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PURE STRATEGY SOLUTIONS IN PROGRESSIVE DISCRETE SILENT DUEL WITH LINEAR ACCURACY AND COMPACTIFIED SHOOTING MOMENTS

V. Romanuke

Vinnytsia Institute of Trade and Economics
of State University of Trade and Economics,
87, Soborna str., Vinnytsia, Ukraine, 21050,
e-mail: romanukevadimv@gmail.com

A zero-sum game defined on a finite subset of the unit square is considered. The game is a progressive discrete silent duel, in which the kernel is skew-symmetric. The two duelists have identical linear accuracy functions varied by the positive accuracy proportionality factor. As the duel starts, time moments of possible shooting become denser in accordance with a pattern, where every next moment is a fraction whose numerator and denominator, being greater by 1, are increased by 1 compared to the preceding moment. Due to the skew-symmetry, both the duelists have the same optimal strategies and the game optimal value is 0. For nontrivial games, where the duelist possesses more than just one moment of possible shooting between the duel beginning and end moments and the accuracy proportionality factor is not less than 1, the single optimal pure strategy is to shoot at the middle of the duel time span. As the factor becomes less than 1, only the two thirds of the duel time span and the duel very end can be optimal pure strategies, for which the factor should be equal to $\frac{1}{2}$ or not exceed the reciprocal of the number of shooting moments decreased by 2. Progressive discrete silent duels with four shooting moments and greater are not solved in pure strategies as the respective accuracy proportionality factor, being less than 1, occupies at least 50 % of interval (0; 1). As the duel size increases, this pure strategy insolvability percentage grows by the same pattern that time moments of possible shooting become denser.

Key words: game theory, silent duel, linear accuracy function, matrix game, pure strategy solution, compactified shooting moments.

1. Introduction

A silent duel is a timing zero-sum game

$$\langle X,\ Y,\ K(x,\ y)\rangle \tag{1}$$

of two players [6, 12, 3, 4], in which kernel $K(x,\ y)$ is usually defined on unit square

$$X \times Y = [0;\ 1] \times [0;\ 1] \tag{2}$$

being the Cartesian product of the players' pure strategy sets representing the standardized duel time span $[0;\ 1]$,

$$x \in X = [0;\ 1]\,,\ \ y \in Y = [0;\ 1]\,. \tag{3}$$

Each of the players, also referred to as the duelists, has a one or few bullets to shoot at any time moment between 0 and 1, where shooting (or firing) a bullet means making a decision or taking an action. In a typical silent duel, which alternatively is called noiseless

[13, 14, 15], each of the two duelists has exactly one bullet, and it is unknown to the duelist whether and when the other duelist has fired its bullet until the end of the duel time span $x = y = 1$ [2, 21, 19, 20]. In fact, the duelist may not fire the bullet during half-open interval $[0;\ 1)$, but if the duelist's bullet has not been fired during $t \in [0;\ 1)$, then it is fired "automatically" at $t = 1$. The duelist is also featured with an accuracy function which is a nondecreasing function of time [7, 11, 10, 8, 23].

It is presumed that both the duelists act within the same conditions (environment). Thus antagonistic game (1) on (2) by (3) becomes symmetric, and its optimal value is 0. The duelist benefits from shooting as late as possible, but only to shoot first [4, 17, 22]. This is modeled, in particular, by a skew-symmetric kernel [15, 5, 20]

$$K(x,\ y) = ax - ay + a^2 xy \operatorname{sign}(y - x), \tag{4}$$

where a is the accuracy proportionality factor, $a > 0$, in the duelists' accuracy functions

$$p_X(x) = ax, \ \ p_Y(y) = ay. \tag{5}$$

As kernel (4) is skew-symmetric, i. e.

$$K(x,\ y) = -K(y,\ x)$$

due to

$$K(y,\ x) = ay - ax + a^2 yx \operatorname{sign}(x - y) = -K(x,\ y),$$

both the duelists have the same optimal strategies [10, 1, 9, 19], although they can be non-symmetric if game (1) is finite [15, 19, 20].

To more realistically simulate duelists' interaction processes, discrete silent duels are considered, in which the duelist can shoot only at specified time moments [6, 11, 3, 14, 15]. The number of such possible shooting moments is finite. The moments of the duel beginning and duel end are included in this number [13, 4, 17, 18]. In a discrete duel the players' pure strategy sets are

$$X = \{x_i\}_{i=1}^{N} = Y = \{y_j\}_{j=1}^{N} = T = \{t_q\}_{q=1}^{N} \subset [0;\ 1]$$
$$\text{by } \ t_q < t_{q+1} \ \ \forall\, q = \overline{1,\ N-1} \ \text{ and } \ t_1 = 0, \ \ t_N = 1 \tag{6}$$

for $N \in \mathbb{N} \backslash \{1\}$, whereupon game (1) is finite being a matrix game

$$\langle X,\ Y,\ \mathbf{K}_N \rangle = \left\langle \{x_i\}_{i=1}^{N},\ \{y_j\}_{j=1}^{N},\ \mathbf{K}_N \right\rangle \tag{7}$$

with skew-symmetric payoff matrix

$$\mathbf{K}_N = [k_{ij}]_{N \times N} = [-k_{ji}]_{N \times N} = -\mathbf{K}_N^{\mathrm{T}} \tag{8}$$

whose entries

$$k_{ij} = K(x_i,\ y_j) = ax_i - ay_j + a^2 x_i y_j \operatorname{sign}(y_j - x_i) \ \text{ for } \ i = \overline{1,\ N} \ \text{ and } \ j = \overline{1,\ N}. \tag{9}$$

The discrete silent duel is called progressive if the density of the duelist's pure strategies between $t_1 = 0$ and $t_N = 1$ progressively grows as the duelist approaches to the duel end $t_N = 1$ [11, 23, 16, 5]. The particular case of when

$$t_q = \sum_{l=1}^{q-1} 2^{-l} = \frac{2^{q-1} - 1}{2^{q-1}} \ \text{ for } \ q = \overline{2,\ N-1} \tag{10}$$

was considered in [19, 20], where pure strategy solutions had been obtained for any $a \geqslant 1$, and specific conditions had been found for $a \in (0;\ 1)$ such, at which the progressive discrete silent duel has a pure strategy solution. Thus, situation

$$\{x_2,\ y_2\} = \left\{\frac{1}{2},\ \frac{1}{2}\right\} \tag{11}$$

is single optimal in duel (7) by (6), (8) – (10) and $a > 1$ for $N \in \mathbb{N}\backslash\{1,\ 2\}$. In the most trivial case, when $N = 2$ and the duelist can shoot only either at the duel beginning or duel end (the progressiveness is just annulled), situation

$$\{x_2,\ y_2\} = \{1,\ 1\} \tag{12}$$

is single optimal by any $a > 0$. Situation (11) remains single optimal by $a = 1$ for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$. The duel by $a = 1$ for $N = 3$ has four optimal situations: (11),

$$\{x_3,\ y_3\} = \{1,\ 1\}\,, \tag{13}$$

$$\{x_3,\ y_2\} = \left\{1,\ \frac{1}{2}\right\}, \tag{14}$$

$$\{x_2,\ y_3\} = \left\{\frac{1}{2},\ 1\right\}. \tag{15}$$

However, it was proved in [19] that situation (11) is never optimal by $a \in (0;\ 1)$.

Furthermore, while situation (13) is the single solution to 3×3 duels by $a \in (0;\ 1)$, the general case of $a \in (0;\ 1)$ is trickier. It was proved in [20] that only one $n \in \left\{\overline{3,\ N-1}\right\}$ exists such that situation

$$\{x_n,\ y_n\} = \left\{\frac{2^{n-1}-1}{2^{n-1}},\ \frac{2^{n-1}-1}{2^{n-1}}\right\} \tag{16}$$

is optimal by

$$a \in \left[\frac{1}{2^{n-1}-1};\ \frac{2^{n-2}}{(2^{n-1}-1)\cdot(2^{n-2}-1)}\right] \subset (0;\ 1) \tag{17}$$

and situation

$$\{x_N,\ y_N\} = \{1,\ 1\} \tag{18}$$

is optimal by

$$a \in \left(0;\ \frac{1}{2^{N-2}-1}\right] \subset (0;\ 1) \tag{19}$$

for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$. If

$$a = \frac{1}{2^{N-2}-1} \tag{20}$$

then situations (18),

$$\{x_{N-1},\ y_{N-1}\} = \left\{\frac{2^{N-2}-1}{2^{N-2}},\ \frac{2^{N-2}-1}{2^{N-2}}\right\}, \tag{21}$$

$$\{x_{N-1},\ y_N\} = \left\{\frac{2^{N-2}-1}{2^{N-2}},\ 1\right\}, \tag{22}$$

$$\{x_N,\ y_{N-1}\} = \left\{1,\ \frac{2^{N-2}-1}{2^{N-2}}\right\} \tag{23}$$

are optimal; apart from situations (18), (21) – (23), there are no other pure strategy solutions in the duel by (20). If

$$a \in \left(0;\ \frac{1}{2^{N-2}-1}\right) \tag{24}$$

then optimal situation (18) is the single one. If

$$a \neq \frac{1}{2^{N-2}-1} \tag{25}$$

and (17) holds, optimal situation (16) is the single one. Finally, if neither (17) nor (19) holds, then the duel does not have a pure strategy solution.

Nevertheless, the density of the duelist's pure strategies between $t_1 = 0$ and $t_N = 1$ may seem growing too quickly if the geometrical progression by (10) is used. A more smooth growth can be modeled as

$$t_q = \sum_{n=1}^{q-1} \frac{1}{n\,(n+1)} = \frac{q-1}{q} \quad \text{for} \quad q = \overline{2,\ N-1}. \tag{26}$$

Therefore, the goal is to study solutions of the progressive discrete silent duel with linear accuracy and compactified shooting moments in accordance with (26) by $N \in \mathbb{N} \backslash \{1,\ 2\}$.

2. Convention

Above all, a pure strategy solution of duel (7) corresponds to a saddle point of matrix (8) with entries (9). Due to the skew-symmetry of matrix (8) with entries (9), as only a zero entry of the matrix can be a saddle point, then a row containing a negative entry does not contain saddle points; neither does the respective column containing the positive entry. In the further consideration, only the inferences on saddle points in definite rows of matrix (8), which imply the same inferences on saddle points in respective columns, will be stated.

Inasmuch as

$$K\,(x_1,\ y_j) = K\,(0,\ y_j) = -ay_j < 0 \quad \forall\, j = \overline{2,\ N}$$

then the first row of matrix (8) with entries (9) is not an optimal strategy of the first duelist. In other words, the first row does not contain saddle points of the matrix (the first column does not contain saddle points either), so situation

$$\{x_1,\ y_1\} = \{0,\ 0\}$$

is never optimal in the duel.

It is clear that a nonnegative row of matrix (8) with entries (9) contains a saddle point on the main diagonal of the matrix. A row whose entries are positive, except for the main diagonal entry, contains a single saddle point which is the single one in such a duel (all the other $N-1$ rows of the respective column contain negative entries).

3. Three moments to shoot

It is convenient to start with the case when the duelist has the fewest number of moments to shoot.

Theorem 1. *The solutions in a progressive discrete silent* 3×3 *duel*

$$\langle X,\ Y,\ \mathbf{K}_3 \rangle = \left\langle \{x_i\}_{i=1}^{3},\ \{y_j\}_{j=1}^{3},\ \mathbf{K}_3 \right\rangle \tag{27}$$

by (6), (8), (9), (26) for $N = 3$ *are the same as the solutions in a progressive discrete silent* 3×3 *duel (27) by (6), (8) – (10) for* $N = 3$*: situation (11) is optimal by* $a \geqslant 1$ *and it is never optimal by* $a \in (0;\ 1)$*, and saddle point (11) is single by* $a > 1$*; the duel by* $a = 1$ *has four optimal situations (11), (13) – (15); situation (13) is single optimal by* $a \in (0;\ 1)$.

Proof. *Due to* $t_2 = \dfrac{1}{2}$ *for both duels, with (10) and (26), the respective payoff matrix*

$$\mathbf{K}_3 = [k_{ij}]_{3\times 3} = \begin{bmatrix} 0 & -\dfrac{a}{2} & -a \\ \dfrac{a}{2} & 0 & \dfrac{a}{2}(a-1) \\ a & -\dfrac{a}{2}(a-1) & 0 \end{bmatrix} \tag{28}$$

is the same. Therefore, the conclusions on the saddle points in (28) can be directly taken from Theorem 1 in [20]. □

4. Second moment optimality

Now, we continue with considering bigger duels by $a \geqslant 1$.

Theorem 2. *A progressive discrete silent duel (7) by (6), (8), (9), (26) for* $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$ *and* $a \geqslant 1$ *has the single optimal situation (11).*

Proof. *Consider the second row of matrix (8), where*

$$K(x_2,\ y_1) = K\left(\frac{1}{2},\ 0\right) = \frac{a}{2} > 0 \tag{29}$$

and

$$K\left(\frac{1}{2},\ y_j\right) = \frac{a}{2} - ay_j + \frac{a^2}{2}y_j = \frac{a}{2} \cdot (1 - 2y_j + ay_j) \quad \forall\, y_j > \frac{1}{2}. \tag{30}$$

If $a = 1$ *then*

$$1 - 2y_j + ay_j = 1 - y_j \geqslant 0$$

and (30) is nonnegative:

$$K\left(\frac{1}{2},\ y_j\right) = \frac{a}{2} \cdot (1 - 2y_j + ay_j) = \frac{1 - y_j}{2} \geqslant 0 \ \ \forall\, y_j > \frac{1}{2}, \tag{31}$$

where

$$K\left(\frac{1}{2},\ y_N\right) = K\left(\frac{1}{2},\ 1\right) = k_{2N} = 0 \tag{32}$$

is the second zero entry after k_{22} in the second row. Due to (29) and (31), situation (11) is a saddle point. However,

$$K\left(x_N,\ y_{N-1}\right) = K\left(1,\ \frac{N-2}{N-1}\right) = 1 - \frac{N-2}{N-1} - \frac{N-2}{N-1} = 1 - 2 \cdot \frac{N-2}{N-1} < 0 \tag{33}$$

due to

$$\frac{1}{2} < \frac{N-2}{N-1}.$$

Inequality (33) implies that the last row and last column of matrix (8) do not contain saddle points. So, situation (11) is single optimal by $a = 1$.

If $a > 1$ then, as $\frac{1}{2} < y_j \leqslant 1$,

$$(1 - y_j)^2 \geqslant 0,$$

$$1 - 2y_j + y_j^2 \geqslant 0,$$

$$2y_j - 1 \leqslant y_j^2,$$

$$\frac{2y_j - 1}{y_j} \leqslant y_j,$$

and

$$\frac{2y_j - 1}{y_j} \leqslant y_j \leqslant 1 < a \ \text{ by } \ y_j \in \left(\frac{1}{2};\ 1\right]. \tag{34}$$

Inequality (34) is followed by inequality

$$1 - 2y_j + ay_j > 0 \tag{35}$$

whence (30) is positive:

$$K\left(\frac{1}{2},\ y_j\right) = \frac{a}{2} \cdot (1 - 2y_j + ay_j) > 0 \ \ \forall\, y_j > \frac{1}{2}. \tag{36}$$

Due to (29) and (36), situation (11) is single optimal by $a > 1$. □

5. Second moment non-optimality

It was proved in [20] that the second shooting moment in a progressive discrete silent duel (7) by (6), (8) – (10) for $N \in \mathbb{N} \backslash \{1,\ 2\}$ is not an optimal strategy by $0 < a < 1$. See whether this property keeps for the duel with the compactified shooting moments by (26).

Theorem 3. *Situation (11) is never optimal in a progressive discrete silent duel (7) by (6), (8), (9), (26) for $N \in \mathbb{N} \backslash \{1,\ 2\}$ and $0 < a < 1$.*

Proof. *For $N \in \mathbb{N} \backslash \{1,\ 2\}$ consider the second row of matrix (8) whose last column entry*

$$K\left(\frac{1}{2},\ 1\right) = \frac{a}{2} - a + \frac{a^2}{2} = \frac{a}{2} \cdot (a - 1) < 0 \ \text{ by } \ 0 < a < 1. \tag{37}$$

Inequality (37) directly implies that the second row of matrix (8) does not contain saddle points by $0 < a < 1$. □

6. Third moment optimality

Due to Theorem 1, situation (13), which consists of the third shooting moments in the progressive discrete silent 3×3 duel, is single optimal in such duels by $a \in (0;\ 1)$. See whether the third shooting moment in bigger duels can be an optimal strategy.

Theorem 4. *In a progressive discrete silent duel (7) by (6), (8), (9), (26) for* $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$ *and* $0 < a < 1$, *situation*

$$\{x_3,\ y_3\} = \left\{\frac{2}{3},\ \frac{2}{3}\right\} \tag{38}$$

is optimal only if $a = \frac{1}{2}$. *Except for the third and last shooting moments* $t_3 = \frac{2}{3}$ *and* $t_N = 1$, *there are no other optimal pure strategies. The* 4×4 *duel with* $a = \frac{1}{2}$ *has four optimal pure strategy situations: situation (38) and situations*

$$\{x_4,\ y_4\} = \{1,\ 1\}, \tag{39}$$

$$\{x_3,\ y_4\} = \left\{\frac{2}{3},\ 1\right\}, \tag{40}$$

$$\{x_4,\ y_3\} = \left\{1,\ \frac{2}{3}\right\}. \tag{41}$$

Proof. *Due to Theorem 3, situation (11) is not optimal, so the first two rows of matrix (8) do not contain saddle points. If situation*

$$\{x_n,\ y_n\} = \left\{\frac{n-1}{n},\ \frac{n-1}{n}\right\} \text{ by } n \in \left\{\overline{3,\ N-1}\right\} \tag{42}$$

is optimal, then, in the n*-th row of matrix (8), inequalities*

$$K(x_n,\ y_j) = ax_n - ay_j - a^2 x_n y_j \geqslant 0 \ \ \forall\, y_j < x_n \tag{43}$$

and

$$K(x_n,\ y_j) = ax_n - ay_j + a^2 x_n y_j \geqslant 0 \ \ \forall\, y_j > x_n \tag{44}$$

must hold. From inequality (43) it follows that

$$\frac{x_n}{1 + ax_n} \geqslant y_j \ \ \forall\, y_j < x_n. \tag{45}$$

As

$$y_j \leqslant \frac{n-2}{n-1} < \frac{n-1}{n} = x_n \tag{46}$$

then inequality (45) is transformed into

$$\frac{n-1}{n} \cdot \frac{1}{1 + a \cdot \frac{n-1}{n}} \geqslant \frac{n-2}{n-1},$$

$$\frac{n-1}{n + a(n-1)} \geqslant \frac{n-2}{n-1},$$

$$n^2 - 2n + 1 \geqslant n^2 - 2n + a(n-1)(n-2),$$

$$1 \geqslant a(n-1)(n-2),$$

whence

$$a \leqslant \frac{1}{(n-1)(n-2)}. \tag{47}$$

From inequality (44) it follows that

$$\frac{x_n}{1 - ax_n} \geqslant y_j \;\; \forall\, y_j > x_n. \tag{48}$$

As

$$1 \geqslant y_j > \frac{n-1}{n} = x_n \tag{49}$$

then inequality (48) is transformed into

$$\frac{n-1}{n} \cdot \frac{1}{1 - a \cdot \dfrac{n-1}{n}} \geqslant 1,$$

$$\frac{n-1}{n - a(n-1)} \geqslant 1. \tag{50}$$

If $0 < a < 1$ *then*

$$n - a(n-1) > 0$$

and inequality (50) is written as

$$n - 1 \geqslant n - a(n-1). \tag{51}$$

Thus, inequality (51) is followed by

$$a \geqslant \frac{1}{n-1}. \tag{52}$$

Inasmuch as

$$\frac{1}{n-1} - \frac{1}{(n-1)(n-2)} = \frac{n-3}{(n-1)(n-2)} \geqslant 0,$$

then points

$$\frac{1}{n-1} \tag{53}$$

and

$$\frac{1}{(n-1)(n-2)} \tag{54}$$

coincide by $n = 3$*, and by* $n > 3$ *point (53) is located to the right with respect to point (54). The latter makes impossible simultaneous inequalities (47) and (52), whichever* a *is. By* $n = 3$ *these inequalities are* $a \leqslant \frac{1}{2}$ and $a \geqslant \frac{1}{2}$*, respectively, that is situation (38) is optimal only if* $a = \frac{1}{2}$ and $n = 3$*. In addition, no other* n*-th row of matrix (8) contains* *saddle points by* $n \in \left\{\overline{4,\ N-1}\right\}$ *implying that there are no optimal pure strategies between the fourth and* $(N-1)$*-th row for* $N \in \mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$*. If*

$$a \in \left(0;\ \frac{1}{2}\right) \cup \left(\frac{1}{2};\ 1\right) \tag{55}$$

then there are no optimal pure strategies between the third and $(N-1)$*-th row for* $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$ *(or, in other words, the first* $N-1$ *rows do not contain saddle points).*

In the third row by $a = \frac{1}{2}$ *we have*

$$k_{31} = K\left(\frac{2}{3},\ 0\right) = \frac{1}{3}, \tag{56}$$

$$k_{32} = K\left(\frac{2}{3},\ \frac{1}{2}\right) = \frac{1}{3} - \frac{1}{4} - \frac{1}{12} = 0, \tag{57}$$

$$k_{3j} = K\left(\frac{2}{3},\ y_j\right) = \frac{1}{3} - \frac{1}{2}y_j + \frac{1}{6}y_j = \frac{1}{3} - \frac{1}{3}y_j > 0\ \ \forall\, y_j \in \left(\frac{2}{3};\ 1\right), \tag{58}$$

$$k_{3N} = K\left(\frac{2}{3},\ 1\right) = 0. \tag{59}$$

So, due to (56) – (59), the nonnegative third row contains another two zero entries (57) and (59) – in the second and last columns, respectively. But the second row does not contain saddle points, and in the last row

$$\begin{gathered} K\left(x_N,\ y_{N-1}\right) = K\left(1,\ \frac{N-2}{N-1}\right) = \\ = \frac{1}{2} - \frac{N-2}{2N-2} - \frac{N-2}{4N-4} = \frac{4-N}{4N-4} < 0 \end{gathered} \tag{60}$$

for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$ *and*

$$K\left(x_N,\ y_{N-1}\right) = 0 \ \text{ for } \ N = 4. \tag{61}$$

So, the last row does not contain saddle points for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$ *and* $a = \frac{1}{2}$*. Therefore, situation (38) is single optimal in this case. In the* 4×4 *duel by* $a = \frac{1}{2}$ *situation (38) is optimal as well, but equality (61) holds, and so the fourth row is nonnegative as*

$$K\left(x_4,\ y_3\right) = 0 = K\left(x_3,\ y_4\right)$$

and

$$k_{41} = K\left(1,\ 0\right) = \frac{1}{2},$$

$$k_{42} = K\left(1,\ \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.$$

So, situations (38) – (41) are optimal for $a = \frac{1}{2}$ *in the* 4×4 *duel.* □

7. Last moment optimality

It was proved in [20] that the last shooting moment in a progressive discrete silent duel (7) by (6), (8) – (10) for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$ is the single optimal pure strategy when the accuracy proportionality factor (being less than 1) drops below value (20). Theorem 1 has already shown that, when the accuracy proportionality factor is less than 1, the last shooting moment is the single optimal pure strategy in a progressive discrete silent 3×3 duel (27), regardless of the compactification (because the second shooting moment is the middle of the duel time span anyway). See whether the last shooting moment in bigger duels can be the single optimal pure strategy when the accuracy proportionality factor drops sufficiently low.

Theorem 5. *In a progressive discrete silent duel (7) by (6), (8), (9), (26) for* $N \in \mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$ *and*

$$a \leqslant \frac{1}{N-2} \tag{62}$$

situation

$$\{x_N,\ y_N\} = \{1,\ 1\} \tag{63}$$

is single optimal. In the 4×4 *duel with*

$$a < \frac{1}{N-2} = \frac{1}{2} \tag{64}$$

situation (63) is single optimal as well.

Proof. Situation (63) is optimal only if the last row of matrix (8) is nonnegative. Thus,

$$k_{Nj} = K(x_N,\ y_j) = K(1,\ y_j) = a(1 - y_j - ay_j) \geqslant 0 \ \ \forall\, y_j < 1 \tag{65}$$

if

$$1 - y_j - ay_j \geqslant 0,$$

whence

$$\frac{1}{1+a} \geqslant y_j \ \ \forall\, y_j < 1. \tag{66}$$

As

$$y_j \leqslant y_{N-1} = \frac{N-2}{N-1} < 1$$

then inequality (66) is transformed into

$$\frac{1}{1+a} \geqslant \frac{N-2}{N-1},$$

$$N - 1 \geqslant N - 2 + a(N-2),$$

whence inequality (62) emerges. So, if (62) holds then situation (63) is optimal, and it is the single optimal pure strategy situation for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$ *because here membership (55) is true and there cannot be any other optimal situations due to Theorem 3 and Theorem 4. Situation (63) is single optimal in the* 4×4 *duel for (64) by the same reason.* □

8. Discussion and conclusion

The solutions of the progressive discrete silent duel with linear accuracy and compactified shooting moments in accordance with (26) by $N \in \mathbb{N} \backslash \{1, 2\}$ do not differ much from those with the faster converging shooting moments by (10). The 3×3 duel remains the same (Theorem 1). In bigger duels with (26) for $a \geqslant 1$, just like in bigger duels with (10) for $a \geqslant 1$, the single optimal strategy of the duelist is the middle of the duel time span, which is the second shooting moment following the duel very beginning (Theorem 2).

The difference between the duels with shooting moments by (10) and shooting moments by (26) exists only for $a \in (0; 1)$. Unlike the duels with the faster converging shooting moments by (10), the duelist in 4×4 and bigger duels with compactified shooting moments by (26) can have only third and last shooting moments as an optimal strategy. In the 4×4 duel by $a \in \left(0; \frac{1}{2}\right)$ the single optimal strategy is to shoot at the duel very end. Only this behavior remains optimal in bigger duels as well, where the accuracy proportionality factor does not exceed the reciprocal of the number of shooting moments decreased by 2 (Theorem 5). If the accuracy proportionality factor is equal to $\frac{1}{2}$, then the duelist in the 4×4 duel possesses two optimal pure strategies – the two thirds of the duel time span and the duel very end (Theorem 4). Bigger duels at such an accuracy proportionality factor have only one pure strategy solution, which is of the two thirds. Therefore, a progressive discrete silent duel with four shooting moments and greater is not solved in pure strategies if

$$a \in \left(\frac{1}{N-2}; 1\right) \backslash \left\{\frac{1}{2}\right\} \quad \text{for} \quad N \in \mathbb{N} \backslash \{1, 2, 3\}. \tag{67}$$

The length of the interval in (67) is $\frac{N-3}{N-2}$. So, it is 50 % probable that a 4×4 duel by $a < 1$ will be not solved in pure strategies. As the duel size increases, this pure strategy insolvability percentage grows by the same pattern that time moments of possible shooting become denser.

The proved assertions contribute the specificity of the compactified shooting moments in discrete silent duels to the games of timing. The specificity slightly broadens the duelist behavior that may include the two thirds of the duel time span, rather than a shooting moment exactly expressed with a finite number of decimal places. However, this specificity, as well as some others, less remarkable, exists only at weaker shooting accuracies.

Progressive discrete silent duels with compactified shooting moments can be further studied for some nonlinearities in the accuracy function. For instance, it can be the quadratic accuracy as a case of the low-accurate duelist [19]. In a symmetric addition, a case of the better-shooting duelist with the square-root accuracy can be studied as well.

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РОЗВ'ЯЗКИ У ЧИСТИХ СТРАТЕГІЯХ ПРОГРЕСУЮЧОЇ ДИСКРЕТНОЇ БЕЗШУМНОЇ ДУЕЛІ З ЛІНІЙНОЮ ВЛУЧНІСТЮ ТА КОМПАКТИЗОВАНИМИ МОМЕНТАМИ ПОСТРІЛУ

В. Романюк

Вінницький торговельно-економічний інститут
Державного торговельно-економічного університету,
вул. Соборна, 87, Вінниця, 21050, Україна
e-mail: *romanukevadimv@gmail.com*

Розглянуто гру з нульовою сумою, яка визначена на скінченній підмножині одиничного квадрата. Ця гра є прогресуючою дискретною безшумною дуеллю, в якій ядро кососиметричне. Два дуелянти мають ідентичні лінійні функції влучності, змінювані додатним коефіцієнтом пропорційності влучності. Як тільки дуель розпочинається, моменти часу можливих пострілів стають щільнішими за правилом, де кожен наступний момент є дробом, чиї чисельник і знаменник, який більше на 1, збільшуються на 1 порівняно з попереднім моментом. Внаслідок кососиметричності обидва дуелянти мають ті самі оптимальні стратегії, а оптимальне значення гри дорівнює 0. Для нетривіальних ігор, де дуелянт володіє понад одним моментом можливого пострілу між початком і закінченням дуелі, а коефіцієнт пропорційності влучності не менше 1, єдиною оптимальною чистою стратегією є постріл у середині інтервалу часу тривання дуелі. Як тільки цей коефіцієнт стає меншим 1, лише дві третини інтервалу часу тривання дуелі та момент закінчення дуелі можуть бути оптимальними чистими стратегіями, для яких коефіцієнт має дорівнювати 1/2 або не перевищувати обернене значення кількості моментів пострілу, зменшеної на 2. Прогресуючі дискретні безшумні дуелі з чотирма моментами пострілу більше не розв'язуються у чистих стратегіях, де відповідний коефіцієнт пропорційності влучності, що менший 1, займає щонайменше 50 % інтервалу (0; 1). Зі збільшенням розміру дуелі цей відсоток, що засвідчує відсутність розв'язків у чистих стратегіях, зростає за тим самим правилом, за яким моменти можливого пострілу стають щільнішими.

Ключові слова: теорія ігор, безшумна дуель, лінійна функція влучності, матрична гра, розв'язок у чистих стратегіях, компактизовані моменти пострілу.