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SIMULTANEOUS APPROXIMATION OF VALUES OF WEIERSTRASS AND JACOBI ELLIPTIC FUNCTIONS IN THE PERIODS AND ALGEBRAIC POINT

Dedicated to the 60th birthday of M. M. Zarichnyi

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Let $\wp(z)$, $\operatorname{sn} z$ be algebraically independent Weierstrass and Jacobi elliptic functions with algebraic invariants and algebraic elliptic module, $(2\omega_1,\ 2\omega_3)$ and $(4K, 2iK')$ be the main periods of $\wp(z)$ and $\operatorname{sn} z$ respectively, α be an algebraic number different from the poles of $\wp(z)$ and $\operatorname{sn} z$. We estimate from below the simultaneous approximation of $\operatorname{sn}(2\omega_1), \operatorname{sn}(\alpha), \wp(4K)$, and $\wp(\alpha)$.

Key words: simultaneous approximation, Weierstrass elliptic function, Jacobi elliptic function.

1. Introduction

Let $\wp(z)$ and $\operatorname{sn} z$ be the elliptic Weierstrass function and the elliptic Jacobi function, respectively. Then $\wp(z)$ satisfies the equation $(\wp'(z))^2 = 4\wp^3(z)$-$g_2\wp(z) - g_3$, the numbers g_2, g_3 are called invariants of $\wp(z)$, $2\omega_1$, $2\omega_3$ is a fixed pair of the main periods $\wp(z)$. The function $\operatorname{sn} z$ satisfies the equation $(\operatorname{sn}' z)^2 = (1 - \operatorname{sn}^2 z)(1 - \varkappa^2 \operatorname{sn}^2 z)$. The number $\varkappa$ is called the elliptic module of $\operatorname{sn} z$, $0 < \varkappa < 1$, the number $\varkappa' = (1 - \varkappa^2)^{1/2}$ is called its additional elliptic module. A pair of main the periods $\operatorname{sn} z$ is $(4K,\ 2iK')$, where $K,\ K'$ are complete elliptic integrals of the first kind corresponding to $\varkappa$, $\varkappa'$ ([1]). In the present article we will consider algebraically independent elliptic functions $\wp(z)$ and $\operatorname{sn} z$ with

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algebraic g_2, g_3 and $\varkappa$. Let the periods $2\omega_1$, $4K$ form a lattice, $2m\omega_1$ and $4mK$ ($m \in \mathbb{Z}$) are different from the poles of $\operatorname{sn} z$ and $\wp(z)$.

For $d(P)$, $L(P)$ denote the degree and the length of a polynomial P with integer coefficients, $d(\alpha)$, $L(\alpha)$ is the degree and the length of algebraic number α [2], α is different from the poles of $\wp(z)$ and $\operatorname{sn} z$. Let ξ_i be approximating algebraic numbers, $n_i = d(\xi_i)$ and $L_i = L(\xi_i)$ be their powers and lengths respectively ($i = 1, \dots, 4$), $n = \deg \mathbb{Q}(g_2, g_3, \varkappa, \alpha, \xi_1, \dots, \xi_4)$.

Theorem 1. *For any algebraic numbers $\xi_1, \dots, \xi_4$ the following inequality holds:*

$$|\wp(4K) - \xi_1| + |\operatorname{sn}(2\omega_1) - \xi_2| + |\wp(\alpha) - \xi_3| + |\operatorname{sn}(\alpha) - \xi_4| > \exp\left(-\Lambda\, n^3 T^2\right), \tag{1}$$

where

$$T = \max\left(\frac{\ln L_1}{n_1} + \dots + \frac{\ln L_4}{n_4} + 1, \ln n\right), \tag{2}$$

$\Lambda > 0$ is a constant that depends only on $g_2, g_3, \varkappa$ and α.

Similar estimates for other numbers can be found in [2]–[6].

2. Auxiliary statements

In the following lemma $c_{10}, \dots, c_{15}$ stand for positive constants that are independent of n, n_i, L_i and λ.

Lemma 1 ([1]). *If z, w, $z + w$ are admissible values, then*

$$\wp(z + w) = \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2 - \wp(z) - \wp(w), \quad \operatorname{sn}(z + w) = \frac{\operatorname{sn} z \operatorname{sn}' w + \operatorname{sn} w \operatorname{sn}' z}{1 - \varkappa^2 \operatorname{sn} z^2 \operatorname{sn} w^2}.$$

Lemma 2. *For each integer $m \geqslant 1$, there exist polynomials $P_{1,s,l}$, $P_{2,s,l}$ with the integer coefficients such that*

$$\frac{d^s}{d\,z^s}((\wp(z)^l) = P_{1,s,l}(g_2, g_3, \wp(z), \wp'(z)), \quad \frac{d^s}{d\,z^s}((\operatorname{sn} z)^l) = P_{2,s,l}(\varkappa^2, \operatorname{sn} z, \operatorname{sn}' z),$$

$$\deg P_{i,s,l} \leqslant c_1(s + l), \quad L(P_{i,s,l}) \leqslant \exp(c_2 s \log(s + l)), \quad i = 1,\, 2.$$

Lemma 3. *For each integer $m \geqslant 1$, there exist polynomials with the integer coefficients $P_{1,m}$, $P_{2,m}$, $Q_{1,m}$, $Q_{2,m}$ such that*

$$\wp(mz) = \frac{P_{1,m}(\wp(z), g_2, g_3)}{Q_{1,m}(\wp(z), g_2, g_3)}, \quad \operatorname{sn} mz = \frac{P_{2,m}(\operatorname{sn} z, \operatorname{sn}' z)}{Q_{2,m}(\varkappa^2, \operatorname{sn} z)},$$

where $L(P_{i,m}), L(Q_{i,m}) \leqslant \exp(c_3 m^2)$, $\deg P_{i,m}$, $\deg Q_{i,m} \leqslant m^2$, $i = 1,\, 2$.

The proof of Lemma 2 and Lemma 3 for the function $\wp(z)$ is, for example, in [1], [2], [8], and proof for $\operatorname{sn} z$ is similar to the proof for $\wp(z)$.

Lemma 4 ([4]). *Let $B, P \in \mathbb{N}$, $Q_{p,b} \in \mathbb{Z}[x_1, \dots, x_n]$, $0 \leqslant b < B$, $0 \leqslant p < P$, $L(Q_{p,b}) \leqslant L$, $\deg_{x_i} Q_{p,b} \leqslant \mathcal{N}_i$; $\alpha_1, \dots, \alpha_n$ be algebraic numbers, $m = \deg \mathbb{Q}(\alpha_1, \dots, \alpha_n)$. If $P > mB$, then the system of linear equations*

$$\sum_{p=0}^{P-1} x_p Q_{p,b}(\alpha_1, \dots, \alpha_n) = 0, \quad 0 \leqslant b < B,$$

has integer rational solutions $A_0, \dots, A_{P-1}$ *such that*

$$0 < \max |A_i| < 1 + (LP)^{\frac{mB}{P-mB}} \left(\prod_{i=1}^{n} (1+\mathcal{N}_i) \left(L(\alpha_i)(1+d(\alpha_i)) \right)^{\frac{\mathcal{N}_i}{d(\alpha_i)}} \right)^{\frac{mB}{P-mB}}.$$

We denote $|f(z)|_D = \sup_{|z|\leqslant D} |f(z)|$.

Lemma 5 ([5]). *Let* $\sigma_1(z)$ *be the Weierstrass* σ*-function which corresponds to* $\wp(z)$. *The functions* $\sigma_1(z)$ *and* $\sigma_1(z)\wp(z)$ *is entire functions and for* $M > 1$

$$|\sigma_1(z)\wp(z)|_M, |\sigma_1(z)|_M \leqslant {c_4}^{M^2}.$$

If ε *is a distance from the nearest to* z_0 *pole of* $\operatorname{sn} z$ *and* $|z_0| \leqslant M$, *then* $|\sigma(z_0)| \geqslant \varepsilon {c_5}^{-M^2}$.

Lemma 6. *Let* $\sigma_2(z)$ *be the Weierstrass* σ*-function which corresponds to the function* $\widetilde{\wp}(z)$ *associated with* $\operatorname{sn}(z)$. *The functions*

$$\sigma_2((z+K)/\sqrt{e_1-e_3}), \quad \sigma_2((z+K)/\sqrt{e_1-e_3})\operatorname{sn}(z)$$

are entire functions and for $M > 1$

$$|\sigma_2((z+K)/\sqrt{e_1-e_3})\operatorname{sn}(z)|_{|z|\leqslant M} \leqslant {c_6}^{M^2}, \quad |\sigma_2((z+K)/\sqrt{e_1-e_3})|_{|z|\leqslant M} \leqslant {c_7}^{M^2}.$$

If δ *is the distance from* z_0 *to the nearest pole of* $\operatorname{sn}(z)$ *and* $|z_0| \leqslant M_0$, *then*

$$|\sigma_2((z+iK')/\sqrt{e_1-e_3})| \geqslant \delta {c_8}^{-M_0^2}.$$

The proof of Lemma 6 is similar to the proof of Lemma 5.

Lemma 7 ([4]). *Let* R_1, $R_2 \in \mathbb{R}$, $8 < 4R_1 < R_2$, $f(z)$ *be analytic in the circle* $|z| \leqslant R_2$, *and* E *is the set of* $\mathcal{D}^2$ *points belonging to the circle* $|z| \leqslant R_1$ *and the distance between them for each pair of points is not less than* ε, $0 < \varepsilon < 1$. *Then*

$$|f(z)|_{|z|\leqslant R_1} \leqslant 2|f(z)|_{|z|\leqslant R_2} \left(\frac{4R_1}{R_2}\right)^{\mathcal{D}^2 S} + 2\mathcal{D}R_1^{-1} \left(\frac{33R_1}{\epsilon\mathcal{D}}\right)^{\mathcal{D}^2 S} \max_{x\in E, 0\leqslant s\leqslant S} \left|\frac{f^{(s)}(x)}{s!}\right|.$$

Lemma 8 ([2]). *Let* $\alpha_1, \dots, \alpha_n$ *be algebraic numbers,* $P \in \mathbb{Z}[x_1, \dots, x_n]$, $\deg_{x_i} P \leqslant \mathcal{N}_i$, $m = \deg \mathbb{Q}(\alpha_1, \dots, \alpha_n)$. *If* $P(\alpha_1, \dots, \alpha_n) \not\equiv 0$, *then*

$$|P(\alpha_1, \dots, \alpha_n)| \geqslant L(P)^{1-m} \prod_{i=1}^{n} L(\alpha_i)^{\frac{-\mathcal{N}_i m}{d(\alpha_i)}}.$$

Lemma 9 ([1], [7]). *Let* $P \in \mathbb{C}[x_1, x_2]$, $P(x_1, x_2) \not\equiv 0$, *be the polynomial of degree not greater than* $\mathcal{D}_1$ *in* x_1 *and* $\mathcal{D}_2$ *in* x_2, $\mathcal{D}_1$, $\mathcal{D}_2 \geqslant 1$, $\wp(z)$ *and* $\operatorname{sn} z$ *are algebraic independent elliptic functions. Then the number of zeros of* $P(\wp(z), \operatorname{sn} z)$, *taking into account their multiplicity, for* $|z| < K$ *does not exceed* $c_9 K^2(\mathcal{D}_1 + \mathcal{D}_2)$.

3. Proof of Theorem 1

The proof of Theorem 1 is based on the second Gelfond's method [7, 8]. Suppose that for a sufficiently large $\lambda \in \mathbb{N}$ we have

$$|\wp(4K) - \xi_1| + |\operatorname{sn}(2\omega_1) - \xi_2| + |\wp(\alpha) - \xi_3| + |\operatorname{sn}(\alpha) - \xi_4| < \exp\left(-\lambda^7 n^3 T^2\right). \tag{3}$$

We denote

$$N^2 = [\lambda^3\, n\, T], \quad S = L = [N^2 \ln \lambda]. \tag{4}$$

Define a function

$$F(z) = \sum_{l_1=0}^{L} \sum_{l_2=0}^{L} C_{l_1,l_2} \wp^{l_1}(z) \operatorname{sn}^{l_2} z, \quad C_{l_1,l_2} = \sum_{\tau=1}^{n} C_{l_1,l_2,\tau}\zeta_\tau, \ C_{l_1,l_2,\tau} \in \mathbb{Z}, \tag{5}$$

where ζ_τ are generating elements of $\mathbb{Q}(g_2, g_3, \varkappa, \alpha, \xi_1, \ldots, \xi_4)$. As in [8], we denote $\varphi_1(z) = \wp(z+\omega_1)$, $\varphi_{2,1}(z) = \operatorname{sn}(z + \frac{K}{2})$, $\varphi_{2,2}(w) = \operatorname{sn}(w + \frac{3K}{2})$. Then from the Lemma 1

$$\wp(z+w) = \frac{1}{4}\left(\frac{\varphi_1'(z) - \varphi_1'(w)}{\varphi_1(z) - \varphi_1(w)}\right)^2 - \varphi_1(z) - \varphi_1(w) = \frac{\Lambda_{1,1}(z,w)}{\Lambda_{1,2}(z,w)}, \tag{6}$$

$$\operatorname{sn}(z+w) = \frac{\varphi_{2,1}(z)\varphi_{2,2}'(w) + \varphi_{2,2}(w)\varphi_{2,1}'(z)}{1 - \varkappa^2 \varphi_{2,1}^2(z)\varphi_{2,2}^2(w)} = \frac{\Lambda_{2,1}(z,w)}{\Lambda_{2,2}(z,w)}. \tag{7}$$

From (6), (7) and Lemma 2 it follows that there exist polynomials $G_{i,s,k,l}(z)$ such that

$$G_{i,s,k,l}(z) = \frac{d^s}{d\, w^s}\left((\Lambda_{i,1}^k(z,w)\Lambda_{i,2}^l(z,w)\right)|_{w=0}, \tag{8}$$

$\deg G_{i,s,k,l} \leqslant 4(k+l)$, $\ln L(G_{i,s,k,l}) \leqslant s\ln(s(k+l) + c_{10}(s+k+l))$.

Applying the technique of [7], [8] one can deduce from (5), (6), (7), (8) the equality

$$F^{(s)}(z) = \frac{d^s}{d\, w^s}(\Lambda_{1,2}^{-L}(z,w)\Lambda_{2,2}^{-L}(z,w)\left(F(z+w)\Lambda_{1,2}^{L}(z,w)\Lambda_{2,2}^{L}(z,w)\right))|_{w=0} =$$

$$= \sum_{t=0}^{s}\binom{s}{t}\frac{d^{s-t}}{d\, w^{s-t}}(\Lambda_{1,2}^{-L}(z,w)\Lambda_{2,2}^{-L}(z,w))|_{w=0}\sum_{l_1=0}^{L}\sum_{l_2=0}^{L} C_{l_1,l_2}\sum_{i=0}^{t}\binom{t}{i} G_{1,t-i,l_1,L-l_1}(z)\times$$

$$\times G_{2,i,l_2,L-l_2}(z) = \sum_{t=0}^{s}\binom{s}{t}\frac{d^{s-t}}{d\, w^{s-t}}(\Lambda_{1,2}^{-L}(z,w)\Lambda_{2,2}^{-L}(z,w))|_{w=0} F_{s,t}(z). \tag{9}$$

Let $\xi_5^2 = 4\xi_1^3 - g_2\xi_1 - g_3$, $\xi_6^2 = (1-\xi_2^2)(1-\varkappa^2\xi_2^2)$, $\xi_7^2 = 4\xi_3^3 - g_2\xi_3 - g_3$, $\xi_8^2 = (1-\xi_4^2)(1-\varkappa^2\xi_4^2)$. Applying Lemma 3, denote by $F_{s,n_1,n_2}(\xi_1,\ldots,\xi_8)$ and $F_{s,t,n_1,n_2}(\xi_1,\ldots,\xi_8)$ the expressions obtained from $F^{(s)}(4n_1K + 2n_2\omega_1 + \alpha)$ and $F_{s,t}(4n_1K + 2n_2\omega_1 + \alpha)$ by the substitution $\wp(4K)$, $\operatorname{sn}(2\omega_1)$, $\wp(\alpha)$, $\operatorname{sn}(\alpha)$, $\wp'(4K)$, $\operatorname{sn}'(2\omega_1)$, $\wp'(\alpha)$, $\operatorname{sn}'(\alpha)$ on $\xi_1,\ldots,\xi_8$. Consider $F_{s,t,n_1,n_2}(\xi_1,\ldots,\xi_8)$ for $1 \leqslant n_1, n_2 \leqslant N$, $0 \leqslant t \leqslant s \leqslant S$ as N^2S of linear forms of nL^2 variables $C_{l_1,l_2,\tau}$. Applying Lemma 4, we choose $C_{l_1,l_2,\tau}$ not all equal to zero such that for $1 \leqslant n_1, n_2 \leqslant N$, $0 \leqslant t \leqslant s \leqslant S$

$$F_{s,t,n_1,n_2}(\xi_1,\ldots,\xi_8) = 0, \quad |C_{l_1,l_2,\tau}| < \exp(c_{11}\lambda^6 \ln\lambda\, n^2 T^3). \tag{10}$$

From (2), (3), (4), (10) we obtain for $1 \leqslant n_1, n_2 \leqslant N$, $0 \leqslant s \leqslant S$

$$|F^{(s)}(4n_1K + 2n_2\omega_1 + \alpha) - F_{s,n_1,n_2}(\xi_1,\ldots,\xi_8)| < \exp(-\frac{1}{2}\lambda^7 n^2 T^3). \tag{11}$$

From (10), (11) for $1 \leqslant n_1, n_2 \leqslant N$, $0 \leqslant s \leqslant S$ it follows

$$|F^{(s)}(4n_1K + 2n_2\omega_1 + \alpha)| < \exp(-\frac{1}{2}\lambda^7 n^2 T^3). \tag{12}$$

We show that (12) holds for $1 \leqslant n_1, n_2 \leqslant N$, $0 \leqslant s \leqslant \lambda S$.

Let $H(z) = F(z)\sigma_1^L(z)\sigma_2^L((z+K)/\sqrt{e_1 - e_3})$. We will select the least $r \in \mathbb{N}$ such that $r > 32N(|K| + |\omega_1| + 1)$, $R = 12r$. From (2), (4), (5), (10) and Lemma 5 it follows

$$|H(z)|_{|z|\leqslant R} < \exp(-\lambda^6 \ln\lambda\, n^2 T^3). \tag{13}$$

From (13) and Lemma 7 we obtain for $0 \leqslant s \leqslant \lambda S$

$$|H^{(s)}(z)|_{|z|\leqslant r} < \exp(-\frac{1}{2}\lambda^6 \ln\lambda\, T^2 \ln T). \tag{14}$$

From Lemma 6 for a sufficiently small ε in the ε-neighborhood of points $4n_1K + \alpha$ function $\sigma_2((z+K)/\sqrt{e_1 - e_3})$ and ε-neighborhood of the points $2n_2\omega_1 + \alpha$ the function $\sigma_1(z)$ has no zeros, thus for $|n_1|, |n_2| \leqslant 32N$ we see that

$$|\sigma_1(z)|_{z\in V(\varepsilon, 4n_1K + 2n_2\omega_1 + \alpha)} > \exp(-c_{12}\lambda^5 \ln\lambda\, n^2 T^3), \tag{15}$$

$$|\sigma_2((z+K)/\sqrt{e_1 - e_3})|_{z\in V(\varepsilon, 4n_1K + 2n_2\omega_1 + \alpha)} > \exp(-c_{13}\lambda^5 \ln\lambda\, n^2 T^3). \tag{16}$$

The conditions (13)–(16) imply that for $1 \leqslant n_1, n_2 \leqslant N$, $0 \leqslant s \leqslant \lambda S$

$$|F^{(s)}(4n_1K_1 + 4n_2K_2 + \alpha)| < \exp\left(-\frac{1}{3}\lambda^6 \ln\lambda\, n^2 T^3\right). \tag{17}$$

From (11) and (17) for $1 \leqslant n_1, n_2 \leqslant N$ and $0 \leqslant s \leqslant \lambda S$ it follows

$$|F_{s,n_1,n_2}(\xi_1, \ldots, \xi_8)| < \exp\left(-\frac{1}{4}\lambda^6 \ln\lambda\, n^2 T^3\right). \tag{18}$$

Considering $F_{s,t,n_1,n_2}(\xi_1, \ldots, \xi_8))$, $0 \leqslant t \leqslant s \leqslant \lambda S$, $1 \leqslant n_1, n_2 \leqslant N$, as the value of the corresponding polynomial in the algebraic points, from Lemma 8 we obtain for $F_{s,t,n_1,n_2}(\xi_1, \ldots, \xi_8)) \neq 0$ the inequality

$$|F_{s,t,n_1,n_2}(\xi_1, \ldots, \xi_8))| > \exp\left(-\lambda^5 \ln\lambda\, n^2 T^3\right). \tag{19}$$

From (9), (19) we obtain for $1 \leqslant n_1, n_2 \leqslant N$, $0 \leqslant s \leqslant \lambda S$

$$|F_{s,n_1,n_2}(\xi_1, \ldots, \xi_8))| > \exp(-2\lambda^5 \ln\lambda\, n^2 T^3). \tag{20}$$

Since (18) and (20) are contradictory,

$F_{s,t,n_1,n_2}(\xi_1, \ldots, \xi_8)) = 0$ for $1 \leqslant n_1, n_2 \leqslant N$, $0 \leqslant t \leqslant s \leqslant \lambda S$. Then for $1 \leqslant n_1, n_2 \leqslant N$, $0 \leqslant s \leqslant \lambda S$

$$F_{s,n_1,n_2}(\xi_1, \ldots, \xi_8) = 0. \tag{21}$$

From (21) it follows that the polynomial $F(z)$ has at least $c_{14}\lambda^7 \ln\lambda\, n^2 T^2$ zeros (taking into account multiplicity), but according to Lemma 9 the number of zeros can be at most $c_{15}\lambda^6 \ln\lambda\, n^2 T^2$, therefore for sufficiently large $\lambda \in \mathbb{N}$ assumption (3) leads to the contradiction which proves the theorem.

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СУМІСНІ НАБЛИЖЕННЯ ЗНАЧЕНЬ ЕЛІПТИЧНИХ ФУНКЦІЙ ВЕЙЄРШТРАССА ТА ЯКОБІ В ПЕРІОДАХ І АЛГЕБРИЧНІЙ ТОЧЦІ

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Нехай $\wp(z)$, $\operatorname{sn} z$ — алгебрично незалежні еліптичні функції Вейєрштрасса та Якобі з алгебричними інваріантами й еліптичним модулем, $(2\omega_1,\ 2\omega_3)$ і $(4K,\ 2iK')$ — пари основних періодів $\wp(z)$ та $\operatorname{sn} z$, α — довільне алгебричне число, відмінне від полюсів $\wp(z)$ і $\operatorname{sn} z$. Отримано оцінку сумісного наближення $\operatorname{sn}(2\omega_1), \operatorname{sn}(\alpha), \wp(4K)$ та $\wp(\alpha)$.

Ключові слова: сумісні наближення, еліптична функція Вейєрштрасса, еліптична функція Якобі.