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REMARKS TO RELATIVE GROWTH OF ENTIRE DIRICHLET SERIES

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Let F and G be entire functions given by Dirichlet series with exponents increasing to $+\infty$ and $\varrho_R[F]_G$ be the R-order of F with respect to a function G. The quantities

$$T_R[F]_G := \varlimsup_{\sigma\to+\infty} \frac{\exp\{M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]_G\sigma\}}, \quad t_R[F]_G := \varliminf_{\sigma\to+\infty} \frac{\exp\{M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]_G\sigma\}}$$

are called the R-type and the lower R-type of F with respect to G. A connection between $T_R[F]_G$, $t_R[F]_G$ and the R-types and the lower R-types of F and G is demonstrated.

Key words: Dirichlet series, relative growth, convergence class

1. Introduction

Let f and g be entire transcendental functions and $M_f(r) = \max\{|f(z)| : |z| = r\}$. For the study of relative growth of the functions f and g Ch. Roy [1] used the order $\varrho_g[f] = \varlimsup\limits_{r\to+\infty} \dfrac{\ln M_g^{-1}(M_f(r))}{\ln r}$ and the lower order $\lambda_g[f] = \varliminf\limits_{r\to+\infty} \dfrac{\ln M_g^{-1}(M_f(r))}{\ln r}$ of the function f with respect to the function g. Research of relative growth of entire functions was continued by T. Biswas and other mathematicians (see, for example, [2], [3]).

Suppose that $\Lambda = (\lambda_n)$ is an increasing to $+\infty$ sequence of non-negative numbers, and Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

2010 Mathematics Subject Classification: 30B50 has the abscissa of absolute convergence $\sigma_a = +\infty$. For $\sigma < +\infty$ we put $M_F(\sigma) = \sup\{|F(\sigma+it)| : t \in \mathbb{R}\}$.

Let L be a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

If $\alpha \in L$ and $\beta \in L$ then for entire Dirichlet series (1) the quantities

$$\varrho_{\alpha,\beta}[F] = \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)}, \quad \lambda_{\alpha,\beta}[F] = \varliminf_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)}$$

are called the generalized order and the generalized lower order of F, respectively. We say that F has the generalized regular growth, if $0 < \lambda_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[F] < +\infty$.

As in [4] we define the generalized order $\varrho_{\alpha,\beta}[F]_G$ and the generalized lower order $\lambda_{\alpha,\beta}[F]_G$ of the function F with respect to a function G, given by an entire Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} g_n \exp\{s\lambda_n\}, \tag{2}$$

as follows

$$\varrho_{\alpha,\beta}[F]_G = \varlimsup_{\sigma\to+\infty} \frac{\beta(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}, \quad \lambda_{\alpha,\beta}[F]_G = \varliminf_{\sigma\to+\infty} \frac{\beta(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}.$$

The following two theorems are proved in [4].

Theorem A. *Let $\alpha \in L$ and $\beta \in L$. Except for the cases when $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = 0$ or $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = +\infty$, the inequality $\varrho_{\alpha,\beta}[F]_G \geq \varrho_{\alpha,\beta}[F]/\varrho_{\alpha,\beta}[G]$ is true and subject to the condition of the regular growth of G this inequality converts into an equality.*

Except for the cases when $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = 0$ or $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = +\infty$, the inequality $\lambda_{\alpha,\beta}[F]_G \leq \lambda_{\alpha,\beta}[F]/\lambda_{\alpha,\beta}[G]$ is true and subject to the condition of the regular growth of G this inequality converts into an equality.

Theorem B. *Let $0 < p < +\infty$ and one of conditions is satisfied:*

a) *$\alpha \in L^0$, $\beta(\ln x) \in L^0$, $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} \to \dfrac{1}{p}$ $(x \to +\infty)$ for each $c \in (0, +\infty)$ and $\ln n = o(\lambda_n)$ $(n \to \infty)$;*

b) *$\alpha \in L_{si}$, $\beta \in L^0$, $\varrho_{\alpha,\beta}[F] < +\infty$, $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ $(x \to +\infty)$ and $\ln n = o(\lambda_n\beta^{-1}(c\alpha(\lambda_n)))$ $(n \to \infty)$ for each $c \in (0, +\infty)$.*

Suppose that $\alpha(\lambda_{n+1}/p) = (1+o(1))\alpha(\lambda_n/p)$ as $n \to \infty$.

If the function G has generalized regular growth and

$$\varkappa_n[G] := \frac{\ln|g_n| - \ln|g_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty, \quad n_0 \leq n \to \infty,$$

then

$$\varrho_{\alpha,\beta}[F]_G = \varlimsup_{n\to\infty} \frac{\beta\left(\dfrac{1}{p}+\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}{\beta\left(\dfrac{1}{p}+\dfrac{1}{\lambda_n}\ln\dfrac{1}{|f_n|}\right)}$$

except for the cases when $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = 0$ *or* $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = +\infty$.

If, moreover, F *has generalized regular growth and* $\varkappa_n[F] \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then*

$$\lambda_{\alpha,\beta}[F]_G = \varliminf_{n\to\infty} \frac{\beta\left(\dfrac{1}{p}+\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}{\beta\left(\dfrac{1}{p}+\dfrac{1}{\lambda_n}\ln\dfrac{1}{|f_n|}\right)}$$

except for the cases when $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = 0$ *or* $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = +\infty$.

If we choose $\alpha(x) = \ln x$ and $\beta(x) = x$ for $x \ge 3$ then from the definition of $\varrho_{\alpha,\beta}[F]$ and $\lambda_{\alpha,\beta}[F]$ we obtain the definition of the R-order $\varrho_R[F]$ and the lower R-order $\lambda_R[F]$. The quantities

$$\varrho_R[F]_G := \varlimsup_{\sigma\to+\infty} \frac{M_G^{-1}(M_F(\sigma))}{\sigma}, \quad \lambda_R[F]_G := \varliminf_{\sigma\to+\infty} \frac{M_G^{-1}(M_F(\sigma))}{\sigma}$$

we will call the R-order and the lower R-order of F with respect to a function G, accordingly. From Theorems A and B we get the following statement.

Corollary 1. *Except for the cases when* $\varrho_R[F] = \varrho_R[G] = 0$ *or* $\varrho_R[F] = \varrho_R[G] = +\infty$, *the inequality* $\varrho_R[F]_G \ge \varrho_R[F]/\varrho_R[G]$ *is true and subject to the condition of the regular growth of G (i. e.* $0 < \lambda_R[G] = \lambda_R[G] < \infty$*) this inequality converts into an equality, and except for the cases when* $\lambda_R[F] = \lambda_R[G] = 0$ *or* $\lambda_R[F] = \lambda_R[G] = +\infty$, *the inequality* $\lambda_R[F]_G \le \lambda_R[F]/\lambda_R[G]$ *is true and subject to the condition of the regular growth of G this inequality converts into an equality.*

If $\ln n = o(\lambda_n \ln \lambda_n)$, $\ln \lambda_{n+1} = (1+o(1)) \ln \lambda_n$ *as* $n\to\infty$, *the function G has the regular growth and* $\varkappa_n[G] \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then* $\varrho_R[F]_G = \varlimsup\limits_{n\to\infty} \dfrac{\ln |g_n|}{\ln |f_n|}$ *except for the cases when* $\varrho_R[F] = \varrho_R[G] = 0$ *or* $\varrho_R[F] = \varrho_R[G] = +\infty$, *and if, moreover, the function F has the regular growth and and* $\varkappa_n[F] \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then* $\lambda_R[F]_G = \varliminf\limits_{n\to\infty} \dfrac{\ln |g_n|}{\ln |f_n|}$ *except for the cases when* $\lambda_R[F] = \lambda_R[G] = 0$ *or* $\lambda_R[F] = \lambda_R[G] = +\infty$.

Here we investigate the growth of F with respect to G in terms of R-types and convergence classes.

2. Relative growth in terms of R-types

Suppose that $\varrho_R[F] = \varrho \in (0, +\infty)$. If we choose $\alpha(x) = x$ and $\beta(x) = e^{\varrho x}$ for $x \ge 1$ then from the definition of $\varrho_{\alpha,\beta}[F]$ and $\lambda_{\alpha,\beta}[F]$ we obtain the definition of the R-type $T_R[F]$ and the lower R-type $t_R[F]$. Similarly, the quantities

$$T_R[F]_G := \varlimsup_{\sigma\to+\infty} \frac{\exp\{M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]_G\sigma\}}, \quad t_R[F]_G := \varliminf_{\sigma\to+\infty} \frac{\exp\{M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]_G\sigma\}}$$

are called the R-type and the lower R-type of F with respect to G, respectively.

The following two statements cannot be derived from Theorems A and B, but are their analogues.

Proposition 1. *Suppose that the function G has regular growth. Then except for the cases when $T_R[F]=T_R[G]=0$ or $T_R[F]=T_R[G]=+\infty$, the inequality $T_R[F]_G\ge$ $\ge (T_R[F]/T_R[G])^{1/\varrho_R[G]}$ is true and subject to the condition of the strongly regular growth of G (i. e. $0<t_R[G]=T_R[G]<\infty$) this inequality converts into an equality, and except for the cases when $t_R[F]=t_R[G]=0$ or $t_R[F]=t_R[G]=+\infty$, the inequality $t_R[F]_G\le (t_R[F]/t_R[G])^{1/\varrho_R[G]}$ and subject to the condition of the strongly regular growth of G this inequality converts into an equality.*

Proof. Since G has the regular growth, from Corollary 1 we get $\varrho_R[F]_G=\varrho_R[F]/\varrho_R[G]$. Therefore,

$$T_R[F]_G=\varlimsup_{\sigma\to+\infty}\frac{\exp\{M_G^{-1}(M_F(\sigma))\}}{\exp\left\{\frac{\varrho_R[F]}{\varrho_R[G]}\sigma\right\}}=\varlimsup_{\sigma\to+\infty}\left(\frac{\exp\{\varrho_R[G]M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]\sigma\}}\right)^{1/\varrho_R[G]},$$

i.e.

$$\begin{aligned}(T_R[F]_G)^{\varrho_R[G]}&=\varlimsup_{x\to+\infty}\frac{\exp\{\varrho_R[G]M_G^{-1}(x)\}}{\exp\{\varrho_R[F]M_F^{-1}(x)\}}\ge\\ &\ge\varlimsup_{x\to+\infty}\frac{\ln x}{\exp\{\varrho_R[F]M_F^{-1}(x)\}}\varliminf_{x\to+\infty}\frac{\exp\{\varrho_R[G]M_G^{-1}(x)\}}{\ln x}=\\ &=\varlimsup_{\sigma\to+\infty}\frac{\ln M_F(\sigma)}{\exp\{\varrho_R[F]\sigma\}}\varliminf_{\sigma\to+\infty}\frac{\exp\{\varrho_R[G]\sigma\}}{\ln M_G(\sigma)}=\\ &=\frac{T_R[F]}{T_R[G]}.\end{aligned}$$

If G has the strongly regular growth then there exists $\lim\limits_{\sigma\to+\infty}\dfrac{\ln M_G(\sigma)}{\exp\{\varrho_R[G]\sigma\}}$ and as above we have

$$\begin{aligned}(T_R[F]_G)^{\varrho_R[G]}&=\varlimsup_{x\to+\infty}\frac{\exp\{\varrho_R[G]M_G^{-1}(x)\}}{\exp\{\varrho_R[F]M_F^{-1}(x)\}}=\\ &=\varlimsup_{x\to+\infty}\frac{\ln x}{\exp\{\varrho_R[F]M_F^{-1}(x)\}}\lim_{x\to+\infty}\frac{\exp\{\varrho_R[G]M_G^{-1}(x)\}}{\ln x}=\\ &=\frac{T_R[F]}{T_R[G]}.\end{aligned}$$

The first part of Proposition 1 is proved.

The proof of second part is similar. Indeed,

$$\begin{aligned}(t_R[F]_G)^{\varrho_R[G]}&=\varliminf_{x\to+\infty}\frac{\exp\{\varrho_R[G]M_G^{-1}(x)\}}{\exp\{\varrho_R[F]M_F^{-1}(x)\}}\le\\ &\le\varliminf_{x\to+\infty}\frac{\ln x}{\exp\{\varrho_R[F]M_F^{-1}(x)\}}\varlimsup_{x\to+\infty}\frac{\exp\{\varrho_R[G]M_G^{-1}(x)\}}{\ln x}=\end{aligned}$$

$$= \lim_{\sigma\to+\infty} \frac{\ln M_F(\sigma)}{\exp\{\varrho_R[F]\sigma\}} \overline{\lim_{\sigma\to+\infty}} \frac{\exp\{\varrho_R[G]\sigma\}}{\ln M_G(\sigma)} =$$
$$= \frac{t_R[F]}{t_R[G]}$$

and if there exists $\lim\limits_{\sigma\to+\infty} \dfrac{\ln M_G(\sigma)}{\exp\{\varrho_R[G]\sigma\}} = t_R[F] = T_R[F]$ then

$$(t_R[F]_G)^{\varrho_R[G]} = \lim_{x\to+\infty} \frac{\ln x}{\exp\{\varrho_R[F]M_F^{-1}(x)\}} \overline{\lim_{x\to+\infty}} \frac{\exp\{\varrho_R[G]M_G^{-1}(x)\}}{\ln x} = \frac{t_R[F]}{t_R[G]}.$$

□

To get the analogue of theorem B we use thefollowing lemma following which can be obtained from Lemma 1 in [4], if we will choose $\alpha(x) = x$ and $\beta(x) = e^{\varrho x}$ for $x \geq 1$.

Lemma 1. *Suppose that for entire Dirichlet series* (1) $\ln n = o(\lambda_n)$ *as* $n \to \infty$. *Then* $T_R[F] = \overline{\lim\limits_{n\to\infty}} \dfrac{\lambda_n}{e\varrho_R[F]} |f_n|^{\varrho_R[F]/\lambda_n}$, *and if, moreover,* $\lambda_{n+1} = (1 + o(1))\lambda_n$ *and* $\varkappa_n[F] \nearrow +\infty$ *as* $n_0 \leq n \to \infty$ *then* $t_R[F] = \lim\limits_{n\to\infty} \dfrac{\lambda_n}{e\varrho_R[F]} |f_n|^{\varrho_R[F]/\lambda_n}$.

Proposition 2. *Suppose that the exponents of entire Dirichlet series* (1) *and* (2) *satisfy the conditions* $\ln n = o(\lambda_n)$ *and* $\lambda_{n+1} = (1 + o(1))\lambda_n$ *as* $n \to \infty$.

If the function G has the strongly regular growth and $\varkappa_n[G] \nearrow +\infty$ *as* $n_0 \leq n \to \infty$ *then*

$$(T_R[F]_G)^{\varrho_R[G]} = k^* := \frac{\varrho_R[G]}{\varrho_R[F]} \overline{\lim_{n\to\infty}} \frac{|f_n|^{\varrho_R[F]/\lambda_n}}{|g_n|^{\varrho_R[G]/\lambda_n}} \tag{3}$$

except for the cases when $T_R[F] = T_R[G] = 0$ *or* $T_R[F] = T_R[G] = +\infty$.

If, moreover, the function F has the strongly regular growth and $\varkappa_n[F] \nearrow +\infty$ *as* $n_0 \leq n \to \infty$ *then*

$$(t_R[F]_G)^{\varrho_R[G]} = k_* := \frac{\varrho_R[G]}{\varrho_R[F]} \lim_{n\to\infty} \frac{|f_n|^{\varrho_R[F]/\lambda_n}}{|g_n|^{\varrho_R[G]/\lambda_n}} \tag{4}$$

except for the cases when $t_R[F] = t_R[G] = 0$ *or* $t_R[F] = t_R[G] = +\infty$.

Proof. By Proposition 1 and Lemma 1 we have

$$(T_R[F]_G)^{\varrho_R[F]_G} = \frac{T_R[F]}{T_R[G]} = \overline{\lim_{n\to\infty}} \frac{\lambda_n}{e\varrho_R[F]} |f_n|^{\varrho_R[F]/\lambda_n} \lim_{n\to\infty} \frac{e\varrho_R[G]}{\lambda_n |f_n|^{\varrho_R[G]/\lambda_n}} \leq$$

$$\leq \overline{\lim_{n\to\infty}} \frac{\lambda_n}{e\varrho_R[F]} |f_n|^{\varrho_R[F]/\lambda_n} \frac{e\varrho_R[G]}{\lambda_n |f_n|^{\varrho_R[G]/\lambda_n}} = \frac{\varrho_R[G]}{\varrho_R[F]} \overline{\lim_{n\to\infty}} \frac{|f_n|^{\varrho_R[F]/\lambda_n}}{|g_n|^{\varrho_R[G]/\lambda_n}} = k^*.$$

On the other hand, let $k^* > 0$. Then for every $\varepsilon \in (0, k^*)$ there exists an increasing to ∞ sequence (n_k) of integers such that

$$\frac{|f_{n_k}|^{\varrho_R[F]/\lambda_{n_k}}}{\varrho_R[F]} > (k^* - \varepsilon) \frac{|g_{n_k}|^{\varrho_R[G]/\lambda_{n_k}}}{\varrho_R[G]}$$

and, thus,

$$\varlimsup_{n\to\infty}\frac{\lambda_n}{e\varrho_R[F]}|f_n|^{\varrho_R[F]/\lambda_n}>(k^*-\varepsilon)\varliminf_{n\to\infty}\frac{\lambda_n|g_n|^{\varrho_R[G]/\lambda_n}}{e\varrho_R[G]}.$$

Since G has the strongly regular growth, that is $T_R[G]=t_R[G]$, hence by Lemma 1 we obtain the inequality $T_R[F]>(k^*-\varepsilon)T_R[G]$, i. e. in view of arbitrariness of ε the inequality $(T_R[F]_G)^{\varrho_R[G]}=\dfrac{T_R[F]}{T_R[G]}\ge k^*$ is true. For $k^*=0$ the last inequality is obvious. The first part of Proposition 2 is proved.

For the proof of the second part we remark that since the function G has the strongly regular growth, by Proposition 1 and Lemma 1

$$(t_R[F]_G)^{\varrho_R[F]_G}=\frac{t_R[F]}{t_R[G]}=\varliminf_{n\to\infty}\frac{\lambda_n}{e\varrho_R[F]}|f_n|^{\varrho_R[F]/\lambda_n}\varlimsup_{n\to\infty}\frac{e\varrho_R[G]}{\lambda_n|f_n|^{\varrho_R[G]/\lambda_n}}\ge$$

$$\ge\varliminf_{n\to\infty}\frac{\lambda_n}{e\varrho_R[F]}|f_n|^{\varrho_R[F]/\lambda_n}\frac{e\varrho_R[G]}{\lambda_n|f_n|^{\varrho_R[G]/\lambda_n}}=\frac{\varrho_R[G]}{\varrho_R[F]}\varliminf_{n\to\infty}\frac{|f_n|^{\varrho_R[F]/\lambda_n}}{|g_n|^{\varrho_R[G]/\lambda_n}}=k_*.$$

On the other hand, if $k_*<+\infty$ then for every $\varepsilon>0$ there exists an increasing to ∞ sequence (n_k) of integers such that

$$\frac{|f_{n_k}|^{\varrho_R[F]/\lambda_{n_k}}}{\varrho_R[F]}<(k_*+\varepsilon)\frac{|g_{n_k}|^{\varrho_R[G]/\lambda_{n_k}}}{\varrho_R[G]}$$

and, thus,

$$\varliminf_{n\to\infty}\frac{\lambda_n}{e\varrho_R[F]}|f_n|^{\varrho_R[F]/\lambda_n}<(k_*+\varepsilon)\varlimsup_{n\to\infty}\frac{e\varrho_R[G]}{\lambda_n|g_n|^{\varrho_R[G]/\lambda_n}},$$

whence by Lemma 1 it follows that $t_R[F]<(k_*+\varepsilon)t_R[G]$ and in view of arbitrariness of ε the inequality $(t_R[F]_G)^{\varrho_R[G]}=\dfrac{t_R[F]}{t_R[G]}\le k_*$ is true. The last inequality is trivial, if $k_*=+\infty$. The proof of Proposition 2 is completed. □

We remark that if $\varrho_R[F]_G=\varrho_R[F]/\varrho_R[G]$ then the R-type and the lower R-type of F with respect to G can be defined also by formulas

$$T_R^*[F]_G:=\varlimsup_{\sigma\to+\infty}\frac{\exp\{\varrho_R[G]M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]\sigma\}},\ t_R^*[F]_G:=\varliminf_{\sigma\to+\infty}\frac{\exp\{\varrho_R[G]M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]\sigma\}}.$$

Then in Proposition 1 the inequalities $T_R[F]_G\ge(T_R[F]/T_R[G])^{1/\varrho_R[G]}$ and $t_R[F]_G\le$ $\le(t_R[F]/t_R[G])^{1/\varrho_R[G]}$ can be replaced by the inequalities $T_R^*[F]_G\ge T_R[F]/T_R[G]$ and $t_R^*[F]_G\le t_R[F]/t_R[G]$, and in formulae (3) and (4) instead of $(T_R[F]_G)^{\varrho_R[G]}$ and $(t_R[F]_G)^{\varrho_R[G]}$ one can put $T_R^*[F]_G$ and $t_R^*[F]_G$.

3. Convergence classes

Generalizing the convergence class introduced by Valiron for entire functions, P. Kamthan [5] showed that if the sequence (λ_n) has a positive finite step, that is $0<h\le\lambda_{n+1}-\lambda_n\le H<\infty$ for $n\ge0$, and $\varkappa_n(F)\uparrow+\infty$ as $n\to\infty$ then in order that

$$\int_0^\infty\frac{\ln M_F(\sigma)}{e^{\varrho\sigma}}d\sigma<+\infty,\qquad(5)$$

it is necessary and sufficient that $\sum\limits_{n=1}^{\infty}|f_n|^{\varrho/\lambda_n}<+\infty$. Giving up a condition on the step of exponents in [6] it is well-proven that if $\ln n=O(\lambda_n)$ as $n\to\infty$ then in order that (5) holds, it is necessary and in the case when $\varkappa_n(F)\nearrow+\infty$ $(n\to\infty)$, it is sufficient that $\sum\limits_{n=1}^{\infty}(\lambda_n-\lambda_{n-1})|f_n|^{\varrho/\lambda_n}<+\infty$.

If $T_R[F]_G=0$, that is in view of the equality $T_R^*[F]_G=(T_R[F]_G)^{\varrho_R[F]_G}$ we get $T_R^*[F]_G=0$, for a characteristic of the growth of F we introduce a convergence class with respect to G by the condition

$$\int\limits_0^{\infty}\frac{\exp\{\varrho_R[G]M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]\sigma\}}d\sigma<+\infty.$$

Proposition 3. *If a function G has the positive lower R-type and the finite R-type then a function F belongs to the convergence class with respect to G if and only if F belongs to the convergence class defined by Kamthan.*

For the proof of this statement it is enough to use the estimations

$$0<t\le\le\frac{\ln M_G(\sigma)}{\exp\{\varrho_R[F]\sigma\}}\le T<+\infty.$$

As in [7], we say that an entire Dirichlet series (1) belongs to the generalized convergence $\alpha\beta$-class, if

$$\int\limits_{\sigma_0}^{\infty}\frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)}d\sigma<+\infty, \tag{6}$$

The following theorem is proved in [7].

Theorem C. *Let α be a concave on $[x_0,+\infty)$ function, $\alpha(e^x)\in L^0$ and a function $\beta\in L^0$ satisfies conditions $x\beta'(x)/\beta(x)\ge h>0$ for $x\ge x_0$ and $\int\limits_{x_0}^{\infty}\frac{\alpha(x)}{\beta(x)}dx<+\infty$. Suppose that $\ln n=o(\lambda_n\beta^{-1}(\alpha(\lambda_n)))$ as $n\to\infty$. Then in order that entire Dirichlet series (1) belongs to the generalized convergence $\alpha\beta$-class, it is necessary and in the case when $\varkappa_n(F)\nearrow+\infty$ $(n\to\infty)$, it is sufficient that*

$$\sum_{n=1}^{\infty}(\alpha(\lambda_n)-\alpha(\lambda_{n-1}))\beta_1\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)<+\infty,\quad \beta_1(x)=\int\limits_x^{\infty}\frac{d\sigma}{\beta(\sigma)}.$$

Finally, in view of the definition of $\varrho_{\alpha\beta}[F]$ by the condition $\varrho_{\alpha\beta}[F]=0$ we say that F belongs to a generalized convergence $\alpha\beta$-class with respect to G, if

$$\int\limits_{\sigma_0}^{\infty}\frac{\beta(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}d\sigma<+\infty.$$

The following statement is true.

Proposition 4. *If a function G has the positive generalized lower order and the finite generalized order then a function F belongs to a generalized convergence $\alpha\beta$-class with respect to G, if and only if F belongs to a generalized convergence $\alpha\beta$-class defined by condition* (6)*.*

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ЗАУВАЖЕННЯ ДО ВІДНОСНОГО ЗРОСТАННЯ ЦІЛИХ РЯДІВ ДІРІХЛЕ

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Нехай F і G — цілі функції, задані рядами Діріхле зі зростаючими до $+\infty$ показниками, а $\varrho_R[F]_G$ — R-порядок функції F стосовно функції G. Величини

$$T_R[F]_G := \varlimsup_{\sigma\to+\infty} \frac{\exp\{M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]_G\sigma\}}, \quad t_R[F]_G := \varliminf_{\sigma\to+\infty} \frac{\exp\{M_G^{-1}(M_F(\sigma))\}}{\exp\{\varrho_R[F]_G\sigma\}}$$

назвемо R-типом і нижнім R-типом функції F відносно функції G. Знайдено зв'язок між $T_R[F]_G$, $t_R[F]_G$ і R-типами та нижніми R-типами функцій F і G.

Ключові слова: ряд Діріхле, відносне зростання, клас збіжності.