & \cdots & s_k \\ b_1 & \cdots & b_k \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} c_1 & \cdots & c_k \\ t_1 & \cdots & t_k \\ d_1 & \cdots & d_k \end{pmatrix},$$

for some $s_1, \ldots, s_k, t_1, \ldots, t_k \in S$ and ordered collections of k distinct elements $(a_1, \ldots, a_k)$, $(b_1, \ldots, b_k)$, $(c_1, \ldots, c_k)$, $(d_1, \ldots, d_k)$ of λ^k. Then the set B consists of the elements of the form

$$\gamma = \begin{pmatrix} b_1 & \cdots & b_k \\ x_1 & \cdots & x_k \\ c_{(1)\sigma} & \cdots & c_{(k)\sigma} \end{pmatrix},$$

where $x_1, \ldots, x_k \in S$ and $\sigma\colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ is a permutation.

First we consider the case when $J_{k,p} = J_{k,jk} = \overline{\left[I_j^k\right]_{\mathscr{I}_\lambda^n(S)}}^{(*)_k}$ for some $j = 1, \ldots, m$. Then

$$J_{k,p-1} = J_{k,jk-1} = \overline{\left[[I_{j-1} \subset I_j]_{k-1}^k\right]_{\mathscr{I}_\lambda^n(S)}}^{(*)_k}$$

and $B \subseteq \left[I_j^k\right]_{\mathscr{I}_\lambda^n(S)}^{(*)_k}$. Since the set B is infinite, there exists $b_{i_0} \in \{b_1, \ldots, b_k\}$ such that there exist infinitely many $\gamma \in B$ such that $\mathbf{d}(\gamma) \ni b_{i_0}$. Without loss of generality we may assume that $b_{i_0} = b_1$. We put $B_0 = \{\gamma \in B \colon b_1 \in \mathbf{d}(\gamma)\}$. Then the set B_0 is infinite and hence the set

$$B_0^S = \left\{ x_1 \in S \colon \begin{pmatrix} b_1 & \cdots & b_k \\ x_1 & \cdots & x_k \\ c_{(1)\sigma} & \cdots & c_{(k)\sigma} \end{pmatrix} \in B_0, \ \sigma \text{ is a permutation of } \{1, \ldots, k\} \right\}$$

is infinite, too. The above implies that there exists a permutation σ_0 of $\{1, \ldots, k\}$ such that infinitely many elements of the form $\begin{pmatrix} b_1 & \cdots & b_k \\ x_1 & \cdots & x_k \\ c_{(1)\sigma_0} & \cdots & c_{(k)\sigma_0} \end{pmatrix}$ belong to B_0. Since $s_1, t_{(1)\sigma_0} \in I_j \setminus I_{j-1}$ and the set $I_j \setminus I_{j-1}$ is strongly ω-unstable we obtain that $a_1 \cdot B_0^S \cup B_0^S \cdot t_{(1)\sigma_0} \nsubseteq I_j \setminus I_{j-1}$, and hence the set $\left[I_j^k\right]_{\mathscr{I}_\lambda^n(S)}^{(*)_k}$ is strongly ω-unstable, as well.

Next we consider the case $J_{k,p} = J_{n,(j-1)k+q} = \overline{\left[[I_{j-1} \subset I_j]_q^k\right]_{\mathscr{I}_\lambda^n(S)}}^{(*)_k}$ for some $j = 1, \ldots, m$. Then

$$J_{k,p-1} = J_{n,(j-1)k+q-1} = \overline{\left[[I_{j-1} \subset I_j]_{q-1}^k\right]_{\mathscr{I}_\lambda^n(S)}}^{(*)_k}$$

and $B \subseteq \left[[I_{j-1} \subset I_j]_q^k\right]_{\mathscr{I}_\lambda^n(S)}^{(*)_k}$. Since the set B is infinite, without loss of generality we may assume that B contains an infinite subset B_0 which consists of elements of the form

$$\gamma = \begin{pmatrix} b_1 & \cdots & b_q & b_{q+1} & \cdots & b_k \\ x_1 & \cdots & x_q & x_{q+1} & \cdots & s_k \\ c_1 & \cdots & c_q & c_{q+1} & \cdots & c_k \end{pmatrix}, \tag{4}$$

where $x_1, \ldots, x_q \in I_j \setminus I_{j-1}$ and $x_{q+1}, \ldots, x_k \in I_{j-1} \setminus I_{j-2}$ for some ordered collections of k distinct elements $(b_1, \ldots, b_k)$ and $(c_1, \ldots, c_k)$ of λ^k. Fix arbitrary elements

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_k \\ s_1 & \cdots & s_k \\ b_1 & \cdots & b_k \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} c_1 & \cdots & c_k \\ t_1 & \cdots & t_k \\ d_1 & \cdots & d_k \end{pmatrix},$$

of the set B. If either $s_u \notin I_j \setminus I_{j-1}$ for some $u \in \{1, \ldots, q\}$ or $s_v \notin I_{j-1} \setminus I_{j-2}$ for some $v \in \{q+1, \ldots, k\}$ then $\alpha B_0 \nsubseteq \left[[I_{j-1} \subset I_j]_q^k\right]_{\mathscr{I}_\lambda^n(S)}^{(*)_k}$. Similarly, $t_u \notin I_j \setminus I_{j-1}$ for some $u \in$

$\{1,\ldots,q\}$ or $t_v \notin I_{j-1}\setminus I_{j-2}$ for some $v\in\{q+1,\ldots,k\}$ then $B_0\beta \nsubseteq \big[[I_{j-1}\subset I_j]_q^k\big]^{(*)_k}_{\mathscr{I}_\lambda^n(S)}$. Hence later we shall assume that $s_u \in I_j\setminus I_{j-1}$ for all $u\in\{1,\ldots,q\}$, $s_v\in I_{j-1}\setminus I_{j-2}$ for all $v\in\{q+1,\ldots,k\}$, $t_u\in I_j\setminus I_{j-1}$ for all $u\in\{1,\ldots,q\}$ and $t_v\in I_{j-1}\setminus I_{j-2}$ for all $v\in\{q+1,\ldots,k\}$. Since the set B_0 is infinite, there exists $i_0\in\{1,\ldots,k\}$ such that there exist infinitely many $\gamma\in B_0$ such that $\mathbf{d}(\gamma)\ni b_{i_0}$. We put $B_1=\{\gamma\in B_0\colon b_{i_0}\in\mathbf{d}(\gamma)\}$. Since the set B_1 is infinite, the following statements hold:

(1) if $i_0\in\{1,\ldots,q\}$ then $s_{i_0}A\cup At_{i_0}\nsubseteq I_j\setminus I_{j-1}$, where

$$A=\left\{x_{i_0}\colon \gamma=\begin{pmatrix} b_1 & \cdots & b_{i_0} & \cdots & b_q & \cdots & b_k\\ x_1 & \cdots & x_{i_0} & \cdots & x_q & \cdots & s_k\\ c_1 & \cdots & c_{i_0} & \cdots & c_q & \cdots & c_k\end{pmatrix}\in B_1\right\},$$

because the set $I_j\setminus I_{j-1}$ is strongly ω-unstable in S;

(2) if $i_0\in\{q+1,\ldots,k\}$ then $s_{i_0}A\cup At_{i_0}\nsubseteq I_{j-1}\setminus I_{j-2}$, where

$$A=\left\{x_{i_0}\colon \gamma=\begin{pmatrix} b_1 & \cdots & b_q & \cdots & b_{i_0} & \cdots & b_k\\ x_1 & \cdots & x_q & \cdots & x_{i_0} & \cdots & s_k\\ c_1 & \cdots & c_q & \cdots & c_{i_0} & \cdots & c_k\end{pmatrix}\in B_1\right\},$$

because the set $I_{j-1}\setminus I_{j-2}$ is strongly ω-unstable in S.

Both above statements imply that

$$\alpha B_1\cup B_1\gamma\nsubseteq \big[[I_{j-1}\subset I_j]_q^k\big]^{(*)_k}_{\mathscr{I}_\lambda^n(S)}$$

and hence

$$\alpha B\cup B\gamma\nsubseteq \big[[I_{j-1}\subset I_j]_q^k\big]^{(*)_k}_{\mathscr{I}_\lambda^n(S)},$$

i.e., the set $\big[[I_{j-1}\subset I_j]_q^k\big]^{(*)_k}_{\mathscr{I}_\lambda^n(S)}$ is strongly ω-unstable in $\mathscr{I}_\lambda^n(S)$. This completes the proof of the theorem. □

Theorem 2 implies the following

Corollary 2. *Let λ be an infinite cardinal, n be a positive integer and let*

$$I_0\subseteq I_1\subseteq I_2\subseteq\cdots\subseteq I_m=S$$

be a strongly tight ideal series for a semigroup S. Then the ideal series (3) *is tight for the semigroup $\mathscr{I}_\lambda^n(S)$.*

The proof of the following theorem is similar to Theorem 2.

Theorem 3. *Let λ be a finite cardinal, n be a positive integer $\leqslant\lambda$ and let*

$$I_0\subseteq I_1\subseteq I_2\subseteq\cdots\subseteq I_m=S$$

be a strongly tight ideal series for a semigroup S. Then the following series

$$
\begin{aligned}
&J_0 = \{0\} \cup \overline{[I_0]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)} \subseteq \\
&\subseteq J_{1,1} = \overline{[I_1]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{1,2} = \overline{[I_2]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)} \subseteq \cdots \subseteq J_{1,m} = \overline{[I_m]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)} = \mathscr{I}_\lambda^1(S) \subseteq \\
&\quad \subseteq J_{2,1} = \overline{[[I_1 \subset I_2]_1^2]}^{(*)_2}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{2,2} = \overline{[[I_1 \subset I_2]_2^2]}^{(*)_2}_{\mathscr{I}_\lambda^n(S)} \subseteq \cdots \subseteq \\
&\quad \subseteq J_{2,2m-1} = \overline{[[I_{m-1} \subset I_m]_1^2]}^{(*)_2}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{2,2m} = \overline{[[I_m]_2^2]}^{(*)_2}_{\mathscr{I}_\lambda^n(S)} \mathscr{I}_\lambda^2(S) \subseteq \quad \cdots \quad \subseteq \\
&\quad \subseteq J_{n,1} = \overline{[[I_0 \subset I_1]_1^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{n,2} = \overline{[[I_0 \subset I_1]_2^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq \\
&\quad \subseteq J_{n,3} = \overline{[[I_0 \subset I_1]_3^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{n,4} = \overline{[[I_0 \subset I_1]_4^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq \cdots \subseteq \\
(5)\qquad &\quad \subseteq J_{n,n-1} = \overline{[[I_0 \subset I_1]_{n-1}^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{n,n} = \overline{[I_1^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq \\
&\quad \subseteq J_{n,n+1} = \overline{[[I_1 \subset I_2]_1^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{n,n+2} = \overline{[[I_1 \subset I_2]_2^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq \\
&\quad \subseteq J_{n,n+3} = \overline{[[I_1 \subset I_2]_3^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{n,n+4} = \overline{[[I_1 \subset I_2]_4^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq \cdots \subseteq \\
&\quad \subseteq J_{n,2n-1} = \overline{[[I_1 \subset I_2]_{n-1}^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{n,2n} = \overline{[I_2^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq \quad \cdots \quad \subseteq \\
&\quad \subseteq J_{n,(m-1)n+1} = \overline{[[I_{m-1} \subset I_m]_1^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{n,(m-1)n+2} = \overline{[[I_{m-1} \subset I_m]_2^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq \\
&\quad \subseteq J_{n,(m-1)n+3} = \overline{[[I_{m-1} \subset I_m]_3^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{n,(m-1)n+4} = \overline{[[I_{m-1} \subset I_m]_4^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq \\
&\quad \subseteq \cdots \subseteq J_{n,mn-1} = \overline{[[I_{m-1} \subset I_m]_{n-1}^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{n,mn} = \overline{[I_m^n]}^{(*)_n}_{\mathscr{I}_\lambda^n(S)} = \mathscr{I}_\lambda^n(S)
\end{aligned}
$$

is a strongly tight ideal series for the semigroup $\mathscr{I}_\lambda^n(S)$.

Theorem 3 implies the following

Corollary 3. *Let* λ *be a finite cardinal,* n *be a positive integer* $\leqslant \lambda$ *and let* $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m = S$ *be a strongly tight ideal series for a semigroup* S. *Then the ideal series* (3) *is tight for the semigroup* $\mathscr{I}_\lambda^n(S)$.

Let $\mathfrak{S}$ be a class of semitopological semigroups. A semigroup $S \in \mathfrak{S}$ is called *H-closed in* $\mathfrak{S}$, if S is a closed subsemigroup of any semitopological semigroup $T \in \mathfrak{S}$ which contains S both as a subsemigroup and as a topological space. The H-closed topological semigroups were introduced by Stepp in [32], and therein they were called *maximal semigroups*. An algebraic semigroup S is called: *algebraically complete in* $\mathfrak{S}$, if S with any Hausdorff topology τ such that $(S,\tau) \in \mathfrak{S}$ is H-closed in $\mathfrak{S}$. We observe that many distinct types of H-closedness of topological and semitopological semigroups is studied in [1]–[10], [16]–[21], [24], [26].

By Proposition 10 from [18] every inverse semigroup S with a tight ideal series is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion. Hence Proposition 6 and Theorems 2, 3 imply the following

Theorem 4. *Let* S *be an inverse semigroup which admits a strongly tight ideal series. Then for every non-zero cardinal* λ *and any positive integer* $n \leqslant \lambda$ *the semigroup* $\mathscr{I}_\lambda^n(S)$

is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion.

We remark that in the case when $n = 1$ the construction of $\mathscr{I}_\lambda^1(S)$ preserves the property to be a semigroup with a tight ideal series, and this follows from the following theorem.

Theorem 5. *Let λ be any non-zero cardinal, n be a positive integer $n \leqslant \lambda$ and let $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m = S$ be a tight ideal series for a semigroup S. Then the series*

$$J_0=\{0\} \subseteq J_1 = \overline{[I_0]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)} \subseteq J_2=\overline{[I_1]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)}\subseteq\cdots\subseteq J_m=\overline{[I_{m-1}]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)} \subseteq J_{m+1}=\mathscr{I}_\lambda^1(S) \tag{6}$$

is a tight ideal series for the semigroup $\mathscr{I}_\lambda^1(S)$ in the case when λ is infinite, and

$$J_0=\{0\} \cup \overline{[I_0]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)} \subseteq J_1=\overline{[I_1]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)} \subseteq \cdots \subseteq J_{m-1}=\overline{[I_{m-1}]}^{(*)_1}_{\mathscr{I}_\lambda^n(S)} \subseteq J_m=\mathscr{I}_\lambda^1(S) \tag{7}$$

is a tight ideal series for the semigroup $\mathscr{I}_\lambda^1(S)$ in the case when λ is finite.

Proof. We consider the case when the cardinal λ is infinite. In the other case the proof is similar.

The semigroup operation of $\mathscr{I}_\lambda^1(S)$ implies that the the set J_k is an ideal in $\mathscr{I}_\lambda^1(S)$ for an arbitrary integer $k \in \{0, 1, \ldots, m+1\}$.

Fix an arbitrary $k \in \{1, \ldots, m+1\}$. Then for any infinite subset B of $J_k \setminus J_{k-1}$ and any $\alpha = \left(\begin{smallmatrix} a \\ s \\ b \end{smallmatrix}\right) \in J_k \setminus J_{k-1}$ the following statements hold.

(1) If $B \cap S_{(i)}^{(i)}$ is infinite for some $i \in \lambda$ then $B \cap S_{(i)}^{(i)} \subseteq [I_{k-1} \setminus I_{k_2}]_{(i)}^{(i)}$. Hence, the semigroup operation of $\mathscr{I}_\lambda^1(S)$ implies that $\alpha B \cup B\alpha \nsubseteq J_k \setminus J_{k-1}$ in the case when $a = b = i$, because the set $I_{k-1} \setminus I_{k_2}$ is ω-unstable in S. Otherwise $0 \in \alpha B \cup B\alpha \nsubseteq J_k \setminus J_{k-1}$.

(2) In the other case the semigroup operation of $\mathscr{I}_\lambda^1(S)$ implies that $0 \in \alpha B \cup B\alpha \nsubseteq J_k \setminus J_{k-1}$.

Both above statements imply that the set $J_k \setminus J_{k-1}$ is ω-unstable in $\mathscr{I}_\lambda^1(S)$, which completes the proof of the theorem. □

5. On a semitopological semigroup $\mathscr{I}_\lambda^n(S)$

For any element $\alpha = \left(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix}\right)$ of the semigroup $\mathscr{I}_\lambda^n$ and any $s \in S$ we denote $\alpha[s] = \left(\begin{smallmatrix} i_1 & \cdots & i_k \\ s & \cdots & s \\ j_1 & \cdots & j_k \end{smallmatrix}\right)$, which is an element of $\mathscr{I}_\lambda^n(S)$. Later in this case we shall say that $\alpha[s]$ is the *s-extension* of α or α is the *$\mathscr{I}_\lambda^n$-restriction* of $\alpha[s]$.

Proposition 11. *Let S be a monoid, λ be any non-zero cardinal, n be an arbitrary positive integer $\leqslant \lambda$, $0 < k \leqslant n$ and $\mathscr{I}_\lambda^n(S)$ be a Hausdorff semitopological semigroup. Then for any ordered collections of k distinct elements $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ of λ^k and any element $\alpha_S \in S_{(b_1,\ldots,b_k)}^{(a_1,\ldots,a_k)}$ there exists an open neighbourhood $U(\alpha_S)$ of α_S such that*

- *$U(\alpha_S) \cap \mathscr{I}_\lambda^{k-1}(S) = \varnothing$ and $U(\alpha_S) \cap \mathscr{I}_\lambda^k(S) \subseteq S_{(b_1,\ldots,b_k)}^{(a_1,\ldots,a_k)}$ in the case when $k \geqslant 2$,*
- *$0 \notin U(\alpha_S)$ and $U(\alpha_S) \cap \mathscr{I}_\lambda^1(S) \subseteq S_{(b_1)}^{(a_1)}$ in the case when $k = 1$.*

Thus $\mathscr{I}_\lambda^k(S)$ is a closed subsemigroup of $\mathscr{I}_\lambda^n(S)$.

Proof. Fix an arbitrary $k \leqslant n$ and an arbitrary $\alpha_S = \left(\begin{smallmatrix} a_1 & \dots & a_k \\ s_1 & \dots & s_k \\ b_1 & \dots & b_k \end{smallmatrix}\right) \in S^{a_1,\dots,a_k}_{b_1,\dots,b_k}$. It is obvious that $\varepsilon_1[1_S] \cdot \alpha_S \cdot \varepsilon_2[1_S] = \alpha_S$, where

$$\varepsilon_1[1_S] = \left(\begin{smallmatrix} a_1 & \dots & a_k \\ 1_S & \dots & 1_S \\ a_1 & \dots & a_k \end{smallmatrix}\right), \qquad \varepsilon_2[1_S] = \left(\begin{smallmatrix} b_1 & \dots & b_k \\ 1_S & \dots & 1_S \\ b_1 & \dots & b_k \end{smallmatrix}\right),$$

and 1_S is the unit element of S.

Simple calculations imply that

$$S^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)} = \\ = \varepsilon_1[1_S] \cdot \mathscr{I}^n_\lambda(S) \cdot \varepsilon_2[1_S] \setminus \bigcup \{\overline{\varepsilon}_1[1_S] \cdot \mathscr{I}^n_\lambda(S) \cdot \overline{\varepsilon}_2[1_S] \colon \overline{\varepsilon}_1 < \varepsilon_1 \text{ and } \overline{\varepsilon}_2 < \varepsilon_2 \text{ in } E(\mathscr{I}^n_\lambda)\}.$$

We observe that eT and Te are closed subset in an arbitrary Hausdorff semitopological semigroup T for any its idempotent e. Since for any idempotent $\varepsilon \in \mathscr{I}^n_\lambda$ the set $\downarrow\varepsilon = \{\iota \in E(\mathscr{I}^n_\lambda) \colon \iota \leqslant \varepsilon\}$ is finite, the set

$$A_{\alpha_S} = \bigcup \{\overline{\varepsilon}_1[1_S] \cdot \mathscr{I}^n_\lambda(S) \cdot \overline{\varepsilon}_2[1_S] \colon \overline{\varepsilon}_1 < \varepsilon_1 \text{ and } \overline{\varepsilon}_2 < \varepsilon_2\}$$

is closed in $\mathscr{I}^n_\lambda(S)$. Fix an arbitrary open neighbourhood $W(\alpha_S)$ of α_S such that $W(\alpha_S) \cap A_{\alpha_S} = \varnothing$. The separate continuity of the semigroup operation on $\mathscr{I}^n_\lambda(S)$ implies that there exists an open neighbourhood $U(\alpha_S)$ of α_S such that $\varepsilon_1[1_S] \cdot U(\alpha_S) \cdot \varepsilon_2[1_S] \subseteq W(\alpha_S)$. The neighbourhood $U(\alpha_S)$ is a requested one. Indeed, if there exists $\beta_S \in \mathscr{I}^k_\lambda(S) \setminus S^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)}$ then $\varepsilon_1[1_S] \cdot \beta_S \cdot \varepsilon_2[1_S] \in A_{\alpha_S}$. □

Remark 4. We observe that in Proposition 11 we may assume that the neighbourhood $U(\alpha_S)$ may be chosen with the property that $\varepsilon_1[1_S] \cdot U(\alpha_S) \cdot \varepsilon_2[1_S] \subseteq S^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)}$.

Proposition 12. *Let S be a monoid, λ be any non-zero cardinal, n be an arbitrary positive integer $\leqslant \lambda$, $0 < k \leqslant n$ and $\mathscr{I}^n_\lambda(S)$ be a Hausdorff semitopological semigroup. Then for any ordered collections of k distinct elements $(a_1, \dots, a_k)$, $(b_1, \dots, b_k)$, $(c_1, \dots, c_k)$, and $(d_1, \dots, d_k)$ of λ^k the subspaces $S^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)}$ and $S^{(c_1,\dots,c_k)}_{(d_1,\dots,d_k)}$ are homeomorphic, and moreover $S^{(a_1,\dots,a_k)}_{(a_1,\dots,a_k)}$ and $S^{(c_1,\dots,c_k)}_{(c_1,\dots,c_k)}$ are topologically isomorphic subsemigroups of $\mathscr{I}^n_\lambda(S)$.*

Proof. Since $\mathscr{I}^n_\lambda(S)$ is a semitopological semigroup, the restrictions of the maps

$${}^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)}\mathfrak{h}^{(c_1,\dots,c_k)}_{(d_1,\dots,d_k)} \colon \mathscr{I}^n_\lambda(S) \to \mathscr{I}^n_\lambda(S), \ \alpha \mapsto \left(\begin{smallmatrix} c_1 & \dots & c_k \\ 1_S & \dots & 1_S \\ a_1 & \dots & a_k \end{smallmatrix}\right) \cdot \alpha \cdot \left(\begin{smallmatrix} b_1 & \dots & b_k \\ 1_S & \dots & 1_S \\ d_1 & \dots & d_k \end{smallmatrix}\right)$$

and

$${}^{(c_1,\dots,c_k)}_{(d_1,\dots,d_k)}\mathfrak{h}^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)} \colon \mathscr{I}^n_\lambda(S) \to \mathscr{I}^n_\lambda(S), \ \alpha \mapsto \left(\begin{smallmatrix} a_1 & \dots & a_k \\ 1_S & \dots & 1_S \\ c_1 & \dots & c_k \end{smallmatrix}\right) \cdot \alpha \cdot \left(\begin{smallmatrix} d_1 & \dots & d_k \\ 1_S & \dots & 1_S \\ b_1 & \dots & b_k \end{smallmatrix}\right)$$

on the subspaces $S^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)}$ and $S^{(c_1,\dots,c_k)}_{(d_1,\dots,d_k)}$, respectively, are mutually inverse, and hence $S^{(a_1,\dots,a_k)}_{(b_1,\dots,b_k)}$ and $S^{(c_1,\dots,c_k)}_{(d_1,\dots,d_k)}$ are homeomorphic subspaces in $\mathscr{I}^n_\lambda(S)$. Also, it is obvious that in the case of subsemigroups $S^{(a_1,\dots,a_k)}_{(a_1,\dots,a_k)}$ and $S^{(c_1,\dots,c_k)}_{(c_1,\dots,c_k)}$ so defined restrictions of maps are topological isomorphisms. □

For any ordered collections of k distinct elements $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ of λ^k we define a map

$$\mathfrak{f}^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}\colon \mathscr{I}^n_\lambda(S)\to\mathscr{I}^n_\lambda(S),\ \alpha\mapsto \begin{pmatrix} a_1 & \ldots & a_k\\ 1_S & \ldots & 1_S\\ a_1 & \ldots & a_k\end{pmatrix}\cdot\alpha\cdot\begin{pmatrix} b_1 & \ldots & b_k\\ 1_S & \ldots & 1_S\\ b_1 & \ldots & b_k\end{pmatrix}.$$

Proposition 11 implies the following corollary.

Corollary 4. *Let S be a monoid, λ be any non-zero cardinal, n be an arbitrary positive integer $\leqslant\lambda$, $0<k\leqslant n$ and $\mathscr{I}^n_\lambda(S)$ be a Hausdorff semitopological semigroup. Then the set*

$$\Uparrow S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}=\left(S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}\right)\left(\mathfrak{f}^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}\right)^{-1}$$

is open in $\mathscr{I}^n_\lambda(S)$ for any ordered collections of k distinct elements $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ of λ^k.

We recall that a topological space X is said to be

- *compact* if each open cover of X has a finite subcover;
- *H-closed* if X is a closed subspace of every Hausdorff topological space in which it contained.

It is well known that every Hausdorff compact space is H-closed, and every regular H-closed topological space is compact (see [12, 3.12.5]).

Lemma 4. *Let S be a monoid, λ be any non-zero cardinal, n be an arbitrary positive integer $\leqslant\lambda$, $0<k\leqslant n$ and $\mathscr{I}^n_\lambda(S)$ be a Hausdorff semitopological semigroup. If $S^{(a)}_{(b)}$ is a closed subset of $\mathscr{I}^n_\lambda(S)$ for any $a,b\in\lambda$ then $S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}$ is a closed subspace of $\mathscr{I}^n_\lambda(S)$ for any ordered collections of k distinct elements $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ of λ^k.*

Proof. For any $a,b\in\lambda$ the map

$$\mathfrak{f}^{(a)}_{(b)}\colon \mathscr{I}^n_\lambda(S)\to\mathscr{I}^n_\lambda(S),\ \alpha\mapsto\begin{pmatrix} a\\ 1_S\\ a\end{pmatrix}\cdot\alpha\cdot\begin{pmatrix} b\\ 1_S\\ b\end{pmatrix}$$

is continuous, because $\mathscr{I}^n_\lambda(S)$ is a semitopological semigroup. This and Proposition 11 imply that

$$S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}=\left(S^{(a_1)}_{(b_1)}\right)\left(\mathfrak{f}^{(a_1)}_{(b_1)}\right)^{-1}\cap\cdots\cap\left(S^{(a_k)}_{(b_k)}\right)\left(\mathfrak{f}^{(a_k)}_{(b_k)}\right)^{-1}\cap\mathscr{I}^k_\lambda(S)$$

a closed subspace of $\mathscr{I}^n_\lambda(S)$. □

Since a continuous image of a compact (an H-closed) space is compact (H-closed) (see [12, Chapter 3]), Proposition 12 and Lemma 4 imply the following corollary.

Corollary 5. *Let S be a monoid, λ be any non-zero cardinal, n be an arbitrary positive integer $\leqslant\lambda$, $0<k\leqslant n$ and $\mathscr{I}^n_\lambda(S)$ be a Hausdorff semitopological semigroup. If the set $S^{(a)}_{(b)}$ is H-closed (compact) in $\mathscr{I}^n_\lambda(S)$ for some $a,b\in\lambda$ then $S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}$ is a closed subspace of $\mathscr{I}^n_\lambda(S)$ for any ordered collections of k distinct elements $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ of λ^k.*

Definition 3. Let $\mathfrak{S}$ be a class of semitopological semigroups. Let $\lambda\geqslant 1$ be a cardinal, n be a positive integer $\leqslant\lambda$, and $(S,\tau)\in\mathfrak{S}$. Let $\tau_{\mathscr{I}}$ be a topology on $\mathscr{I}^n_\lambda(S)$ such that

a) $(\mathscr{I}^n_\lambda(S),\tau_{\mathscr{I}})\in\mathfrak{S}$;

b) the topological subspace $\left(S^{(a)}_{(a)}, \tau_B|_{S_{\alpha,\alpha}}\right)$ is naturally homeomorphic to (S,τ) for some $a \in \lambda$, i.e., the map $\mathfrak{H}\colon S \to \mathscr{I}_\lambda^n(S)$, $s \mapsto \left(\begin{smallmatrix} a\\ s\\ a\end{smallmatrix}\right)$ is a topological embedding.

Then $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$ is called a *topological* $\mathscr{I}_\lambda^n$*-extension of* (S,τ) *in* $\mathfrak{S}$.

Lemma 5. *Let (S,τ) be a semitopological monoid, λ be any non-zero cardinal, n be an arbitrary positive integer $\leqslant \lambda$, $0 < k \leqslant n$ and $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$ be a topological $\mathscr{I}_\lambda^n$-extension of (S,τ) in the class of semitopological semigroups. Let $U_1(s_1),\ldots,U_k(s_k)$ be open neighbourhoods of the points $s_1,\ldots,s_k$ in (S,τ), respectively. Then the following sets*

$$\Uparrow[U_1(s_1)]^{(a_1)}_{(b_1)} = \left([U_1(s_1)]^{(a_1)}_{(b_1)}\right)\left(\mathfrak{f}^{(a_1)}_{(b_1)}\right)^{-1}, \; \ldots, \; \Uparrow[U_k(s_k)]^{(a_k)}_{(b_k)} = \left([U_k(s_k)]^{(a_k)}_{(b_k)}\right)\left(\mathfrak{f}^{(a_k)}_{(b_k)}\right)^{-1},$$

and

$$\Uparrow[U_1(s_1),\ldots,U_k(s_k)]^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)} = \Uparrow[U_1(s_1)]^{(a_1)}_{(b_1)} \cap \ldots \cap \Uparrow[U_k(s_k)]^{(a_k)}_{(b_k)},$$

are open neighbourhoods of the points

$$\left(\begin{smallmatrix} a_1\\ s_1\\ b_1\end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix} a_k\\ s_k\\ b_k\end{smallmatrix}\right), \qquad \text{and} \qquad \left(\begin{smallmatrix} a_1 & \cdots & a_k\\ s_1 & \cdots & s_k\\ b_1 & \cdots & b_k\end{smallmatrix}\right)$$

in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$, respectively, for any ordered collections of k distinct elements $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ of λ^k.

Proof. Since $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$ is a topological $\mathscr{I}_\lambda^n$-extension of (S,τ) in the class of Hausdorff semitopological semigroups, there exist open neighbourhoods $W_1,\ldots,W_k$ of of the points $\left(\begin{smallmatrix} a_1\\ s_1\\ b_1\end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix} a_k\\ s_k\\ b_k\end{smallmatrix}\right)$ in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$, respectively, such that

$$W_1 \cap S^{(a_1)}_{(b_1)} = [U_1(s_1)]^{(a_1)}_{(b_1)}, \qquad \ldots, \qquad W_k \cap S^{(a_k)}_{(b_k)} = [U_k(s_k)]^{(a_k)}_{(b_k)}.$$

Then the requested statement of the lemma follows from the separate continuity of the semigroup operation in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$. □

Theorem 6. *Let (S,τ) be a Hausdorff compact semitopological monoid, λ be any non-zero cardinal, n be an arbitrary positive integer $\leqslant \lambda$, $0 < k \leqslant n$ and $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$ be a compact topological $\mathscr{I}_\lambda^n$-extension of (S,τ) in the class of Hausdorff semitopological semigroups. Then the subspace $S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}$ of $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$ is compact and moreover it is homeomorphic to the power S^k with the product topology by the mapping*

$$\mathfrak{H}\colon S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)} \to S^k, \; \left(\begin{smallmatrix} a_1 & \cdots & a_k\\ s_1 & \cdots & s_k\\ b_1 & \cdots & b_k\end{smallmatrix}\right) \mapsto (s_1,\ldots,s_k),$$

for any ordered collections of k distinct elements $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ of λ^k.

Proof. Since the monoid (S,τ) is compact, Corollary 5 implies that $S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}$ a closed subset of of $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$. Then compactness of of $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$ implies that $S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}$ is compact, as well.

It is obvious that the above defined map $\mathfrak{H}\colon S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)} \to S^k$ is a bijection. Also, Lemma 5 implies that the map $\mathfrak{H}$ is continuous, and it is a homeomorphism, because S^k and $S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}$ are compacta. □

Proposition 11 and Theorem 6 imply the following corollary.

Corollary 6. *Let (S,τ) be a Hausdorff compact semitopological monoid, λ be any non-zero cardinal, n be an arbitrary positive integer $\leqslant \lambda$, $0 < k \leqslant n$ and $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$ be a compact topological $\mathscr{I}_\lambda^n$-extension of (S,τ) in the class of Hausdorff semitopological semigroups. Then $S_{(b_1,\ldots,b_k)}^{(a_1,\ldots,a_k)}$ is an open-and-closed subset of $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$ for any ordered collections of k distinct elements $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ of λ^k, and the space $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}})$ is the topological sum of such sets with isolated zero.*

Remark 5. Since by Theorem of [21] an infinite semigroup of matrix units and hence an infinite semigroup $\mathscr{I}_\lambda^n$ do not embed into compact Hausdorff topological semigroups, Corollary 6 describes compact topological $\mathscr{I}_\lambda^n$-extensions of compact semigroups (S,τ) in the class of Hausdorff topological semigroups.

Example 2. Let (S,τ_S) be a compact Hausdorff semitopological monoid. On the semigroup $\mathscr{I}_\lambda^n(S)$ we define a topology $\tau_{\mathscr{I}}^{\mathbf{c}}$ in the following way. Put

$$\mathscr{P}_k^{\mathbf{c}}(0) = \Big\{ \mathscr{I}_\lambda^n(S) \setminus \Uparrow S_{(b_1,\ldots,b_k)}^{(a_1,\ldots,a_k)} : (a_1,\ldots,a_k) \text{ and } (b_1,\ldots,b_k) \text{ are ordered collections of } k \text{ distinct elements of } \lambda^k \Big\},$$

for any $k = 1,\ldots,n$, and

$$\mathscr{P}^{\mathbf{c}}(a,s,b) = \Big\{ \Uparrow [U(s)]_{(b)}^{(a)} : U(s) \text{ is an open neighbourhood of } s \text{ in } (S,\tau_S) \Big\},$$

for some $\left(\begin{smallmatrix} a \\ s \\ b \end{smallmatrix}\right) \in \mathscr{I}_\lambda^n(S) \setminus \{0\}$.

The topology $\tau_{\mathscr{I}}^{\mathbf{c}}$ on $\mathscr{I}_\lambda^n(S)$ is generated by the family

$$\mathscr{P}^{\mathbf{c}} = \{\mathscr{P}_k^{\mathbf{c}}(0) \colon k = 1,\ldots,n\} \cup \Big\{ \mathscr{P}^{\mathbf{c}}(a,s,b) \colon \left(\begin{smallmatrix} a \\ s \\ b \end{smallmatrix}\right) \in \mathscr{I}_\lambda^n(S) \setminus \{0\} \Big\},$$

as a subbase.

Remark 6. Lemma 5 and the definition of the topology $\tau_{\mathscr{I}}^{\mathbf{c}}$ on $\mathscr{I}_\lambda^n(S)$ implies that the following statements hold.

(1) For any $k = 1,\ldots,n$ and every ordered collection $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ of k distinct elements of λ^k the set $\Uparrow S_{(b_1,\ldots,b_k)}^{(a_1,\ldots,a_k)}$ is closed in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$.

(2) For any element $\alpha_S = \left(\begin{smallmatrix} a_1 & \cdots & a_k \\ s_1 & \cdots & s_k \\ b_1 & \cdots & b_k \end{smallmatrix}\right)$ of $\mathscr{I}_\lambda^n(S)$ and any open neighbourhoods $U_1(s_1),\ldots,U_k(s_k)$ of the points $s_1,\ldots,s_k$ in (S,τ) the set

$$\Uparrow [U_1(s_1),\ldots,U_k(s_k)]_{(b_1,\ldots,b_k)}^{(a_1,\ldots,a_k)} \setminus \left(\Uparrow S_{(b_1^1,\ldots,b_{l_1}^1)}^{(a_1^1,\ldots,a_{l_1}^1)} \cup \cdots \cup \Uparrow S_{(b_1^p,\ldots,b_{l_p}^p)}^{(a_1^p,\ldots,a_{l_p}^p)} \right)$$

such that $\alpha_S \notin \Uparrow S_{(b_1^1,\ldots,b_{l_1}^1)}^{(a_1^1,\ldots,a_{l_1}^1)} \cup \cdots \cup \Uparrow S_{(b_1^p,\ldots,b_{l_p}^p)}^{(a_1^p,\ldots,a_{l_p}^p)}$, is an open neighbourhood of the point α_S in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$. Here $\{a_1,\ldots,a_k\} \subsetneqq \left\{a_1^j,\ldots,a_{l_j}^j\right\}$ and $\{b_1,\ldots,b_k\} \subsetneqq \left\{b_1^j,\ldots,b_{l_j}^j\right\}$ for all $j = 1,\ldots,p$.

Theorem 7. *If (S,τ_S) is a compact Hausdorff semitopological monoid then $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$ is a compact Hausdorff semitopological semigroup.*

Proof. It is obvious that the topology $\tau^{\mathbf{c}}_{\mathscr{I}}$ is Hausdorff.

By the Alexander Subbase Theorem (see [12, 3.12.2]) it is sufficient to show that every open cover of $\mathscr{I}^n_\lambda(S)$ which consists of elements of the subbase $\mathscr{P}^{\mathbf{c}}$ has a finite subcover.

We shall show that the space $(\mathscr{I}^n_\lambda(S),\tau^{\mathbf{c}}_{\mathscr{I}})$ is compact by induction. In the case when $n=1$, Corollary 13 from [23] implies that the space $\left(\mathscr{I}^1_\lambda(S),\tau^{\mathbf{c}}_{\mathscr{I}}\right)$ is compact. Next we shall show the step of induction: $\left(\mathscr{I}^{k-1}_\lambda(S),\tau^{\mathbf{c}}_{\mathscr{I}}\right)$ is compact implies $\left(\mathscr{I}^{k}_\lambda(S),\tau^{\mathbf{c}}_{\mathscr{I}}\right)$ is compact, too, for $k=2,\ldots,n$. Without loss of generality we may assume that $k=n$.

Let $\mathscr{U}$ be an arbitrary open cover of $(\mathscr{I}^n_\lambda(S),\tau^{\mathbf{c}}_{\mathscr{I}})$ which consists of elements of the subbase $\mathscr{P}^{\mathbf{c}}$. The assumption of induction implies that there exists a finite subfamily $\mathscr{U}_{n-1}$ of $\mathscr{U}$ which is a subcover of $\mathscr{I}^{n-1}_\lambda(S)$. Fix an arbitrary element $V_0=\mathscr{I}^n_\lambda(S)\setminus {\Uparrow} S^{(a_1,\ldots,a_p)}_{(b_1,\ldots,b_p)}\in\mathscr{U}_{n-1}$ which contains the zero 0 of $\mathscr{I}^n_\lambda(S)$. Then $p\in\{1,\ldots,n\}$.

We observe that an arbitrary element U_0 of the family $\{\mathscr{P}^{\mathbf{c}}_k(0)\colon k=1,\ldots,n\}$ contains the set $S^{(a_1,\ldots,a_p)}_{(b_1,\ldots,b_p)}$ if and only if $U_0\cap S^{(a_1,\ldots,a_p)}_{(b_1,\ldots,b_p)}\neq\varnothing$. This implies that only one of the following conditions holds:

(1) there does not exist an element of $\mathscr{U}_{n-1}$ from the family $\{\mathscr{P}^{\mathbf{c}}_k(0)\colon k=1,\ldots,n\}$ which contains the set $S^{(a_1,\ldots,a_p)}_{(b_1,\ldots,b_p)}$;

(2) there exists $W_0\in\mathscr{U}_{n-1}\cap\{\mathscr{P}^{\mathbf{c}}_k(0)\colon k=1,\ldots,n\}$ such that $S^{(a_1,\ldots,a_p)}_{(b_1,\ldots,b_p)}\subseteq W_0$.

Suppose that condition (1) holds. First we consider the case when $p<n$. By Theorem 6, the set $S^{(a_1,\ldots,a_p)}_{(b_1,\ldots,b_p)}$ is compact, and hence there exists finitely many elements ${\Uparrow}[U(s_1)]^{(c_1)}_{(d_1)},\ldots,{\Uparrow}[U(s_m)]^{(c_m)}_{(d_m)}$ in the family $\mathscr{U}_{n-1}\cap\mathscr{P}^{\mathbf{c}}\setminus\{\mathscr{P}^{\mathbf{c}}_k(0)\colon k=1,\ldots,n\}$ such that

$$S^{(a_1,\ldots,a_p)}_{(b_1,\ldots,b_p)}\subseteq{\Uparrow}[U(s_1)]^{(c_1)}_{(d_1)}\cup\cdots\cup{\Uparrow}[U(s_m)]^{(c_m)}_{(d_m)}.$$

It is obvious that $\left\{U_0,{\Uparrow}[U(s_1)]^{(c_1)}_{(d_1)},\ldots,{\Uparrow}[U(s_m)]^{(c_m)}_{(d_m)}\right\}$ is a finite cover of $(\mathscr{I}^n_\lambda(S),\tau^{\mathbf{c}}_{\mathscr{I}})$.

Next, we consider case $p=n$. We identify the set $S^{(a_1,\ldots,a_n)}_{(b_1,\ldots,b_n)}$ and the power S^n by the mapping

$$\mathfrak{H}\colon S^{(a_1,\ldots,a_n)}_{(b_1,\ldots,b_n)}\to S^n,\ \begin{pmatrix} a_1 & \ldots & a_n\\ s_1 & \ldots & s_n\\ b_1 & \ldots & b_n\end{pmatrix}\mapsto(s_1,\ldots,s_n). \tag{8}$$

The semigroup operation of $\mathscr{I}^n_\lambda(S)$ implies that ${\Uparrow}[U(s)]^{(c)}_{(d)}\cap S^{(a_1,\ldots,a_n)}_{(b_1,\ldots,b_n)}\neq\varnothing$ if and only if $c=a_i$ and $d=b_i$ for some $i=1,\ldots,n$. Then by (8) for every $i=1,\ldots,n$ we have that

$$\left({\Uparrow}[U(s)]^{(a_i)}_{(b_i)}\cap S^{(a_1,\ldots,a_n)}_{(b_1,\ldots,b_n)}\right)\mathfrak{H}=S\times\cdots\times\underbrace{U(s)}_{i\text{–th}}\times\cdots\times S\subseteq S^n. \tag{9}$$

Then the subbase $\mathscr{P}^{\mathbf{c}}$ on $\mathscr{I}^n_\lambda(S)$ and map (8) determine the product topology on S^n from the space S, and hence the space S^n is compact.

Suppose that $S^{(a_1,\ldots,a_n)}_{(b_1,\ldots,b_n)}$ is not compact. Then $S^{(a_1,\ldots,a_n)}_{(b_1,\ldots,b_n)}$ has a cover $\mathscr{W}$ which consists of the open sets of the form ${\Uparrow}[U(s)]^{(c)}_{(d)}$ and $\mathscr{W}$ does not have a finite subcover. Then the cover $\mathscr{W}_{S^n}$ of S^n which is determined by formula (9) from the family $\mathscr{W}$ has no finite subcover, too. This contradicts the compactness of S^n.

Hence in case (1) the cover $\mathscr{U}$ of $\mathscr{I}_\lambda^n(S)$ has a finite subcover.

Suppose that condition (2) holds. Then $W_0 = \mathscr{I}_\lambda^n(S) \setminus {\Uparrow}S_{(d_1,\ldots,d_q)}^{(c_1,\ldots,c_q)}$ with $q \leqslant n$. If ${\Uparrow}S_{(d_1,\ldots,d_q)}^{(c_1,\ldots,c_q)} \cap {\Uparrow}S_{(b_1,\ldots,b_p)}^{(a_1,\ldots,a_p)} = \varnothing$ then $\{V_0, W_0\}$ is a cover of $\mathscr{I}_\lambda^n(S)$. In the other case there exists a smallest positive integer p_1 such that $\max\{p+1,q\} \leqslant p_1 \leqslant n$ and two ordered p_1-collections of distinct elements $(e_1,\ldots,e_{p_1})$ and $(f_1,\ldots,f_{p_1})$ of the power λ^{p_1} such that

$$ {\Uparrow}S_{(d_1,\ldots,d_q)}^{(c_1,\ldots,c_q)} \cap {\Uparrow}S_{(b_1,\ldots,b_p)}^{(a_1,\ldots,a_p)} = {\Uparrow}S_{(f_1,\ldots,f_{p_1})}^{(e_1,\ldots,e_{p_1})}. $$

Then for the open set $U_1 = U_0 \cup W_0 = \mathscr{I}_\lambda^n(S) \setminus {\Uparrow}S_{(f_1,\ldots,f_{p_1})}^{(e_1,\ldots,e_{p_1})}$ either condition (1) or condition (2) holds.

Since $p+1 \leqslant p_1 \leqslant n$, we repeating finitely many items the above procedure we get that the space $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$ is compact.

Next we shall show that the topology $\tau_{\mathscr{I}}^{\mathbf{c}}$ is shift-continuous on $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$. We consider all possible cases.

(i) $0 \cdot 0 = 0$. Then for any open neighbourhood U_0 of zero in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$ we have that

$$ U_0 \cdot 0 = 0 \cdot U_0 = \{0\} \subseteq U_0. $$

(ii) $\alpha \cdot 0 = 0$. Then for any open neighbourhoods U_0 and U_α of zero and α in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$, respectively, we have that

$$ U_\alpha \cdot 0 = \{0\} \subseteq U_0. $$

Let $W_0 = \mathscr{I}_\lambda^n(S) \setminus \left({\Uparrow}S_{(b_1^1,\ldots,b_{p_1}^1)}^{(a_1^1,\ldots,a_{p_1}^1)} \cup \cdots \cup {\Uparrow}S_{(b_1^k,\ldots,b_{p_k}^k)}^{(a_1^k,\ldots,a_{p_k}^k)} \right)$ be a basic neighbourhood of 0 in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$. Without loss of generality we may assume that $p_1,\ldots,p_k \leqslant |\mathbf{d}(\alpha)|$. Put

$$ \mathbf{B} = \left\{ S_{(b)}^{(a)} \colon a \in \mathbf{d}(\alpha) \quad \text{and} \quad b \in \left\{ b_1^1,\ldots,b_{p_1}^1,\ldots,b_1^k,\ldots,b_{p_k}^k \right\} \right\}. $$

Then the family $\mathbf{B}$ is finite and $\alpha \cdot U_0 \subseteq W_0$ for $U_0 = \mathscr{I}_\lambda^n(S) \setminus \bigcup_{S_{(b)}^{(a)} \in \mathbf{B}} {\Uparrow}S_{(b)}^{(a)}$.

(iii) $0 \cdot \alpha = 0$. Then for any open neighbourhoods U_0 and U_α of zero and α in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$, respectively, we have that

$$ 0 \cdot U_\alpha = \{0\} \subseteq U_0. $$

Let $W_0 = \mathscr{I}_\lambda^n(S) \setminus \left({\Uparrow}S_{(b_1^1,\ldots,b_{p_1}^1)}^{(a_1^1,\ldots,a_{p_1}^1)} \cup \cdots \cup {\Uparrow}S_{(b_1^k,\ldots,b_{p_k}^k)}^{(a_1^k,\ldots,a_{p_k}^k)} \right)$ be a basic neighbourhood of 0 in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$. Without loss of generality we may assume that $p_1,\ldots,p_k \leqslant |\mathbf{d}(\alpha)|$. Put

$$ \mathbf{B} = \left\{ S_{(b)}^{(a)} \colon b \in \mathbf{r}(\alpha) \quad \text{and} \quad a \in \left\{ a_1^1,\ldots,a_{p_1}^1,\ldots,a_1^k,\ldots,a_{p_k}^k \right\} \right\}. $$

Then the family $\mathbf{B}$ is finite and $U_0 \cdot \alpha \subseteq W_0$ for $U_0 = \mathscr{I}_\lambda^n(S) \setminus \bigcup_{S_{(b)}^{(a)} \in \mathbf{B}} {\Uparrow}S_{(b)}^{(a)}$.

(iv) $\alpha \cdot \beta = 0$. Fix an arbitrary open neighbourhood W_0 of 0 in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$. Without loss of generality we may assume that $W_0 = \mathscr{I}_\lambda^n(S) \setminus \left({\Uparrow}S_{(b_1)}^{(a_1)} \cup \cdots \cup {\Uparrow}S_{(b_k)}^{(a_k)} \right)$. Since $\alpha \cdot \beta = 0$ we have that $\mathbf{r}(\alpha) \cap \mathbf{d}(\beta) = \varnothing$. We put

$$ \mathbf{B}_\alpha = \left\{ S_{(b)}^{(a)} \colon a \in \{a_1,\ldots,a_k\}, b \in \mathbf{d}(\beta), \text{ and } \alpha \notin {\Uparrow}S_{(b)}^{(a)} \right\} $$

and

$$\mathbf{B}_\beta=\left\{S^{(a)}_{(b)}\colon b\in\{b_1,\ldots,b_k\}, a\in\mathbf{r}(\alpha), \text{ and } \beta\notin{\Uparrow}S^{(a)}_{(b)}\right\}.$$

Let $S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}$ and $S^{(c_1,\ldots,c_p)}_{(d_1,\ldots,d_p)}$, $1\leqslant k,p\leqslant n$, such that $\alpha\in S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}$ and $\beta\in S^{(c_1,\ldots,c_p)}_{(d_1,\ldots,d_p)}$. Then the families $\mathbf{B}_\alpha$ and $\mathbf{B}_\beta$ are finite, and hence by Remark 6(2) the sets

$$V_\alpha=S^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}\setminus\bigcup_{S^{(a)}_{(b)}\in\mathbf{B}_\alpha}{\Uparrow}S^{(a)}_{(b)} \qquad\text{and}\qquad V_\beta=S^{(c_1,\ldots,c_p)}_{(d_1,\ldots,d_p)}\setminus\bigcup_{S^{(a)}_{(b)}\in\mathbf{B}_\beta}{\Uparrow}S^{(a)}_{(b)}$$

are open neighbourhoods of the points α and β in $(\mathscr{I}_\lambda^n(S),\tau^{\mathbf{c}}_{\mathscr{I}})$, respectively, such that

$$V_\alpha\cdot\beta\subseteq W_0 \qquad\text{and}\qquad \alpha\cdot V_\beta\subseteq W_0.$$

(v) $\alpha\cdot\beta=\gamma\neq 0$ and $\mathbf{r}(\alpha)=\mathbf{d}(\beta)$. Without loss of generality we may assume that $\alpha=\left(\begin{smallmatrix} a_1 & \ldots & a_k\\ s_1 & \ldots & s_k\\ b_1 & \ldots & b_k\end{smallmatrix}\right)$ and $\beta=\left(\begin{smallmatrix} b_1 & \ldots & b_k\\ t_1 & \ldots & t_k\\ c_1 & \ldots & c_k\end{smallmatrix}\right)$, and hence we have that $\gamma=\left(\begin{smallmatrix} a_1 & \ldots & a_k\\ s_1t_1 & \ldots & s_kt_k\\ c_1 & \ldots & c_k\end{smallmatrix}\right)$. Then for any open neighbourhood

$$U_\gamma={\Uparrow}[U_1(s_1t_1),\ldots,U_k(s_kt_k)]^{(a_1,\ldots,a_k)}_{(c_1,\ldots,c_k)}\setminus\left({\Uparrow}S^{(a^1_1,\ldots,a^1_{l_1})}_{(b^1_1,\ldots,b^1_{l_1})}\cup\cdots\cup{\Uparrow}S^{(a^p_1,\ldots,a^p_{l_p})}_{(b^p_1,\ldots,b^p_{l_p})}\right)$$

of γ in $(\mathscr{I}_\lambda^n(S),\tau^{\mathbf{c}}_{\mathscr{I}})$ we have that

$${\Uparrow}[V_1(s_1),\ldots,V_k(s_k)]^{(a_1,\ldots,a_k)}_{(b_1,\ldots,b_k)}\cdot\beta\subseteq{\Uparrow}[U_1(s_1t_1),\ldots,U_k(s_kt_k)]^{(a_1,\ldots,a_k)}_{(c_1,\ldots,c_k)}\cap S^{(a_1,\ldots,a_k)}_{(c_1,\ldots,c_k)}\subseteq U_\gamma$$

and

$$\alpha\cdot{\Uparrow}[V_1(t_1),\ldots,V_k(t_k)]^{(b_1,\ldots,b_k)}_{(c_1,\ldots,c_k)}\subseteq{\Uparrow}[U_1(s_1t_1),\ldots,U_k(s_kt_k)]^{(a_1,\ldots,a_k)}_{(c_1,\ldots,c_k)}\cap S^{(a_1,\ldots,a_k)}_{(c_1,\ldots,c_k)}\subseteq U_\gamma,$$

where $V_1(s_1),\ldots,V_k(s_k),V_1(t_1),\ldots,V_k(t_k)$ are open neighbourhoods of the points $s_1,\ldots,s_k,t_1,\ldots,t_k$ in (S,τ_S), respectively, such that

$$V_1(s_1)\cdot t_1\subseteq U_1(s_1t_1),\ \ldots,\ V_k(s_k)\cdot t_k\subseteq U_k(s_kt_k)$$

and

$$s_1\cdot V_1(t_1)\subseteq U_1(s_1t_1),\ \ldots,\ s_k\cdot V_k(t_k)\subseteq U_k(s_kt_k).$$

(vi) $\alpha\cdot\beta=\gamma\neq 0$ and $\mathbf{r}(\alpha)\subsetneqq\mathbf{d}(\beta)$. Without loss of generality we may assume that $\alpha=\left(\begin{smallmatrix} a_1 & \ldots & a_k\\ s_1 & \ldots & s_k\\ b_1 & \ldots & b_k\end{smallmatrix}\right)$ and $\beta=\left(\begin{smallmatrix} b_1 & \ldots & b_k & b_{k+1} & \ldots & b_{k+j}\\ t_1 & \ldots & t_k & t_{k+1} & \ldots & t_{k+j}\\ c_1 & \ldots & c_k & c_{k+1} & \ldots & c_{k+j}\end{smallmatrix}\right)$, where $1\leqslant j\leqslant n-k$, and hence we have that $\gamma=\left(\begin{smallmatrix} a_1 & \ldots & a_k\\ s_1t_1 & \ldots & s_kt_k\\ c_1 & \ldots & c_k\end{smallmatrix}\right)$. Then for any open neighbourhood

$$U_\gamma={\Uparrow}[U_1(s_1t_1),\ldots,U_k(s_kt_k)]^{(a_1,\ldots,a_k)}_{(c_1,\ldots,c_k)}\setminus\left({\Uparrow}S^{(a^1_1,\ldots,a^1_{l_1})}_{(b^1_1,\ldots,b^1_{l_1})}\cup\cdots\cup{\Uparrow}S^{(a^p_1,\ldots,a^p_{l_p})}_{(b^p_1,\ldots,b^p_{l_p})}\right)$$

of the point γ in $(\mathscr{I}_\lambda^n(S),\tau^{\mathbf{c}}_{\mathscr{I}})$ we have that

$$\alpha\cdot{\Uparrow}[V_1(t_1),\ldots,V_k(t_k)]^{(b_1,\ldots,b_k)}_{(c_1,\ldots,c_k)}\subseteq{\Uparrow}[U_1(s_1t_1),\ldots,U_k(s_kt_k)]^{(a_1,\ldots,a_k)}_{(c_1,\ldots,c_k)}\cap S^{(a_1,\ldots,a_k)}_{(c_1,\ldots,c_k)}\subseteq U_\gamma,$$

where $V_1(t_1),\ldots,V_k(t_k)$ are open neighbourhoods of the points $t_1,\ldots,t_k$ in (S,τ_S), respectively, such that

$$s_1\cdot V_1(t_1)\subseteq U_1(s_1t_1),\ldots,s_k\cdot V_k(t_k)\subseteq U_k(s_kt_k).$$

Fix an arbitrary open neighbourhood U_γ of the point γ in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$. Then Lemma 5 implies that without loss of generality we may assume that

$$U_\gamma = {\Uparrow}[U_1(s_1t_1), \ldots, U_k(s_kt_k)]_{(c_1,\ldots,c_k)}^{(a_1,\ldots,a_k)} \setminus \left({\Uparrow}S_{(c_1,\ldots,c_k,y_1)}^{(a_1,\ldots,a_k,x_1)} \cup \cdots \cup {\Uparrow}S_{(c_1,\ldots,c_k,y_p)}^{(a_1,\ldots,a_k,x_p)}\right)$$

for some $x_1, \ldots, x_p \in \lambda \setminus \{a_1, \ldots, a_k\}$ and $y_1, \ldots, y_p \in \lambda \setminus \{c_1, \ldots, c_k\}$. We put

$$\mathbf{B}_\alpha = \left\{S_{(b_1,\ldots,b_k,b)}^{(a_1,\ldots,a_k,a)} : a \in \{x_1, \ldots, x_p\} \quad \text{and} \quad b \in \{b_{k+1}, \ldots, b_{k+j}\}\right\}.$$

It is obvious that the family $\mathbf{B}_\alpha$ is finite. Then $V_\alpha \cdot \beta \subseteq U_\gamma$ for

$$V_\alpha = {\Uparrow}[V_1(s_1), \ldots, V_k(s_k)]_{(b_1,\ldots,b_k)}^{(a_1,\ldots,a_k)} \setminus \bigcup_{S_{(b_1,\ldots,b_k,b)}^{(a_1,\ldots,a_k,a)} \in \mathbf{B}_\alpha} {\Uparrow}S_{(b_1,\ldots,b_k,b)}^{(a_1,\ldots,a_k,a)},$$

where $V_1(s_1), \ldots, V_k(s_k)$ are open neighbourhoods of the points $s_1, \ldots, s_k$ in (S, τ_S), respectively, such that

$$V_1(s_1) \cdot t_1 \subseteq U_1(s_1t_1), \ \ldots, \ V_k(s_k) \cdot t_k \subseteq U_k(s_kt_k).$$

(vii) $\alpha \cdot \beta = \gamma \neq 0$ and $\mathbf{d}(\beta) \subsetneqq \mathbf{r}(\alpha)$. In this case the proof of separate continuity of the semigroup operation is similar to case (vi).

$(viii)$ $\alpha \cdot \beta = \gamma \neq 0$, $\mathbf{d}(\gamma) \subsetneqq \mathbf{d}(\alpha)$ and $\mathbf{r}(\gamma) \subsetneqq \mathbf{r}(\beta)$. Without loss of generality we may assume that

$$\alpha = \begin{pmatrix} a_1 & \ldots & a_k & a_{k+11} & \ldots & a_{k+m} \\ s_1 & \ldots & s_k & s_{k+11} & \ldots & s_{k+m} \\ b_1 & \ldots & b_k & b_{k+11} & \ldots & b_{k+m} \end{pmatrix}, \quad \beta = \begin{pmatrix} b_1 & \ldots & b_k & b_{k+1} & \ldots & b_{k+j} \\ t_1 & \ldots & t_k & t_{k+1} & \ldots & t_{k+j} \\ c_1 & \ldots & c_k & c_{k+1} & \ldots & c_{k+j} \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} a_1 & \ldots & a_k \\ s_1t_1 & \ldots & s_kt_k \\ c_1 & \ldots & c_k \end{pmatrix},$$

where $1 \leqslant j, m \leqslant n-k$. We put $\varepsilon = \begin{pmatrix} b_1 & \ldots & b_k \\ 1_S & \ldots & 1_S \\ b_1 & \ldots & b_k \end{pmatrix}$, where 1_S is the unit element of S. It is obvious that $\gamma = \alpha \cdot \varepsilon \cdot \beta$. Hence, in this case the separate continuity of the semigroup operation at the point $\alpha \cdot \beta$ in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$ follows from cases (vi) and (vii).

The previous statements of this section imply that $\tau_{\mathscr{I}}^{\mathbf{c}} \subseteq \tau_{\mathscr{I}}$ for any compact shift-continuous Hausdorff topology $\tau_{\mathscr{I}}$ on $\mathscr{I}_\lambda^n(S)$, and hence $\tau_{\mathscr{I}}^{\mathbf{c}}$ is the unique requested compact shift-continuous Hausdorff topology on $\mathscr{I}_\lambda^n(S)$. □

Corollary 7. *If (S, τ_S) is a compact Hausdorff semitopological inverse monoid with continuous inversion then $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$ is a compact Hausdorff semitopological inverse semigroup with continuous inversion.*

Proof. Since $W_0^{-1} = \mathscr{I}_\lambda^n(S) \setminus \left({\Uparrow}S_{(a_1^1,\ldots,a_{p_1}^1)}^{(b_1^1,\ldots,b_{p_1}^1)} \cup \cdots \cup {\Uparrow}S_{(a_1^k,\ldots,a_{p_k}^k)}^{(b_1^k,\ldots,b_{p_k}^k)}\right)$ for an arbitrary basic neighbourhood $W_0 = \mathscr{I}_\lambda^n(S) \setminus \left({\Uparrow}S_{(b_1^1,\ldots,b_{p_1}^1)}^{(a_1^1,\ldots,a_{p_1}^1)} \cup \cdots \cup {\Uparrow}S_{(b_1^k,\ldots,b_{p_k}^k)}^{(a_1^k,\ldots,a_{p_k}^k)}\right)$ of zero, inversion is continuous at zero in $(\mathscr{I}_\lambda^n(S), \tau_{\mathscr{I}}^{\mathbf{c}})$.

Also, for an arbitrary element $\alpha = \begin{pmatrix} a_1 & \ldots & a_k \\ s_1 & \ldots & s_k \\ b_1 & \ldots & b_k \end{pmatrix}$ of $\mathscr{I}_\lambda^n(S)$ and any its open neighbourhood

$$V_\alpha = {\Uparrow}[V_1(s_1), \ldots, V_k(s_k)]_{(b_1,\ldots,b_k)}^{(a_1,\ldots,a_k)} \setminus \left({\Uparrow}S_{(b_1^1,\ldots,b_{l_1}^1)}^{(a_1^1,\ldots,a_{l_1}^1)} \cup \cdots \cup {\Uparrow}S_{(b_1^p,\ldots,b_{l_p}^p)}^{(a_1^p,\ldots,a_{l_p}^p)}\right)$$

we have that $(V_\alpha)^{-1} \subseteq U_{\alpha^{-1}}$ for the neighbourhood

$$U_{\alpha^{-1}} = \Uparrow \left[U_1(s_1^{-1}), \ldots, V_k(s_k^{-1})\right]^{(b_1,\ldots,b_k)}_{(a_1,\ldots,a_k)} \setminus \left(\Uparrow S^{(b_1^1,\ldots,b^1_{l_1})}_{(a_1^1,\ldots,a^1_{l_1})} \cup \cdots \cup \Uparrow S^{(b_1^p,\ldots,b^p_{l_p})}_{(a_1^p,\ldots,a^p_{l_p})}\right)$$

of α^{-1} in $(\mathscr{I}_\lambda^n(S), \tau^{\mathbf{c}}_{\mathscr{I}})$ with

$$(V_1(s_1))^{-1} \subseteq U_1(s_1^{-1}), \ldots, (V_k(s_k))^{-1} \subseteq U_k(s_k^{-1}).$$

This completes the proof of the corollary. □

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РОЗШИРЕННЯ НАПІВГРУП СИМЕТРИЧНИМИ ІНВЕРСНИМИ НАПІВГРУПАМИ ОБМЕЖЕНОГО СКІНЧЕННОГО РАНГУ

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Вивчаємо напівгрупове розширення $\mathscr{I}_\lambda^n(S)$ напівгрупи S симетричною інверсною напівгрупою обмеженого скінченного рангу n. Описуємо ідемпотенти та регулярні елементи напівгрупи $\mathscr{I}_\lambda^n(S)$, доводимо, що напівгрупа $\mathscr{I}_\lambda^n(S)$ є регулярною, ортодоксальною, інверсною або стійкою тоді і тільки тоді, коли такою напівгрупою є моноїд S. Описані відношення Ґріна на напівгрупі $\mathscr{I}_\lambda^n(S)$ для довільного моноїда S. Вводимо поняття напівгрупи з сильними щільними ідеальними рядами і доводимо, що для довільного нескінченного кардинала λ та довільного натурального числа n напівгрупа $\mathscr{I}_\lambda^n(S)$ має сильний щільний ідеальний ряд за умови, коли моноїд S також має сильний щільний ідеальний ряд. На завершення доводимо, що для кожного компактного гаусдорфового напівтопологічного моноїда (S,τ_S) існує єдине його компактне топологічне розширення $(\mathscr{I}_\lambda^n(S),\tau_{\mathscr{I}}^{\mathsf{c}})$ в класі гаусдорфових напівтопологічних напівгруп.

Ключові слова: інверсна напівгрупа, симетрична інверсна напівгрупа скінченних перетворень, відношення Ґріна, напівгрупа зі щільними ідеальними рядами, напівтопологічна напівгрупа, компактна напівгрупа.