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# FINITE ELEMENT ANALYSIS OF GREEN-LINDSAY THERMOPIEZOELECTRICITY TIME-HARMONIC PROBLEM

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Using Green-Lindsay thermopiezoelectricity model with so-called relaxation parameters, which influence on the way of interaction of mechanical, electrical and thermal fields in pyroelectric materials, we formulate initial boundary value problem and corresponding variational problem for this model in terms of vector of elastic displacements, electric potential and temperature increment. Then, under assumption of a harmonic loading with known beforehand angular frequency, we construct a mixed variational problem on amplitudes of harmonic waves in pyroelectrics and prove the existence of unique and robust solution of this problem under conditions suitable for practical applications. After that, the aforementioned problem is discretized using Galerkin-scheme with standard for Sobolev spaces basis functions constructed via finite element method (FEM). A priori estimates of FEM-approximations errors are found. The estimates show the dependence of FEM-approximations convergence velocity both on the order of polynomial basis function and on regularity of solution of the problem. Finally, we demonstrate the results of numerical experiment, which show the influence of Green-Lindsay relaxation parameters on the characteristics of harmonic wave, which emerge in pyroelectric bar under harmonic heat loading with preset angular frequency.

*Key words*: pyroelectric specimen, thermopiezoelectricity, Green-Lindsay model, initial boundary value problem, variational problem, Galerkin-discretization, finite element method.

## 1. INTRODUCTION

Pyroelectric and piezoelectric materials are nowadays widely utilized in various modern devices [12, 17, 18]. That is why computer modelling of their behaviour becomes more and more relevant. During the last decades many researchers contributed to developing various mathematical models, which are then used as a basis for that computer modelling. Firstly, Mindlin [10] proposed the classic theory of linear thermopiezoelectricity, where the interaction of thermal, electrical and mechanical fields in pyroelectric materials was studied. Nowacki [1, 11] performed further development of this theory. In parallel, a classic thermoelasticity model, which is obtained by eliminating the electrical field from the scope, was studied by other researcher. The main drawback of the latter theory (and therefore the classic thermopiezoelectricity theory too) is the assumption of infinite speed of heat propagation in the materials. To overcome it, Green and Lindsay [8] proposed a modified theory of thermoelasticity (GL-theory), where heat conduction equation became hyperbolic with introduction of two so-called "relaxation time"-parameters. Similar generalizations of the thermoelasticity model can be found in [2]. Later, the modifications of thermoelasticity model were extended to the scope of thermopiezoelectricity. Nowadays a set of generalization theories for thermoelasticity and thermopiezoelectricity is known, namely Lord-Shulman, Chandrasekharaiah-Tzou, Green-

Naghdi, etc. A comprehensive review of the existing generalization theories can be found in [7, 9]. Researchers used different techniques for solving the generalized thermopiezoelectricity problems, see [13, 16].

In authors` previous works [3, 5, 6, 14] the classic thermopiezoelectricity problem was considered. In article [4] Green-Lindsay theory of thermopiezoelectricity for dynamical problems was investigated. In our paper [15] forced vibrations of pyroelectric materials under Lord-Shulman theory were studied. In this article, similar techniques as in [15] are applied to GL-theory of forced vibrations of pyroelectric materials.

In section 2 the initial boundary value problem of Green-Lindsay thermopiezoelectricity is described. Section 3 is dedicated to construction of the corresponding variational problem and variational problem for the special case of forced vibrations. In section 4 we prove the well-posedness of the constructed variational problem. Section 5 describes how Galerkin-discretization allows us to build a numerical scheme for solving this variational problem. Section 6 shows the results of numerical experiments. Finally, in section 7 the conclusions are made.

## 2. PROBLEM STATEMENT

Let $\Omega$ be the bounded connected domain of points $\mathbf{x}=(x_1,\ldots,x_d)\in\Re^d$ with Lipschitz-continuous boundary $\partial\Omega=\Gamma$, and $\mathbf{n}=\{n_i\}_{i=1}^d$ is unit outer normal vector, $n_i=\cos(n,x_i)$. Also let us consider time interval $[0,T]$, $0<T<+\infty$. Like in classic thermopiezoelectricity problem, our goal is to find vector of elastic displacements $\mathbf{u}=\{u_i(\mathbf{x},t)\}_{i=1}^d$, electric potential $p=p(\mathbf{x},t)$, and temperature increment $\theta=\theta(\mathbf{x},t)$, which satisfy the following equations in $\Omega\times(0,T]$ (here and everywhere below the ordinary summation by repetitive indices is expected):

$$\rho(u_i''-f_i)-\sigma_{ji,j}=0, \tag{1}$$

$$D'_{k,k}+J_{k,k}=0, \tag{2}$$

$$\rho(T_0S'-w)+q_{i,i}=0, \tag{3}$$

The above expressions (1)-(3) are equation of motion, differentiated Maxwell`s equation and heat conduction equation and below we will explain the meaning of each notation more thoroughly. Here $\sigma_{ij}$ is a stress tensor, which is defined by the following constitutive equation:

$$\sigma_{ij}=c_{ijkm}[\varepsilon_{km}(\mathbf{u})-\alpha_{km}(\theta+t_1\theta')]+a_{ijkm}\varepsilon_{km}(\mathbf{u}')-e_{kij}E_k(p). \tag{4}$$

Constitutive equation for electric displacement $D_k$ is shown below:

$$D_k=\chi_{km}E_m(p)+e_{kij}\varepsilon_{ij}(u)+\pi_k(\theta+t_1\theta'). \tag{5}$$

Entropy density $S$ is defined via

$$\rho S=\rho c_v T_0^{-1}(\theta+t_0\theta')+c_{ijkm}\alpha_{km}\varepsilon_{ij}(\mathbf{u})+\pi_k E_k(p). \tag{6}$$

Here, in the equations (4-6), parameters $t_1\ge t_0>0$ are of time dimension and were introduced by Green and Lindsay in [8] to eliminate the effect of infinite speed of heat propagation from the classic heat conduction problem. These parameters are also known as so-called “relaxation times” and their values are always taken as less than $10^{-10}\,\mathrm{s}$. Setting $t_1=t_0=0$ we come to the classic thermopiezoelectricity model.

Vector $q_i$ describes heat flux in different directions and it relation with temperature increment is determined via classic Fourier`s law:

$$q_i = -\lambda_{ij}\theta_{,j}. \tag{7}$$

Vector $J_k$ is the electrical current density, generated by a free electrical charge density. We assume that pyroelectric material is not an ideal dielectric, and the electric current runs through the pyroelectric specimen and satisfies standard Ohm`s law, i.e.

$$J_k = z_{km}E_m(p). \tag{8}$$

Strain tensor $\varepsilon_{km}$ and electrical field vector $E_k$ are assumed to satisfy the relations

$$\begin{aligned} &\varepsilon_{km}(\mathbf{u}) = \tfrac{1}{2}(u_{k,m} + u_{m,k}), \\ &E_k(p) = -p_{,k}, \end{aligned} \tag{9}$$

where comma in the subscript stands for the partial derivative by the spatial variable, i.e. $g_{,k} = \partial g / \partial x_k$.

Notation $\rho$ is a mass density of pyroelectric material, $c_v$ is its specific heat and $T_0$ is a fixed uniform reference temperature of the specimen. Notation $f_i$ is a vector of volume mechanical forces and $w$ represents volume heat forces. Tensors $a_{ijkm}$ and $c_{ijkm}$ describe the viscosity and elasticity properties of a pyroelectric material with the common properties of symmetry and ellipticity. Notation $e_{kij}$ depicts a piezoelectricity coefficients tensor with symmetric properties:

$$e_{kij} = e_{kji}. \tag{10}$$

Coefficients $z_{km}$, $\chi_{ij}$, $\lambda_{ij}$, $\alpha_{km}$ define the symmetrical and elliptical electrical conductivity, dielectric susceptibility, heat conductivity and thermal expansion coefficients respectively. Notation $\pi_k$ describes pyroelectricity coefficients, which satisfy the inequality[11]:

$$\chi_{km}y_k y_m + 2\pi_k y_k \xi + \rho c_v \xi^2 \geq 0, \quad \forall \xi, y_k \in \Re. \tag{11}$$

To finalize the formulation of the initial boundary value problem of Green-Lindsay thermopiezoelectricity, the system of partial differential equations (1)-(3) is then complemented by boundary conditions

$$\begin{cases} u_i = 0 \quad on \quad \Gamma_u \times [0,T], \ \Gamma_u \subset \Gamma, \ mes(\Gamma_u) > 0, \\ \sigma_{ij}n_j = \bar{\sigma}_i \quad on \quad \Gamma_\sigma \times [0,T], \quad \Gamma_\sigma = \Gamma \setminus \Gamma_u, \\ p = 0 \quad on \quad \Gamma_p \times [0,T], \ \Gamma_p \subset \Gamma, \ mes(\Gamma_p) > 0, \\ (D'_k + J_k)n_k = 0 \quad on \ \Gamma_d \times [0,T], \ \Gamma_d \subset \Gamma, \ \Gamma_d \cap \Gamma_p = \varnothing, \\ \int_{\Gamma_e} (D'_k + J_k)n_k d\gamma = I \quad on \ \Gamma_e \times [0,T], \ \Gamma_e = \Gamma \setminus (\Gamma_d \cap \Gamma_p), \\ E_k(p) - n_k E_m(p) n_m = 0 \quad on \ \Gamma_e \times [0,T], \\ \theta = 0 \quad on \quad \Gamma_\theta \times [0,T], \ \Gamma_\theta \subset \Gamma, \quad mes(\Gamma_\theta) > 0, \\ q_i n_i = \bar{h} \quad on \quad \Gamma_q \times [0,T], \quad \Gamma_q = \Gamma \setminus \Gamma_\theta, \end{cases} \tag{12}$$

and the initial conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \ \mathbf{u}'|_{t=0} = \mathbf{v}_0, \ p|_{t=0} = p_0, \ \theta|_{t=0} = \theta_0, \ \theta'|_{t=0} = \theta_{10} \quad in \ \Omega. \tag{13}$$

Here $\bar{\boldsymbol{\sigma}} = \{\bar{\sigma}_i(\mathbf{x},t)\}_{i=0}^{d}$, $I = I(\mathbf{x},t)$ $\bar{h} = \bar{h}(\mathbf{x},t)$ represent vector of mechanical loading, external electric current, and applied heat flux correspondingly.

3. VARIATIONAL PROBLEM STATEMENT

Let us introduce the spaces of admissible elastic displacements, electric potentials and temperature increments (relatively to the initial temperature $T_0$) respectively:

$$\begin{aligned} &\mathbf{V} = \left\{\mathbf{v} \in [H^1(\Omega)]^d : \mathbf{v} = 0 \ on \ \Gamma_u\right\}, \\ &P = \left\{r \in H^1(\Omega) \ : \ r = 0 \ \ on \ \Gamma_p, \ r = const \ on \ \Gamma_e\right\} \\ &Z = \left\{\zeta \in H^1(\Omega) : \zeta = 0 \ on \ \Gamma_\theta\right\}. \end{aligned} \tag{14}$$

Here symbol $H^m(\Omega)$ means a standard Sobolev space. We denote $\mathbf{\Phi} := \mathbf{V} \times P \times Z$, the dual space $\mathbf{\Phi}' := \mathbf{V}' \times P' \times Z'$ and $\mathbf{H} = [L^2(\Omega)]^d$. Then the initial boundary value problem of Green-Lindsay thermopiezoelectricity (1)-(9), (12)-(13) can be rewritten in the following variational formulation:

$$\left\{\begin{aligned} &\text{given } \boldsymbol{\psi}_0 = (u_0, p_0, \theta_0) \in \mathbf{\Phi},\ \mathbf{v}_0 \in \mathbf{H},\ \theta_{10} \in L^2(\Omega) \text{ and} (l, r, \mu) \in L^2(0,T;\mathbf{\Phi}'); \\ &\text{find } \boldsymbol{\psi} = \{\mathbf{u}(x,t), p(x,t), \theta(x,t)\} \in L^2(0,T;\mathbf{\Phi}) \text{ such that} \\ &m(\mathbf{u}''(t), \mathbf{v}) + a(\mathbf{u}'(t), \mathbf{v}) + c(\mathbf{u}(t), \mathbf{v}) - e(p(t), \mathbf{v}) - \\ &-\gamma(\theta(t) + t_1\theta'(t), \mathbf{v}) = < l(t), \mathbf{v} >, \\ &\chi(p'(t), \xi) + e(\xi, u'(t)) + z(p(t), \xi) + \pi(\theta'(t) + t_1\theta''(t), \xi) = < r(t), \xi >, \\ &s(\theta'(t) + t_0\theta''(t), \eta) + k(\theta(t), \eta) + \pi(\eta, p'(t)) + \\ &+\gamma(\eta, \mathbf{u}'(t)) = < \mu(t), \eta > \ \ \forall\, t \in \left(0, T\right], \\ &m(\mathbf{u}'(0) - \mathbf{v}_0, \mathbf{v}) = 0, \quad c(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in V, \\ &\chi(p(0) - p_0, \xi) = 0 \quad \forall \xi \in P, \\ &s(\theta(0) + t_0\theta'(0) - (\theta_0 + t_0\theta_{10}), \eta) = 0 \ \ \forall \eta \in Z. \end{aligned}\right. \tag{15}$$

The introduced bilinear and linear forms are as follows:

$$\begin{aligned} &m(\mathbf{u}, \mathbf{v}) := \int_\Omega \rho u_i v_i dx = \int_\Omega \rho \mathbf{u}.\mathbf{v} dx,\ a(\mathbf{u}, \mathbf{v}) := \int_\Omega a_{ijkm}\varepsilon_{ij}(\mathbf{u})\varepsilon_{km}(\mathbf{v}) dx, \\ &c(\mathbf{u}, \mathbf{v}) := \int_\Omega c_{ijkm}\varepsilon_{ij}(\mathbf{u})\varepsilon_{km}(\mathbf{v}) dx,\ < l, \mathbf{v} > := \int_\Omega \rho f_i v_i dx + \int_{\Gamma_\sigma} \bar{\sigma}_i v_i d\gamma, \\ &\gamma(\eta, \mathbf{v}) := \int_\Omega \eta c_{ijkm}\alpha_{km}\varepsilon_{ij}(\mathbf{v}) dx,\ e(\xi, \mathbf{v}) := \int_\Omega e_{kij} E_k(\xi)\varepsilon_{ij}(\mathbf{v}) dx \quad \forall \mathbf{u}, \mathbf{v} \in V, \\ &\chi(p, \xi) := \int_\Omega \chi_{km} E_k(p) E_m(\xi) dx, \quad z(p, \xi) := \int_\Omega z_{km} E_k(p) E_m(\xi) dx, \\ &\pi(\eta, \xi) = \int_\Omega \eta \pi_k E_k(\xi) dx,\ < r, \xi > := I\xi|_{\Gamma_e} \quad \forall p, \xi \in P, \\ &s(\theta, \eta) = \int_\Omega \rho c_v T_0^{-1} \theta\eta dx,\ k(\theta, \eta) = \int_\Omega T_0^{-1}\lambda_{ij}\theta_{,i}\eta_{,j} dx, \\ &< \mu, \eta > := \int_\Omega T_0^{-1}\rho w \eta dx - \int_{\Gamma_h} T_0^{-1}\bar{h}\eta d\gamma \quad \forall \eta, \theta \in Z. \end{aligned} \tag{16}$$

We suppose that the harmonic loadings with angular frequency $\omega$ are applied to the piezoelectric specimen:

$$\begin{aligned} &l(t) = (l_1 + il_2)e^{-i\omega t}, \\ &r(t) = (r_1 + ir_2)e^{-i\omega t}, \\ &\mu(t) = (\mu_1 + i\mu_2)e^{-i\omega t}, \ \ \forall\, t \in (0,T]. \end{aligned} \tag{17}$$

Then the approximate solutions of problem (15) can be looked for in the form of the following expansions:

$$\begin{aligned} &\mathbf{u}(x,t) \cong (\mathbf{u}_1(x) + i\mathbf{u}_2(x))e^{-i\omega t}, \\ &p(x,t) \cong (p_1(x) + ip_2(x))e^{-i\omega t}, \\ &\theta(x,t) \cong (\theta_1(x) + i\theta_2(x))e^{-i\omega t}, \end{aligned} \tag{18}$$

where $\mathbf{u}_1(x)$, $\mathbf{u}_2(x)$, $p_1(x)$, $p_2(x)$, $\theta_1(x)$, $\theta_2(x)$ are the unknown amplitudes of vector of mechanical displacements, electric potential and temperature increment respectively.

Substituting expressions (17) and (18) into variational problem (15) and neglecting its initial conditions, we obtain the variational problem for force harmonic vibrations of piezoelectric specimen:

$$\begin{cases} \text{given } \omega > 0,\, t_1 \ge t_0 > 0,\, (l_1, l_2, r_1, r_2, \mu_1, \mu_2) \in \mathbf{W}' = \mathbf{\Phi}' \times \mathbf{\Phi}'; \\ \text{find } \boldsymbol{\psi} = (\mathbf{u}_1, p_1, \theta_1, \mathbf{u}_2 p_2, \theta_2) \in \mathbf{W} = \mathbf{\Phi} \times \mathbf{\Phi} \text{ such that} \\ \forall\, (\mathbf{v}_1, \xi_1, \eta_1, \mathbf{v}_2, \xi_2, \eta_2) \in \mathbf{W} = \mathbf{\Phi} \times \mathbf{\Phi} \\ -\omega^2 m(\mathbf{u}_1, \mathbf{v}_2) + \omega a(\mathbf{u}_2, \mathbf{v}_2) + c(\mathbf{u}_1, \mathbf{v}_2) - e(p_1, \mathbf{v}_2) - \\ -\gamma(\theta_1, \mathbf{v}_2) - \omega t_1 \gamma(\theta_2, \mathbf{v}_2) = < l_1, \mathbf{v}_2 >, \\ -\omega^2 m(\mathbf{u}_2, \mathbf{v}_1) - \omega a(\mathbf{u}_1, \mathbf{v}_1) + c(\mathbf{u}_2, \mathbf{v}_1) - e(p_2, \mathbf{v}_1) - \\ -\gamma(\theta_2, \mathbf{v}_1) + \omega t_1 \gamma(\theta_1, \mathbf{v}_1) = < l_2, \mathbf{v}_1 >, \\ \omega\chi(p_2, \xi_1) + \omega e(\xi_1, \mathbf{u}_2) + z(p_1, \xi_1) + \omega\pi(\theta_2, \xi_1) - \omega^2 t_1 \pi(\theta_1, \xi_1) = < r_1, \xi_1 >, \\ -\omega\chi(p_1, \xi_2) - \omega e(\xi_2, \mathbf{u}_1) + z(p_2, \xi_2) - \omega\pi(\theta_1, \xi_2) - \omega^2 t_1 \pi(\theta_2, \xi_2) = < r_2, \xi_2 >, \\ \omega s(\theta_2, \eta_1) - \omega^2 t_0 s(\theta_1, \eta_1) + k(\theta_1, \eta_1) + \omega\pi(\eta_1, p_2) + \omega\gamma(\eta_1, \mathbf{u}_2)) = < \mu_1, \eta_1 >, \\ -\omega s(\theta_1, \eta_2) - \omega^2 t_0 s(\theta_2, \eta_2) + k(\theta_2, \eta_2) - \omega\pi(\eta_2, p_1) - \omega\gamma(\eta_2, \mathbf{u}_1)) = < \mu_2, \eta_2 > . \end{cases} \tag{19}$$

Now we will transform the last two equations of the variational problem (19) by using the linear combination of the admissible functions $\eta_1$, $\eta_2$. We will obtain:

$$\begin{aligned} &\text{given } \omega > 0,\, t_1 \ge t_0 > 0,\, (l_1, l_2, r_1, r_2, \mu_1, \mu_2) \in \mathbf{W}' = \mathbf{\Phi}' \times \mathbf{\Phi}'; \\ &\text{find } \boldsymbol{\psi} = (\mathbf{u}_1, p_1, \theta_1, \mathbf{u}_2 p_2, \theta_2) \in \mathbf{W} = \mathbf{\Phi} \times \mathbf{\Phi} \text{ such that} \\ &\forall\, (\mathbf{v}_1, \xi_1, \eta_1, \mathbf{v}_2, \xi_2, \eta_2) \in \mathbf{W} = \mathbf{\Phi} \times \mathbf{\Phi} \\ &-\omega^2 m(\mathbf{u}_1, \mathbf{v}_2) + \omega a(\mathbf{u}_2, \mathbf{v}_2) + c(\mathbf{u}_1, \mathbf{v}_2) - e(p_1, \mathbf{v}_2) - \\ &-\gamma(\theta_1, \mathbf{v}_2) - \omega t_1 \gamma(\theta_2, \mathbf{v}_2) = < l_1, \mathbf{v}_2 >, \\ &-\omega^2 m(\mathbf{u}_2, \mathbf{v}_1) - \omega a(\mathbf{u}_1, \mathbf{v}_1) + c(\mathbf{u}_2, \mathbf{v}_1) - e(p_2, \mathbf{v}_1) - \\ &-\gamma(\theta_2, \mathbf{v}_1) + \omega t_1 \gamma(\theta_1, \mathbf{v}_1) = < l_2, \mathbf{v}_1 >, \\ &\omega\chi(p_2, \xi_1) + \omega e(\xi_1, \mathbf{u}_2) + z(p_1, \xi_1) + \omega\pi(\theta_2, \xi_1) - \omega^2 t_1 \pi(\theta_1, \xi_1) = < r_1, \xi_1 >, \\ &-\omega\chi(p_1, \xi_2) - \omega e(\xi_2, \mathbf{u}_1) + z(p_2, \xi_2) - \omega\pi(\theta_1, \xi_2) - \omega^2 t_1 \pi(\theta_2, \xi_2) = < r_2, \xi_2 >, \end{aligned} \tag{20}$$

$$\omega s(\theta_2,\eta_1+\omega t_1\eta_2)-\omega^2 t_0 s(\theta_1,\eta_1+\omega t_1\eta_2)+k(\theta_1,\eta_1+\omega t_1\eta_2)+$$
$$+\omega\pi(\eta_1+\omega t_1\eta_2,p_2)+\omega\gamma(\eta_1+\omega t_1\eta_2,\mathbf{u}_2))=<\mu_1,\eta_1+\omega t_1\eta_2>$$
$$-\omega s(\theta_1,\eta_2-\omega t_1\eta_1)-\omega^2 t_0 s(\theta_2,\eta_2-\omega t_1\eta_1)+k(\theta_2,\eta_2-\omega t_1\eta_1)-$$
$$-\omega\pi(\eta_2-\omega t_1\eta_1,p_1)-\omega\gamma(\eta_2-\omega t_1\eta_1,\mathbf{u}_1))=<\mu_2,\eta_2-\omega t_1\eta_1>.$$

Having added all the equations of the problem (20) we introduce the linear form $\chi_\omega:\mathbf{W}\to\Re$:

$$\begin{gathered}<\chi_\omega,\mathbf{w}>=-<l_2,\mathbf{v}_1>+\omega^{-1}[<r_1,\xi_1>+<\mu_1,\eta_1+\omega t_1\eta_2>]+\\ +<l_1,\mathbf{v}_2>+\omega^{-1}[<r_2,\xi_2>+<\mu_2,\eta_2-\omega t_1\eta_1>]\ \ \forall\ \mathbf{w}=(\mathbf{v}_1,\xi_1,\eta_1,\mathbf{v}_2,\xi_2,\eta_2)\in\mathbf{W}.\end{gathered}\tag{21}$$

and the bilinear form $\Pi_\omega:\mathbf{W}\times\mathbf{W}\to\Re$:

$$\begin{gathered}\Pi_\omega(\boldsymbol{\psi},\mathbf{w})=-\omega^2[m(\mathbf{u}_1,\mathbf{v}_2)-m(\mathbf{u}_2,\mathbf{v}_1)]+\omega[a(\mathbf{u}_1,\mathbf{v}_1)+a(\mathbf{u}_2,\mathbf{v}_2)]+\\
+[c(\mathbf{u}_1,\mathbf{v}_2)-c(\mathbf{u}_2,\mathbf{v}_1)]+[e(p_2,\mathbf{v}_1)-e(p_1,\mathbf{v}_2)+e(\xi_1,\mathbf{u}_2)-e(\xi_2,\mathbf{u}_1)]+\\
+[\gamma(\theta_2,\mathbf{v}_1)-\gamma(\theta_1,\mathbf{v}_2)+\omega t_1\gamma(\theta_1,\mathbf{v}_1)-\omega t_1\gamma(\theta_1,\mathbf{v}_1)+\\
+\gamma(\eta_1+\omega t_1\eta_2,\mathbf{u}_2)-\gamma(\eta_2-\omega t_1\eta_1,\mathbf{u}_1)]+\\
+[\pi(\theta_2,\xi_1)-\omega t_1\gamma(\theta_1,\xi_1)-\pi(\theta_1,\xi_2)-\omega t_1\gamma(\theta_2,\xi_2)+\\
+\pi(\eta_1+\omega t_1\eta_2,p_2)-\pi(\eta_2-\omega t_1\eta_1,p_1)]+\\
+[\chi(p_2,\xi_1)-\chi(p_1,\xi_2)]+\omega^{-1}[z(p_1,\xi_1)+z(p_2,\xi_2)]+\\
+\omega^{-1}[k(\theta_1,\eta_1+\omega t_1\eta_2)+k(\theta_2,\eta_2-\omega t_1\eta_1)]+\\
+[s(\theta_2,\eta_1+\omega t_1\eta_2)-\omega t_0 s(\theta_1,\eta_1+\omega t_1\eta_2)-s(\theta_1,\eta_2-\omega t_1\eta_1)-\omega t_0 s(\theta_2,\eta_2-\omega t_1\eta_1)]\\
\forall\ \boldsymbol{\psi}=(\mathbf{u}_1,p_1,\theta_1,\mathbf{u}_2,p_2,\theta_2)\in\mathbf{W}\ \forall\ \mathbf{w}=(\mathbf{v}_1,\xi_1,\eta_1,\mathbf{v}_2,\xi_2,\eta_2)\in\mathbf{W}.\end{gathered}\tag{22}$$

Then variational problem for forced harmonic vibrations of pyroelectric can be rewritten as follows:

$$\begin{cases}\text{given }\ \omega>0,\ t_1\ge t_0>0,<\chi_\omega,\mathbf{w}>\in\mathbf{W}'=\boldsymbol{\Phi}'\times\boldsymbol{\Phi}';\\ \text{find }\ \boldsymbol{\psi}=(\mathbf{u}_1,p_1,\theta_1,\mathbf{u}_2,p_2,\theta_2)\in\mathbf{W}=\boldsymbol{\Phi}\times\boldsymbol{\Phi}\text{ such that}\\ \Pi_\omega(\boldsymbol{\psi},\mathbf{w})=<\chi_\omega,\mathbf{w}>\ \ \forall\ \mathbf{w}=(\mathbf{v}_1,\xi_1,\eta_1,\mathbf{v}_2,\xi_2,\eta_2)\in\mathbf{W}.\end{cases}\tag{23}$$

## 4. WELL-POSEDNESS OF THE VARIATIONAL PROBLEM

Let us introduce a scalar product on the space $\mathbf{W}$ in the following way:

$$\begin{gathered}((\mathbf{y},\mathbf{w}))=\sum_{i=1}^{2}\left[a(\mathbf{u}_i,\mathbf{v}_i)+z(p_i,\xi_i)+k(\theta_i,\eta_i)\right]\\ \forall\ \mathbf{y}=(\mathbf{u}_1,p_1,\theta_1,\mathbf{u}_2,p_2,\theta_2)\in\mathbf{W},\ \forall\ \mathbf{w}=(\mathbf{v}_1,\xi_1,\eta_1,\mathbf{v}_2,\xi_2,\eta_2)\in\mathbf{W}.\end{gathered}\tag{24}$$

We also introduce a norm generated by the scalar product (24):

$$|||\,\mathbf{y}\,|||^2=(\mathbf{y},\mathbf{y})\ \forall\mathbf{y}\in\mathbf{W}.\tag{25}$$

Then we can easily notice the following estimations:

$$\begin{gathered}|\,\Pi_\omega(\mathbf{y},\mathbf{w})\,|\le M_1(\omega)\,|||\,\mathbf{y}\,|||\cdot|||\,\mathbf{w}\,|||,\\ M_1(\omega)=C\max\{\omega^{-1},1,\omega,\omega^2\},\ \forall\mathbf{y},\mathbf{w}\in\mathbf{W},\end{gathered}\tag{26}$$

and

$$\begin{gathered}|<\chi_\omega,\mathbf{w}>|\le M_2(\omega)\,\|\,\chi_\omega\,\|_*\cdot|||\,\mathbf{w}\,|||,\\ M_2(\omega)=C\max\{\omega^{-1},1\},\ \forall\mathbf{w}\in\mathbf{W}.\end{gathered}\tag{27}$$

Here and everywhere the symbol $C$ means a positive constant value, which is not dependent on solutions of variational problem (23).

Consider now the expression for $\Pi_\omega(\mathbf{w},\mathbf{w})$ in order to confirm that bilinear form $\Pi_\omega : \mathbf{W}\times\mathbf{W}\to\Re$ is $\mathbf{W}$-elliptic:

$$\begin{aligned}&\Pi_\omega(\mathbf{w},\mathbf{w}) = \omega[a(\mathbf{u}_1,\mathbf{u}_1)+a(\mathbf{u}_2,\mathbf{u}_2)]+\omega^{-1}[z(p_1,p_1)+z(p_2,p_2)]+\\&+\omega^{-1}[k(\theta_1,\theta_1)+k(\theta_2,\theta_2)]+\omega(t_1-t_0)[s(\theta_1,\theta_1)+s(\theta_2,\theta_2)]\geq\\&\geq\omega[a(\mathbf{u}_1,\mathbf{u}_1)+a(\mathbf{u}_2,\mathbf{u}_2)]+\omega^{-1}[z(p_1,p_1)+z(p_2,p_2)]+\\&+\omega^{-1}[k(\theta_1,\theta_1)+k(\theta_2,\theta_2)]\geq\alpha(\omega)\cdot|||\,\mathbf{w}\,|||^2,\\&\alpha(\omega)=\min\{\omega^{-1},\omega\}\ \forall\mathbf{w}\in\mathbf{W}.\end{aligned}\tag{28}$$

Since the statements (26)-(28) are held and they are actually the conditions of Lions-Lax-Milgram theorem, the following theorem is then correct:

**Theorem 4.1**. For each $\omega>0$ and $t_1\geq t_0>0$ the variational problem (23) has a unique solution $\boldsymbol{\psi}\in\mathbf{W}$, which satisfies the relation:

$$|||\,\boldsymbol{\psi}\,|||\leq\alpha^{-1}(\omega)M_2(\omega)\,\|\,\chi_\omega\,\|_*\,. \tag{29}$$

## 5. GALERKIN DISCRETIZATION

Standard Galerkin scheme implies looking for solution $\boldsymbol{\psi}\in\mathbf{W}$ of variational problem (23) in some finite-dimensional subspace $\mathbf{W}_h:=\boldsymbol{\Phi}_h\times\boldsymbol{\Phi}_h$, $\boldsymbol{\Phi}_h\subset\boldsymbol{\Phi}$, $\dim\mathbf{W}_h=N(h)<+\infty$. Thus, the Galerkin-discretized variational problem (23) looks in the following way:

$$\begin{aligned}&\text{given}\,\omega>0,\ \chi_\omega\in\mathbf{W}',\ \mathbf{W}_h\subset\mathbf{W},\dim\mathbf{W}_h<+\infty;\\&\text{find}\,\boldsymbol{\psi}_h=(\mathbf{u}_{1h},p_{1h},\theta_{1h},\mathbf{u}_{2h},p_{2h},\theta_{2h})\in\mathbf{W}_h\ \text{such that}\\&\Pi_\omega(\boldsymbol{\psi}_h,\boldsymbol{\varphi})=<\chi_\omega,\boldsymbol{\varphi}>\ \ \forall\,\boldsymbol{\varphi}\in\mathbf{W}_h.\end{aligned}\tag{30}$$

Since problem (23) is well-posed, the same thing we can say about problem (30). In the space $\mathbf{W}$ we select some basis functions $\{\mathbf{w}_i\}_{i=1}^\infty$. For each natural number $m\geq1$, $h=1/m$ a sequence of approximation spaces $\mathbf{W}_h$ and operators of orthogonal projection $Pr_h:\mathbf{W}\to\mathbf{W}_h$ are defined so that a set $\{\mathbf{w}_i\}_{i=1}^m$ is a basis of $\mathbf{W}_h$, and $((\boldsymbol{\psi}-Pr_h\boldsymbol{\psi},\mathbf{w}))=0\ \ \forall\,\boldsymbol{\psi}\in\mathbf{W},\forall\,\mathbf{w}_h\in\mathbf{W}_h$. Now variational problem (23) is replaced by a sequence of the following problems:

$$\begin{aligned}&\text{given}\,\omega>0,\ \chi_\omega\in\mathbf{W}',\ h>0,\ \mathbf{W}_h\subset\mathbf{W},\dim\mathbf{W}_h=m<+\infty;\\&\text{find}\,\boldsymbol{\psi}_h=(\mathbf{u}_{1h},p_{1h},\theta_{1h},\mathbf{u}_{2h},p_{2h},\theta_{2h})\in\mathbf{W}_h\ \text{such that}\\&\Pi_\omega(\boldsymbol{\psi}_h,\boldsymbol{\varphi})=<\chi_\omega,\boldsymbol{\varphi}>\ \ \forall\,\boldsymbol{\varphi}\in\mathbf{W}_h.\end{aligned}\tag{31}$$

**Theorem 5.1**. Let $\boldsymbol{\psi}\in\mathbf{W}$ be a solution of problem (23) with parameter $\omega>0$. Then a sequence of Galerkin approximations $\{\boldsymbol{\psi}_h\}\subset\mathbf{W}$ is unambiguously defined by the solutions of the problems (31) and has the following properties:

$$|||\,\boldsymbol{\psi}-\boldsymbol{\psi}_h\,|||\leq\alpha^{-1}M_1(\omega)\inf_{\mathbf{w}\in\mathbf{W}_h}|||\,\boldsymbol{\psi}-\mathbf{w}\,|||\ \forall\,h>0; \tag{32}$$

$$\lim_{h\to0}|||\,\boldsymbol{\psi}-\boldsymbol{\psi}_h\,|||=0. \tag{33}$$

*Proof*. The correctness of the inequality (32) is based on the fact that

$$\Pi_\omega(\boldsymbol{\psi}-\boldsymbol{\psi}_h,\mathbf{w})=0\ \forall\ \mathbf{w}\in\mathbf{W}_h, \tag{34}$$

and the estimation

$$\begin{aligned}&\alpha\,|||\,\boldsymbol{\psi}-\boldsymbol{\psi}_h\,|||^2\le\Pi_\omega(\boldsymbol{\psi}-\boldsymbol{\psi}_h,\boldsymbol{\psi}-\boldsymbol{\psi}_h)=\Pi_\omega(\boldsymbol{\psi}-\boldsymbol{\psi}_h,\boldsymbol{\psi}-\mathbf{w})\le\\&\le M_1(\omega)\,|||\,\boldsymbol{\psi}-\boldsymbol{\psi}_h\,|||\;|||\,\boldsymbol{\psi}-\mathbf{w}\,|||\ \forall\ \mathbf{w}\in\mathbf{W}_h.\end{aligned} \tag{35}$$

Taking into account the density of sequence of spaces $\{\mathbf{W}_h\}$ in the separable space $\mathbf{W}$:

$$\lim_{h\to 0}|||\,\mathbf{w}-Pr_h\mathbf{w}\,|||=0\ \forall\ \mathbf{w}\in\mathbf{W}. \tag{36}$$

Therefore, basing on the equality

$$\inf_{\mathbf{w}\in\mathbf{W}_h}|||\,\boldsymbol{\psi}-\mathbf{w}\,|||=|||\,\boldsymbol{\psi}-Pr_h\boldsymbol{\psi}\,||| \tag{37}$$

and inequality (32) we can conclude the correctness of (33), when $\omega>0$.

**Theorem 5.2**. on the convergence of FEM approximations. Let $\boldsymbol{\psi}\in\mathbf{W}$ be a solution of problem (23) and there exists a natural number $k\ge 1$ such that $\boldsymbol{\psi}\in\mathbf{W}\cap[H^{k+1}(\Omega)]^{2(d+1)}$. Let approximations $\boldsymbol{\psi}_h$ be defined by solving problem (31) in the spaces $\mathbf{W}_h\subset\mathbf{W}$, which are constructed with making use of piecewise-polynomial functions of FEM and have the following property:

for each $\boldsymbol{\varphi}\in\mathbf{W}\cap[H^{k+1}(\Omega)]^{2(d+1)}$, $k\ge 1$ there exists $\boldsymbol{\varphi}_h\in\mathbf{W}_h$ and $C=const>0$ such that $\|\,\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{m,\Omega}\le C{\cdot}h^{k+1-m}\|\,\boldsymbol{\varphi}\|_{k+1,\Omega}$, $0\le m\le k$, where $h$ is the diameter of finite element mesh and $k$ is the greatest degree of full polynomial of $d$ variables, which is precisely defined by base functions of $\mathbf{W}_h$ on each finite element.

Then the convergence of sequence $\boldsymbol{\varphi}_h\in\mathbf{W}_h$ is characterized by the estimation:

$$|||\,\boldsymbol{\psi}-\boldsymbol{\psi}_h\,|||\le C{\cdot}h^k\,\|\,\boldsymbol{\psi}\,\|_{k+1,\Omega}, \tag{38}$$

where $C=const>0$ is not dependent on values we are looking for.

*Proof*. The estimation (38) is implied from the inequality (32), the equivalence of norms $|||\cdot|||$ and $\|\cdot\|_{1,\Omega}$ on $\mathbf{W}$ and the density properties defined in the theorem`s body.

$$|||\,\boldsymbol{\psi}-\boldsymbol{\psi}_h\,|||\le\alpha^{-1}M_1(\omega)\inf_{\mathbf{w}\in\mathbf{W}_h}|||\,\boldsymbol{\psi}-\mathbf{w}\,|||\doteq\|\,\boldsymbol{\psi}-\mathbf{w}\|_{1,\Omega}\le C{\cdot}h^k\,\|\,\boldsymbol{\psi}\,\|_{k+1,\Omega}\,. \tag{39}$$

Let us now pay a deeper attention to the aforementioned selection of finite-dimensional subspace $\mathbf{W}_h\subset\mathbf{W}$. Taking into account the definition of $\mathbf{W}_h$, that is $\mathbf{W}_h=\mathbf{V}_h\times P_h\times Z_h\times\mathbf{V}_h\times P_h\times Z_h$, where

$$\begin{aligned}&\mathbf{V}_h\subset\mathbf{V},\ P_h\subset P,\ Z_h\subset Z,\\&\dim\mathbf{V}_h<+\infty,\ \dim P_h<+\infty,\ \dim Z_h<+\infty,\end{aligned} \tag{40}$$

we can write the expansions of solution amplitudes as following:

$$\begin{aligned}&\mathbf{u}_{\alpha h}=\sum_{i=0}^{N}\mathbf{U}_{\alpha i}\boldsymbol{\varphi}_i^{\mathrm{V}}(x),\ p_{\alpha h}=\sum_{i=0}^{N}\mathbf{P}_{\alpha i}\varphi_i^P(x),\\&\theta_{\alpha h}=\sum_{i=0}^{N}\boldsymbol{\Theta}_{\alpha i}\varphi_i^Z(x),\ \alpha=1,2,\end{aligned} \tag{41}$$

where $\boldsymbol{\varphi}_i^{\mathrm{V}}(x),\varphi_i^P(x),\varphi_i^Z(x)$ are the basis functions of spaces $\mathbf{V}_h,P_h,Z_h$ respectively. Then we obtain the system of linear equations for finding nodal values of the unknown amplitudes:

$$\begin{bmatrix} \omega\mathbf{A} & -\mathbf{B} & 0 & \mathbf{E}^T & -\omega t_1\mathbf{Y}^T & \mathbf{Y}^T \\ \mathbf{B} & \omega\mathbf{A} & -\mathbf{E}^T & 0 & -\mathbf{Y}^T & -\omega t_1\mathbf{Y}^T \\ 0 & \mathbf{E} & \omega^{-1}\mathbf{Z} & \mathbf{G} & -\omega t_1\mathbf{\Pi}^T & \mathbf{\Pi}^T \\ -\mathbf{E} & 0 & -\mathbf{G} & \omega^{-1}\mathbf{Z} & -\mathbf{\Pi}^T & -\omega t_1\mathbf{\Pi}^T \\ \omega t_1\mathbf{Y} & \mathbf{Y} & \omega t_1\mathbf{\Pi} & \mathbf{\Pi} & \mathbf{D} & -\mathbf{H} \\ -\mathbf{Y} & \omega t_1\mathbf{Y} & -\mathbf{\Pi} & \omega t_1\mathbf{\Pi} & \mathbf{H} & \mathbf{D} \end{bmatrix} \times \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{\Theta}_1 \\ \mathbf{\Theta}_2 \end{bmatrix} = \begin{bmatrix} -\mathbf{L}_2 \\ \mathbf{L}_1 \\ \omega^{-1}\mathbf{R}_1 \\ \omega^{-1}\mathbf{R}_2 \\ \omega^{-1}\mathbf{F}_1 + t_1\mathbf{F}_2 \\ \omega^{-1}\mathbf{F}_2 - t_1\mathbf{F}_1 \end{bmatrix}, \qquad (42)$$

where $\mathbf{B} = -\omega^2\mathbf{M} + \mathbf{C},\ \mathbf{D} = \omega^{-1}\mathbf{K} + \omega(t_1 - t_0)\mathbf{S},\ \mathbf{H} = -(1 + \omega^2 t_0 t_1)\mathbf{S} + t_1\mathbf{K}$.

Here the elements of the matrices and vectors are computed using the bilinear and linear forms defined in (16), for example $\mathbf{A} = \{a_{ij}\} = \{a(\boldsymbol{\varphi}_i^{\mathbf{V}}, \boldsymbol{\varphi}_j^{\mathbf{V}})\}$. The matrix of the system of equations (42) is positively defined, but not the symmetric one. More precisely, it can be represented as the sum of positively defined symmetric matrix and a skew-symmetric one.

## 6. NUMERICAL EXPERIMENTS

We consider a piezoelectric bar with length $L = 10^{-8}\,m$ made of PZT-4 ceramics. A harmonic heat loading with angular frequency $\omega = 3\cdot 10^6\,rad/s$ is applied to the right edge of the bar. So, the boundary conditions for thermal field are:

$$\theta_1(0) = 0\,K,\ \theta_2(0) = 0\,K,\ \bar{h}_1(L) = 100\,J\cdot m^{-2}s^{-1},\ \bar{h}_2(L) = 0\,J\cdot m^{-2}s^{-1}. \qquad (43)$$

On the left edge of the bar the boundary conditions for mechanical and electric fields are homogeneous and of Dirichlet type:

$$u_1(0) = 0\,m,\quad u_2(0) = 0\,m,\ p_1(0) = 0\,V,\ p_2(0) = 0\,V. \qquad (44)$$

On the right edge of the bar the boundary conditions for mechanical and electric fields are homogeneous and of Neumann type:

$$\bar{\sigma}_1(L) = 0\,N\cdot m^{-2},\ \bar{\sigma}_2(L) = 0\,N\cdot m^{-2},\ I_1(L) = 0\,A,\ I_2(L) = 0\,A. \qquad (45)$$

We take the coefficients of PZT-4 as in [16]:

$$\begin{aligned} &\rho = 7500\,kg/m^3, && c_v = 350\,J/kg\cdot K, \\ &\lambda = 1.1\,W/m\cdot K, && c = 115\times 10^9\,N/m^2, \\ &e = 15.1\,C/m^2, && \pi = 2.7\times 10^{-4}\,C/K\cdot m^2, \\ &\chi = 6.46\times 10^{-9}\,C^2/N\cdot m^2, && \alpha = 3.13\times 10^{-5}\,K^{-1}. \end{aligned} \qquad (46)$$

Also we take $z = 5\times 10^{-12}\,\Omega^{-1}\cdot m^{-1}$, $a = 40\,m^2\cdot s^{-1}$ and $T_0 = 298\,K$. Unfortunately, the exact values of relaxation time parameters $t_0$, $t_1$ are unknown for majority of materials, PZT-4 ceramics inclusive. However, it is experimentally determined that these values can vary between $10^{-10}\,s$ for gases and $10^{-14}\,s$ for metals. To demonstrate the effective influence of $t_0$ and $t_1$ on solutions, we will perform a set of numerical experiments with such pairs of relaxation time parameters:

$$\begin{aligned} &t_1 = 10^{-10}\,s, && t_0 = 10^{-11}\,s, \\ &t_1 = 6\cdot 10^{-11}\,s, && t_0 = 5\cdot 10^{-11}\,s, \\ &t_1 = 3\cdot 10^{-11}\,s, && t_0 = 2\cdot 10^{-11}\,s, \\ &t_1 = 10^{-11}\,s, && t_0 = 10^{-12}\,s. \end{aligned} \qquad (47)$$

For discretization by spatial variable we divide the interval $[0, L]$ into $N = 256$ finite elements with piecewise linear solution approximations on them.

The experiments show that almost all the solution amplitudes do not depend on relaxation times $t_0$ and $t_1$ and practically coincide with the solution amplitude of the classical thermopiezoelectricity problem. The only difference is in the values of temperature increment amplitude $\theta_2$, which is shown on Figure 1. Concerning Fig. 1, it is worth noting that amplitude $\theta_2$ at relaxation time parameters $t_1 = 10^{-11}\, s$ and $t_0 = 10^{-12}\, s$ coincides with the solution of classical thermopiezoelectricity forced vibration problem and making $t_0$ and $t_1$ even lesser does not give any effect on the solution amplitude $\theta_2$.

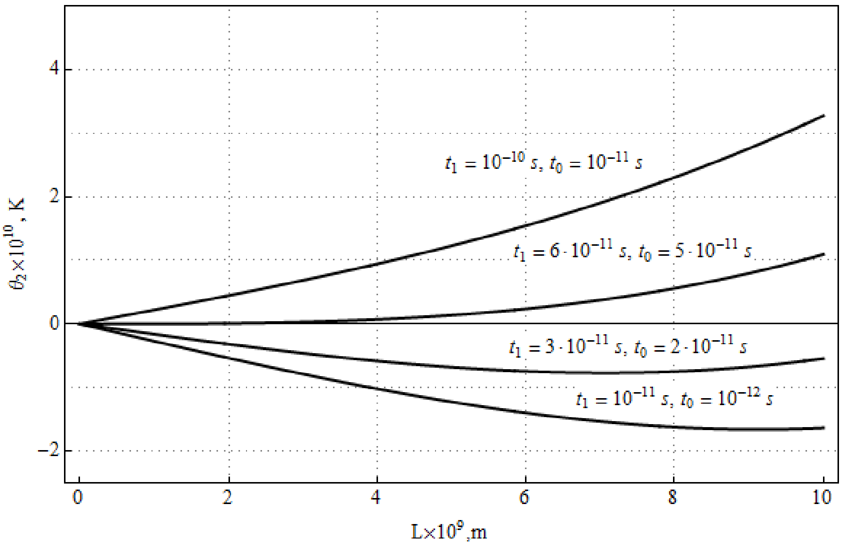


Fig. 1. Temperature increment amplitude $\theta_2$ depending on relaxation times $t_0$ and $t_1$

## 7. CONCLUSIONS

The harmonic vibrations of the pyroelectric materials have been studied under generalized Green-Linsay thermopiezoelectricity theory. We have formulated the variational problem for this special case and proved its well-posedness. Then this special variational problem has been discretized using standard Galerkin-method. The bases of approximation spaces of the discretized problem have been constructed using finite element method. The rate of convergence of FEM-approximations has been studied theoretically. The numerical experiment of applying a harmonic heat loading to the pyroelectric bar has been set up and studied. The results of the experiment showed that in some special cases the "relaxation time"-parameters have a significant influence on the nodal values of solution amplitudes.

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## СКІНЧЕННОЕЛЕМЕНТНИЙ АНАЛІЗ ГАРМОНІЙНИХ ХВИЛЬ У МОДЕЛІ ТЕРМОП'ЄЗОЕЛЕКТРИКИ ГРІНА-ЛІНДСЕЯ

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Використовуючи модель термоп'єзоелектрики Гріна-Ліндсея (G-L) з так званими параметрами релаксації, які впливають на характер взаємодії механічного, електричного та теплового полів у піроелектриках, подано формулювання початково-крайової і відповідної їй варіаційної задач цієї моделі в термінах вектора пружних зміщень, електричного потенціалу, приросту температури. Далі, за допущенням щодо гармонійного навантаження із відомою круговою частотою, побудовано змішану варіаційну задачу про амплітуди гармонійних хвиль у піроелектрику і за придатних для практичних застосувань умов доведено існування єдиного стійкого розв'язку розглядуваної задачі. Після цього згадана задача дискретизується схемою Гальоркіна зі стандартними для просторів Соболєва базисними функціями методу скінченних елементів (МСЕ). В нормах цих самих просторів знайдено апріорні оцінки похибок апроксимацій МСЕ, які виявляють залежність швидкості їхньої збіжності від порядку поліноміальних базисних функцій та від запасу регулярності шуканого розв'язку. Наведено результати числового експерименту, який демонструє вплив значень параметрів релаксації Гріна-Ліндсея на характеристику гармонійних хвиль, які виникають у піроелектричному стрижні, що піддається тепловому навантаженню заданої частоти.

*Ключові слова*: піроелектрик, термоп'єзоелектрика, модель Гріна-Ліндсея, початково-крайова задача, варіаційна задача, дискретизація Гальоркіна, метод скінченних елементів.