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A FUNCTIONAL REPRESENTATION OF THE CAPACITY MULTIPLICATION MONAD

Dedicated to the 60th birthday of M. M. Zarichnyi

Taras RADUL

Ivan Franko National University of Lviv,
Universitetska Str., 1, 79000, Lviv, Ukraine
Casimirus the Great University of Bydgoszcz, Poland
e-mail: tarasradul@yahoo.co.uk

Functional representations of the capacity monad based on the max and min operations were considered in [10] and [7]. Nykyforchyn considered in [8] some alternative monad structure for the possibility capacity functor based on the max and usual multiplication operations. We show that such a capacity monad (which we call the capacity multiplication monad) has a functional representation, i.e. the space of capacities on a compactum X can be naturally embedded (with preservation of the monad structure) in some space of functionals on $C(X, I)$. We also describe this space of functionals in terms of properties of functionals.

Key words: Banach space, locally convex space, approximation, Schrödinger operator

1. Introduction

Functional representations of monads (i.e. natural embeddings into $\mathbb{R}^{C(X,S)}$ which preserves a monad structure where S is a subset of $\mathbb{R}$) were considered in [11] and [12]. Some functional representations of hyperspace monad were constructed in [13] and [14].

Capacities (non-additive measures, fuzzy measures) were introduced by Choquet in [1] as a natural generalization of additive measures. They found numerous applications (see for example [2],[4],[16]). Categorical and topological properties of spaces of upper-semicontinuous capacities on compact Hausdorff spaces were investigated in [9].

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In particular, the capacity functor was constructed. This functor is a functorial part of a capacity monad $\mathbb{M}$ based on the max and min operations.

The space of capacities MX can be naturally embedded in $\mathbb{R}^{C(X)}$ by means of the Choquet integral. In other words, the Choquet integral provides some functional representation of the functor M. However, this representation does not preserve the monad structure. Nykyforchyn using the Sugeno integral provided a functional representation of capacities as functionals on the space $C(X,I)$ which preserves the monad structure [7]. Some modification of the Sugeno integral yields a functional representation of capacities as functionals on the space $C(X)$ [10].

Let us remark that the min operation is a triangular norm on the unit interval I. Another important triangular norm is the multiplication operation. Nykyforchyn constructed a capacity monad based on the max and multiplication operations [8]. (Let us remark that recently Zarichnyi proposed to use triangular norms to construct monads [20]). The main aim of this paper is to find a representation of the monad from [8]. We use a fuzzy integral based on the max and multiplication operations for this purpose.

2. Capacities and monads

By Comp we denote the category of compact Hausdorff spaces (compacta) and continuous maps. For each compactum X we denote by $C(X)$ the Banach space of all continuous functions $\phi : X \to \mathbb{R}$ with the usual sup-norm: $\|\phi\| = \sup\{|\phi(x)| \mid x \in X\}$. We also consider on $C(X)$ the natural partial order.

In what follows, all spaces and maps are assumed to be in Comp except for $\mathbb{R}$, the spaces $C(X)$ and functionals defined on $C(X)$ with X compact Hausdorff.

We recall some categorical notions (see [15] and [17] for more details). We define them only for the category Comp. The central notion is the notion of monad (or triple) in the sense of S.Eilenberg and J.Moore.

A *monad* [3] $\mathbb{T} = (T,\eta,\mu)$ in the category Comp consists of an endofunctor $T : \mathsf{Comp} \to \mathsf{Comp}$ and natural transformations $\eta : \mathrm{Id}_{\mathsf{Comp}} \to T$ (unity), $\mu : T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$. (By $\mathrm{Id}_{\mathsf{Comp}}$ we denote the identity functor on the category Comp and T^2 is the superposition $T \circ T$ of T.)

Let $\mathbb{T} = (T,\eta,\mu)$ be a monad in the category Comp. The pair (X,ξ) where $\xi : TX \to X$ is a map is called a $\mathbb{T}$-*algebra* if $\xi \circ \eta X = id_X$ and $\xi \circ \mu X = \xi \circ T\xi$. Let (X,ξ), (Y,ξ') be two $\mathbb{T}$-algebras. A map $f : X \to Y$ is called a morphism of $\mathbb{T}$-algebras if $\xi' \circ Tf = f \circ \xi$.

A natural transformation $\psi : T \to T'$ is called a *morphism* from a monad $\mathbb{T} = (T,\eta,\mu)$ into a monad $\mathbb{T}' = (T',\eta',\mu')$ if $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ \eta T' \circ T\psi$. If all of the components of ψ are monomorphisms then the monad $\mathbb{T}$ is called a *submonad* of $\mathbb{T}'$ and ψ is called a *monad embedding.*

Let A be a subset of X. By $\mathcal{F}(X)$ we denote the family of all closed subsets of X. Put $I = [0,1]$.

We follow a terminology from [9]. A function $\nu : \mathcal{F}(X) \to I$ is called an *upper-semicontinuous capacity* on X if the following three properties hold for each closed subsets F and G of X:

(1) $\nu(X) = 1$, $\nu(\emptyset) = 0$,
(2) if $F \subset G$, then $\nu(F) \le \nu(G)$,

(3) if $\nu(F) < a$, then there exists an open set $O \supset F$ such that $\nu(B) < a$ for each compactum $B \subset O$.

A capacity ν is extended in [9] to all open subsets $U \subset X$ by the formula

$$\nu(U) = \sup\{\nu(K) \mid K \text{ is a closed subset of } X \text{ such that } K \subset U\}.$$

It was proved in [9] that the space MX of all upper-semicontinuous capacities on a compactum X is a compactum as well, if a topology on MX is defined by a subbase that consists of all sets of the form $O_-(F,a) = \{c \in MX \mid c(F) < a\}$, where F is a closed subset of X, $a \in [0,1]$, and $O_+(U,a) = \{c \in MX \mid c(U) > a\}$, where U is an open subset of X, $a \in [0,1]$. Since all capacities under consideration here are upper-semicontinuous, in the following we call the elements of MX simply capacities.

A capacity $\nu \in MX$ for a compactum X is called a necessity (possibility) capacity if for each family $\{A_t\}_{t\in T}$ of closed subsets of X (such that $\bigcup_{t\in T} A_t$ is a closed subset of X) we have $\nu(\bigcap_{t\in T} A_t) = \inf_{t\in T} \nu(A_t)$ $(\nu(\bigcup_{t\in T} A_t) = \sup_{t\in T} \nu(A_t))$. (See [19] for more details.) We denote by $M_\cap X$ $(M_\cup X)$ the subspace of MX consisting of all necessity (possibility) capacities. Since X is compact and ν is upper-semicontinuous, $\nu \in M_\cap X$ if and only if ν satisfies the simpler requirement that $\nu(A \cap B) = \min\{\nu(A), \nu(B)\}$.

If ν is a capacity on a compactum X, then the function $\kappa X(\nu)$ defined on the family $\mathcal{F}(X)$ by the formula $\kappa X(\nu)(F) = 1 - \nu(X \setminus F)$, is a capacity as well. It is called the dual capacity (or conjugate capacity) to ν. The mapping $\kappa X : MX \to MX$ is a homeomorphism and an involution [9]. Moreover, ν is a necessity capacity if and only if $\kappa X(\nu)$ is a possibility capacity. This implies in particular that $\nu \in M_\cup X$ if and only if ν satisfies the simpler requirement that $\nu(A \cup B) = \max\{\nu(A), \nu(B)\}$. It is easy to check that $M_\cap X$ and $M_\cup X$ are closed subsets of MX.

The assignment M extends to the capacity functor M in the category of compacta, if the map $Mf : MX \to MY$ for a continuous map of compacta $f : X \to Y$ is defined by the formula $Mf(c)(F) = c(f^{-1}(F))$ where $c \in MX$ and F is a closed subset of X. This functor was completed to the monad $\mathbb{M} = (M, \eta, \mu)$ [9], where the components of the natural transformations are defined as follows: $\eta X(x)(F) = 1$ if $x \in F$ and $\eta X(x)(F) = 0$ if $x \notin F$;

$$\mu X(\mathcal{C})(F) = \sup\{t \in [0,1] \mid \mathcal{C}(\{c \in MX \mid c(F) \geq t\}) \geq t\},$$

where $x \in X$, F is a closed subset of X and $\mathcal{C} \in M^2(X)$ (see [9] for more details).

It was shown in [5] that $M_\cup$ and $M_\cap$ are subfunctors of M and if we take the corresponding restrictions of the functions μX, we obtain submonads $\mathbb{M}_\cup$ and $\mathbb{M}_\cap$ of the monad $\mathbb{M}$.

The semicontinuity of capacities yields that we can change sup for max in the definition of the map μX. More precisely, existing of max follows from Lemma 3.7 [9]. For a closed set $F \subset X$ and for $t \in I$ put $F_t = \{c \in MX \mid c(F) \geq t\}$. We can rewrite the definition of the map μX as follows

$$\mu X(\mathcal{C})(F) = \max\{\mathcal{C}(F_t) \wedge t \mid t \in (0,1]\}.$$

Let us remark that the operation $\wedge$ is a triangular norm. It seems natural to consider another triangular norm instead of $\wedge$. Define the map $\mu^\bullet X : M^2X \to MX$ by the formula

$$\mu^\bullet X(\mathcal{C})(F) = \max\{\mathcal{C}(F_t) \cdot t \mid t \in (0,1]\}.$$

(Existence of max also follows from Lemma 3.7 [9].)

Proposition 1. *The natural transformation $\mu^\bullet$ does not satisfy the property $\mu^\bullet\circ\mu^\bullet M=\mu^\bullet\circ M\mu^\bullet$.*

Proof. Consider $X=\{a,b\}$, where $\{a,b\}$ is a two-point discrete space. Define $\mathcal{A}_1\in M^2X$ as follows $\mathcal{A}_1(\alpha)=1$ if and only if $\alpha\supset\{a\}_{\frac12}$ and $\mathcal{A}_1(\alpha)=0$ otherwise for $\alpha\in\mathcal{F}(MX)$. Define $\mathcal{A}_2\in M^2X$ as follows $\mathcal{A}_2(\alpha)=1$ if and only if $\alpha=MX$, $\mathcal{A}_2(\alpha)=\frac12$ if and only if $\alpha\supset\{a\}_1$ and $\mathcal{A}_1(\alpha)=0$ otherwise for $\alpha\in\mathcal{F}(MX)$. Now, define $\daleth\in M^3(X)$ by the formula
$$\daleth(\Lambda)=\frac12\eta M^2X(\mathcal{A}_1)(\Lambda)+\frac12\eta M^2X(\mathcal{A}_2)(\Lambda)$$
for $\Lambda\in\mathcal{F}(M^2X)$.

We have
$$\mu^\bullet X\circ M(\mu^\bullet X)(\daleth)(\{a\})=\max\{\daleth((\mu^\bullet X)^{-1}(\{a\}_t))\cdot t\mid t\in(0,1]\}.$$
It is easy to see that $\mu^\bullet X(\mathcal{A}_1)(\{a\})=\mu^\bullet X(\mathcal{A}_2)(\{a\})=\frac12$. Then $\daleth((\mu^\bullet X)^{-1}(\{a\}_{\frac12}))\cdot\frac12=1\cdot\frac12=\frac12$. Hence we obtain $\mu^\bullet X\circ\mu^\bullet MX(\daleth)(\{a\})\ge\frac12$.

On the other hand
$$\begin{aligned}\mu^\bullet X\circ\mu^\bullet MX(\daleth)(\{a\})&=\max\{\mu^\bullet MX(\daleth)(\{a\}_t))\cdot t\mid t\in(0,1]\}=\\&=\max\{\max\{\daleth((\{a\}_t)_s)\cdot s\mid s\in(0,1]\}\cdot t\mid t\in(0,1]\}.\end{aligned}$$
The function $\delta(s,t)=\daleth((\{a\}_t)_s)$ is nonincreasing on both variables. We have $\delta(s,t)=0$ for each (s,t) such that $s>\frac12$ and $t>\frac12$. Moreover $\delta(1,\frac12)=\delta(\frac12,1)=\frac12$. Hence
$$\mu^\bullet X\circ\mu^\bullet MX(\daleth)(\{a\})=\max\{\max\{\daleth((\{a\}_t)_s)\cdot s\mid s\in(0,1]\}\cdot t\mid t\in(0,1]\}=\frac14.$$
□

Remark 1. Since the triple $\mathbb{M}^\bullet=(M,\eta,\mu^\bullet)$ does not form a monad, the problem of uniqueness of the monad $\mathbb{M}$ stated in [9] is still open.

But things may turn out differently if we restrict the map $\mu^\bullet X$ to the set $M_\cup(M_\cup X)\subset M(MX)$. It is easy to see that for such restriction we can consider the sets A_t in the definition of the map $\mu^\bullet X$ as subsets of $M_\cup X$. It was deduced from some general facts that the triple $\mathbb{M}^\bullet_\cup=(M_\cup,\eta,\mu^\bullet)$ is a monad [8]. For the sake of completeness we give here a direct proof.

Lemma 1. *We have $\mu^\bullet X(M_\cup(M_\cup X))\subset M_\cup X$ for each compactum X.*

Proof. Consider any $\mathcal{A}\in M_\cup(M_\cup X)$ and B, $C\in\mathcal{F}(X)$. Since B_t and C_t are subsets of $M_\cup X$, we have $(C\cup B)_t=C_t\cup B_t$. Then
$$\begin{aligned}\mu^\bullet X(\mathcal{A})(B\cup C)&=\max\{\mathcal{A}((C\cup B)_t)\cdot t\mid t\in(0,1]\}=\\&=\max\{\mathcal{A}(C_t\cup B_t)\cdot t\mid t\in(0,1]\}=\\&=\max\{\max\{\mathcal{A}(C_t)\cdot t\mid t\in(0,1]\},\max\{\mathcal{A}(B_t)\cdot t\mid t\in(0,1]\}=\\&=\max\{\mu^\bullet X(\mathcal{A})(B),\mu^\bullet X(\mathcal{A})(C)\}.\end{aligned}$$
□

We will use the notation $\mu^\bullet X$ also for the restriction $\mu^\bullet X|_{M^2_\cup X}$.

Theorem 1. *The triple* $\mathbb{M}^\bullet_\cup = (M_\cup, \eta, \mu^\bullet)$ *is a monad.*

Proof. It is easy to check that η and $\mu^\bullet$ are well-defined natural transformations of the corresponding functors. Let us check two monad properties.

Take any compactum X, $\nu \in M_\cup X$ and $A \in \mathcal{F}(X)$. Then we have

$$\begin{aligned}\mu^\bullet X \circ \eta M_\cup X(\nu)(A) &= \max\{\eta \mathbb{M}_\cup X(\nu)(A_t)\cdot t \mid t \in (0,1]\} = \\ &= \nu(A) \text{ and } \mu^\bullet X \circ M_\cup(\eta X)(\nu)(A) = \\ &= \max\{M_\cup(\eta X)(\nu)(A_t)\cdot t \mid t \in (0,1]\} = \\ &= \max\{\nu((\eta X)^{-1}(A_t))\cdot t \mid t \in (0,1]\} = \\ &= \max\{\nu(A)\cdot t \mid t \in (0,1]\} = \nu(A).\end{aligned}$$

We obtain the equality $\mu^\bullet \circ M_\cup \eta = \mu^\bullet \circ \eta M_\cup = \mathbf{1}_{M_\cup}$.

Now, consider any $\gimel \in M^3_\cup(X)$ and $A \in \mathcal{F}(X)$. Put

$$a = \mu^\bullet X \circ M_\cup(\mu^\bullet X)(\gimel)(A) = \max\{\gimel((\mu^\bullet X)^{-1}(A_t))\cdot t \mid t \in (0,1]\}$$

and

$$\begin{aligned}b &= \mu^\bullet X \circ \mu^\bullet M_\cup X(\gimel)(\{a\}) = \\ &= \max\{\mu^\bullet M_\cup X(\gimel)(A_t))\cdot t \mid t \in (0,1]\} = \\ &= \max\{\max\{\gimel((A_t)_s)\cdot s \mid s \in (0,1]\}\cdot t \mid t \in (0,1]\}.\end{aligned}$$

There exists $t_0 \in (0,1]$ such that $a = \gimel((\mu^\bullet X)^{-1}(A_{t_0}))\cdot t_0$. We have

$$\begin{aligned}(\mu^\bullet X)^{-1}(A_{t_0}) &= \left\{\mathcal{A} \in M^2_\cup(X) \mid \mu^\bullet X(\mathcal{A}) \ge t_0\right\} = \\ &= \left\{\mathcal{A} \in M^2_\cup(X) \mid \text{ there exists } c \in (0,1] \text{ such that } \mathcal{A}(A_c)\cdot c \ge t_0\right\} = \\ &= \left\{\mathcal{A} \in M^2_\cup(X) \mid \text{ there exists } c \in (0,1] \text{ such that } \mathcal{A}(A_c) \ge \frac{t_0}{c}\right\}.\end{aligned}$$

Since $\gimel$ is a possibility capacity, there exists $\mathcal{A}_0 \in M^2_\cup(X)$ and $c_0 \in (0,1]$ such that $\mathcal{A}_0(A_{c_0}) \ge \frac{t_0}{c_0}$ and $\gimel((\mu^\bullet X)^{-1}(A_{t_0})) = \gimel(\{\mathcal{A}_0\})$. But then we have

$$a \le \gimel((A_{c_0})_{\frac{t_0}{c_0}})\cdot t_0 = \gimel((A_{c_0})_{\frac{t_0}{c_0}})\cdot \frac{t_0}{c_0}\cdot c_0 \le b.$$

On the other hand choose p_0, $z_0 \in (0,1]$ such that $b = \gimel((A_{p_0})_{z_0})\cdot p_0\cdot z_0$. Since $\gimel$ is a possibility capacity, there exists $\mathcal{B}_0 \in (A_{p_0})_{z_0}$ such that $\gimel((A_{p_0})_{z_0}) = \gimel(\{\mathcal{B}_0\})$. We have $\mathcal{B}_0(A_{p_0}) \ge z_0$, hence $\mu^\bullet X(\mathcal{B}_0)(A) \ge z_0\cdot p_0$. Then we obtain

$$b = \gimel(\{\mathcal{B}_0\})\cdot p_0\cdot z_0 \le \gimel((\mu^\bullet X)^{-1}(A_{p_0\cdot z_0}))\cdot p_0\cdot z_0 \le a.$$

□

3. Functional representation of the monad $\mathbb{M}^\bullet_\cup$

A monad $\mathcal{F} = (F, \eta, \mu)$ is called an *IL-monad* if there exists a map $\xi : FI \to I$ such that the pair (I, ξ) is an $\mathcal{F}$-algebra and for each $X \in \mathsf{Comp}$ there exists a point-separating family of F-algebras morphisms $\{f_\alpha : (FX, \mu X) \to (I, \xi) \mid \alpha \in A\}$ [12].

There was defined a monad $\mathbb{V}_I$ in [12], which is universal in the class of IL-monads. By $V_I X$ we denote the power $I^{C(X,I)}$. For a map $\phi \in C(X, I)$ we denote by π_ϕ or $\pi(\phi)$

the corresponding projection $\pi_\phi : V_I X \to I$. For each map $f : X \to Y$ we define the map $V_I f : V_I X \to V_I Y$ by the formula $\pi_\phi \circ V_I f = \pi_{\phi\circ f}$ for $\phi \in C(Y, I)$. For a compactum X, we define components hX and mX of natural transformations by $\pi_\phi \circ hX = \phi$ and $\pi_\phi \circ mX = \pi(\pi_\phi)$ for all $\phi \in C(X, I))$. The triple $\mathbb{V}_I = (V_I, h, m)$ forms a monad in the category Comp and for each monad $\mathcal{F}$ there exists a monad embedding $l : \mathcal{F} \to \mathbb{V}_I$ if and only if $\mathcal{F}$ is IL-monad [12]. Moreover, for a compactum X the map $lX : FX \to V_I X$ is defined by the conditions $\pi_\phi \circ lX = \xi \circ F\phi$ for each $\psi \in C(X, I)$.

Theorem 2. *The monad $\mathbb{M}^\bullet_\cup$ is an IL-monad.*

Proof. Define the map $\xi : M_\cup I \to I$ by the formula $\xi(\nu) = \max\{\nu([t, 1] \cdot t \mid t \in (0, 1]\}$. We can check that the pair (I, ξ) is an $\mathbb{M}^\bullet_\cup$-algebra by the same but simpler arguments as in the proof of Theorem 1.

Consider any compactum X and two distinct capacities ν, $\beta \in M_\cup X$. Then there exists $A \in \mathcal{F}(X)$ such that $\nu(A) \neq \beta(A)$. We can suppose that $\nu(A) < \beta(A)$. Since ν and β are possibility capacities, there exist a, $b \in A$ such that $\nu(\{a\}) = \nu(A)$ and $\beta(\{b\}) = \beta(A)$. Choose a point $t \in (\nu(A), \beta(A))$. Put $B = \{x \in X \mid \nu(\{x\}) \geq t\}$. Since ν is a possibility capacity and $\nu(X) = 1$, B is not empty. Since ν is upper semicontinuous, B is closed. Evidently, $B \cap A = \emptyset$. Choose a function $\varphi \in C(X, I)$ such that $\varphi(B) \subset \{0\}$ and $\varphi(A) \subset \{1\}$. Then

$$\begin{aligned}\pi_\varphi \circ lX(\nu) = \xi \circ M_\cup\varphi(\nu) = \\ &= \max\{M_\cup\varphi(\nu)([s, 1] \cdot s \mid s \in (0, 1]\} = \\ &= \max\{\nu(\varphi^{-1}[s, 1]) \cdot s \mid s \in (0, 1]\} \leq \\ &\leq t < \beta(A) \leq \beta(\varphi^{-1}\{1\}) \cdot 1 \leq \\ &\leq \pi_\varphi \circ lX(\beta)\end{aligned}$$

It is easy to check that

$$\pi_\phi \circ lX = \xi \circ \mathbb{M}_\cup\phi : \mathbb{M}_\cup X \to I$$

is a morphism of $\mathbb{M}^\bullet_\cup$-algebras . □

Hence we obtain a monad embedding $l : \mathbb{M}^\bullet_\cup \to \mathbb{V}_I$ such that

$$\pi_\varphi \circ lX(\nu) = \max\{\nu(\varphi^{-1}[s, 1]) \cdot s \mid s \in (0, 1]\}$$

for each compactum X, $\nu \in M_\cup X$ and $\varphi \in C(X, I)$.

Let X be any compactum. For any $c \in I$ we will denote by c_X the constant function on X taking the value c. Following the notations of idempotent mathematics (see e.g., [6]) we use the notation $\oplus$ in I and $C(X, I)$ as an alternative for max. We will use the notation $\nu(\varphi) = \pi_\varphi \circ lX(\nu)$ for $\nu \in V_I X$ and $\varphi \in C(X, I)$.

Consider the subset $SX \subset V_I X$ consisting of all functionals ν satisfying the following conditions

(1) $\nu(1_X) = 1$;
(2) $\nu(\lambda \cdot \varphi) = \lambda \cdot \nu(\varphi)$ for each $\lambda \in I$ and $\varphi \in C(X, I)$;
(3) $\nu(\psi \oplus \varphi) = \nu(\psi) \oplus \nu(\varphi)$ for each ψ, $\varphi \in C(X, I)$.

Let us remark that properties 1 and 2 yield that $\nu(c_X) = c$ for each $\nu \in SX$ and $c \in I$.

Theorem 3. $lX(M_\cup X) = SX$.

Proof. Consider any $\nu \in M_\cup X$. Put $\upsilon = lX(\nu)$. Then we have

$$\upsilon(1_X) = \max\{\nu((1_X)^{-1}[s,1]) \cdot s \mid s \in (0,1]\} = \max\{\nu(X) \cdot s \mid s \in (0,1]\} = 1.$$

Take any $c \in I$ and $\varphi \in C(X,I)$. For $c = 0$, Property 2 is trivial. For $c > 0$ we have

$$\begin{aligned}\upsilon(c\varphi) &= \max\left\{\nu((c\varphi)^{-1}[s,1]) \cdot s \mid s \in (0,1]\right\} = \\ &= \max\left\{\nu(\varphi^{-1}\left[\tfrac{s}{c},1\right]) \cdot \tfrac{s}{c} \mid s \in (0,1]\right\} \cdot c = \\ &= c \cdot \upsilon(\varphi).\end{aligned}$$

Consider any ψ and $\varphi \in C(X,I)$. We have

$$\begin{aligned}\upsilon(\psi \oplus \varphi) &= \max\left\{\nu((\psi \oplus \varphi)^{-1}[s,1]) \cdot s \mid s \in (0,1]\right\} = \\ &= \max\left\{\nu(\psi^{-1}[s,1] \cup \varphi^{-1}[s,1]) \cdot s \mid s \in (0,1]\right\} = \\ &= \max\left\{(\nu(\psi^{-1}[s,1]) \oplus \nu(\varphi^{-1}[s,1])) \cdot s \mid s \in (0,1]\right\} = \\ &= \upsilon(\psi) \oplus \upsilon(\varphi).\end{aligned}$$

We obtained $lX(M_\cup X) \subset SX$.

Take any $\upsilon \in SX$. For $A \in \mathcal{F}(X)$ put

$$\Upsilon_A = \{\varphi \in C(X,I) \mid \varphi(a) = 1 \text{ for each } a \in A\}.$$

Define $\nu : \mathcal{F}(X) \to I$ as follows $\nu(A) = \inf\{\upsilon(\varphi) \mid \varphi \in \Upsilon_A\}$ if $A \neq \emptyset$ and $\nu(\emptyset) = 0$. It is easy to see that ν satisfies Conditions 1 and 2 from the definition of capacity.

Let $\nu(A) < \eta$ for some $\eta \in I$ and $A \in \mathcal{F}(X)$. Then there exists $\varphi \in \Upsilon_A$ such that $\upsilon(\varphi) = \chi < \eta$. Choose $\varepsilon > 0$ such that $(1+\varepsilon)\chi < \eta$. Put $\delta = \frac{1}{1+\varepsilon}$ and $\psi = \min\{\delta_X, \varphi\}$. Then $\upsilon(\psi) \le \upsilon(\varphi) = \chi$ and $\upsilon((1+\varepsilon)\psi) \le (1+\varepsilon)\chi < \eta$. Put $U = \varphi^{-1}(\delta, 1]$. Evidently, U is an open set and $U \supset A$. But for each compact $K \subset U$ we have $(1+\varepsilon)\psi \in \Upsilon_K$. Hence $\nu(K) < \eta$.

Finally take any A, $B \in \mathcal{F}(X)$. Evidently, $\nu(A \cup B) \ge \nu(A) \oplus \nu(B)$. Suppose that $\nu(A \cup B) > \nu(A) \oplus \nu(B)$. Then there exists $\varphi \in \Upsilon_A$ and $\psi \in \Upsilon_B$ such that $\nu(A \cup B) > \upsilon(\varphi) \oplus \upsilon(\psi) = \upsilon(\varphi \oplus \psi)$. However, $\varphi \oplus \psi \in \Upsilon_{A \cup B}$ and we obtain a contradiction. Hence $\nu \in M_\cup X$.

Let us show that $lX(\nu) = \upsilon$. Take any $\varphi \in C(X,I)$. Denote $\varphi_t = \varphi^{-1}[t,1]$. Then

$$\begin{aligned}lX(\nu)(\varphi) &= \max\left\{\inf\{\upsilon(\chi) \mid \chi \in \Upsilon_{\varphi_t}\} \cdot t \mid t \in (0,1]\right\} = \\ &= \max\left\{\inf\{\upsilon(t\chi) \mid \chi \in \Upsilon_{\varphi_t}\} \mid t \in (0,1]\right\}.\end{aligned}$$

For each $t \in (0,1]$ put $\chi_t = \min\{\frac{1}{t}\varphi, 1_X\} \in \Upsilon_{\varphi_t}$. We have $t\chi \le \varphi$, hence $\upsilon(t\chi) \le \upsilon(\varphi)$. Then we have $\inf\{\upsilon(t\chi) \mid \chi \in \Upsilon_{\varphi_t}\} \le \upsilon(\varphi)$ for each $t \in (0,1]$, hence $lX(\nu)(\varphi) \le \upsilon(\varphi)$.

Suppose that $lX(\nu)(\varphi) < \upsilon(\varphi)$. Choose any $a \in (lX(\nu)(\varphi), \upsilon(\varphi))$. Then for each $t \in (0,1]$ there exists $\chi_t \in \Upsilon_{\varphi_t}$ such that $\upsilon(t\chi_t) < a$. Choose $\varepsilon > 0$ such that $(1+\varepsilon)a < \upsilon(\varphi)$. Put $\delta = \frac{1}{1+\varepsilon}$. Choose $n \in \mathbb{N}$ such that $\delta^n < \upsilon(\varphi)$. Put $\psi_{n+1} = \delta^n_X$ and $\psi_i = \delta^{i-1}\chi_{\delta^i}$ for $i \in \{1, \ldots, n\}$. We have $\upsilon(\psi_i) < \upsilon(\varphi)$ for each $i \in \{1, \ldots, n+1\}$. Put $\psi = \bigoplus_{i=1}^{n+1} \psi_i$. Then $\upsilon(\psi) = \bigoplus_{i=1}^{n+1} \upsilon(\psi_i) < \upsilon(\varphi)$. On the other hand $\varphi \le \psi$ and we obtain a contradiction. $\square$

Hence we obtain, in fact, that the monad $\mathbb{M}^{\bullet}_{\cup}$ is isomorphic to a submonad of $\mathbb{V}_I$ with functorial part acting on compactum X as SX. Let us remark that this monad is one of monads generated by t-norms considered by Zarichnyi [20]. Thus the following question seems to be natural: can we generalize the results of this paper to any continuous t-norms?

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ФУНКЦІОНАЛЬНЕ ЗОБРАЖЕННЯ МОНАДИ ЄМНОСТЕЙ НА ОСНОВІ МНОЖЕННЯ

Тарас РАДУЛ

Львівський національний університет імені Івана Франка,
вул. Університетська 1, 79000, Львів
and Casimirus the Great University of Bydgoszcz, Poland
e-mail: tarasradul@yahoo.co.uk

Функціональне зображення монади ємностей розглядалось в [10] і [7]. Никифорчин розглянув в [8] альтернативну структуру монади для певного підфунктора функтора ємностей, базовану на операціях максимуму та звиклого множення. Ми показуємо, що ця монада має функціональне зображення, тобто простір ємностей на компакті X може бути природно вкладеним (зі збереженням структури монади) в деякий простір функціоналів на $C(X, I)$. Ми також описуємо цей простір функціоналів в термінах властивостей функціоналів.

Ключові слова: монада, ємність, нечіткий інтеграл, трикутна норма.