BACKWARD HEAT EQUATION SOLUTION
VIA LIE-ALGEBRAIC DISCRETE APPROXIMATIONS

Adriana Kindybaliuk[1], Arkadii Kindybaliuk[2], M. Prytula[3]

[1]Kyiv National University of construction and architecture,
Povitroflotsky Avenue 31, Kyiv, 03680,
[2,3]Shape, Poliova 21, Kyiv, 03056, kindybaluk.arkadii@outlook.com
[3]Ivan Franko National University of Lviv,
Universytetska Str., 1, Lviv, 79000, e-mail: mykola.prytula@gmail.com

We consider numerical solving the Cauchy problem for backward heat equation via the generalized method of Lie-algebraic discrete approximations. Discretization of the equation is performed by all variables in equation leading to a factorial rate of convergence in the case of quasi representations for differential operator are built by means of Lagrange interpolation. The rank of a finite dimensional operator is determined and approximation properties are investigated. Error estimations and the factorial rate of convergence are proven.

Ключові слова: generalized method of Lie algebraic discrete approximations, backward heat equation, finite dimensional quasi representation, Lagrange's polynomial, factorial convergence.

1. INTRODUCTION

Backward heat equation has applications in different fields: image processing, signal processing, eliminating of diffusion. Hence effective numerical solution is an important problem besides the variety of different approaches [16]. In this paper we propose a numerical scheme built via generalized method of Lie-algebraic discrete approximations for backward heat equation [20] and prove the factorial rate of convergence of numerical scheme.

The generalized method of Lie algebraic method was introduced in [3] and developed in [4, 18, 19], which is based on classic Lie algebraic method of discrete approximations. The history of classical approach, open questions in this field and further development guidelines are analyzed in [1]. Key findings of classical approach may be found in [6, 7, 9 – 16, 21, 22].

The main problem analyzed in these papers is the Cauchy problem for evolution equation which is considered in a bounded domain $\Omega := (a, b) \subset R$ with time limit $T < +\infty$ and cylinder $Q_T = \Omega \times (0, T]$:

$$\begin{cases} \text{find function } u = u(x_1, x_2, \ldots, x_q, t) \text{ such, that} \\ u_t = K(t, x; \partial)u + f(x, t), \ x \in \Omega \subset R^q, \ t > 0, \\ u\big|_{t=0} = \varphi(x) \in B, \end{cases} \quad (1)$$

where B denotes some functional Banach space, linear operator K is assumed to be a formal polynomial of elements from the Lie algebra $\{x, \partial/\partial x, 1\}$ and can be represented as

$$K = a_k \frac{\partial^k}{\partial x^k} + a_{k+1} \frac{\partial^{k+1}}{\partial x^{k+1}} + \ldots + a_{k+p} \frac{\partial^{k+p}}{\partial x^{k+p}},$$

where $a_{k+i} \in R$ for all $i = \overline{0, p}$ and $a_k \neq 0$ and $k \geq 1$. Similarly, as in Calogero's method, the Heisenberg-Weyl algebra $G = \overset{q}{\underset{j=1}{\oplus}} \{x_j, \partial / \partial x_j, 1\}$ has been used as a basic algebraic tool for constructing the corresponding discrete approximations $X_j^{(n)}, Z_j^{(n)}, I^{(n)} \in \overset{q}{\underset{j=1}{\otimes}} R^{n_j}$.

Using q-dimensional Lagrange interpolation scheme problem (1) is reduced to the Cauchy problem in the following form:

$$\begin{cases} \text{find function } u_{(n)} = u_{(n)}(t) \text{ such, that} \\ \dfrac{du_{(n)}}{dt} = K_{(n)} u_{(n)} + f_{(n)} \text{ , } t > 0, \\ u_{(n)}\big|_{t=0} = \varphi_{(n)} \in B_{(n)} \end{cases} \tag{2}$$

where $K_{(n)}$ denotes finite dimensional quasi representation of differential operator K, and $B_{(n)}$ denotes finite dimensional space of approximations. System (2) is solved by means of Euler's or Runge-Kutta's method [1, 7, 9, 12].

Since reduced problem (2) is solved making use of some numerical algorithm the rate of time convergence is constrained by the convergence rate of the method based on, hence Lie-algebraic discrete approximations for spacial variables rate of convergence is factorial [7, 9].

This restriction led to development of the Generalized Method of Lie-algebraic discrete approximations proposed in [3], convergence rate for the time variable becomes factorial [18].

The main idea of Generalized Method of Lie-algebraic discrete approximations is the following. We take the Banach spaces $V = C_{x,t}^{k+p,1}(Q_T) \cap C(\overline{Q}_T)$, $C = C(Q_T)$ and formulate the Cauchy problem

$$\begin{cases} \text{find function } u = u(x,t) \in V \text{ such, that} \\ u_t = Ku + f, \; \forall (x,t) \in Q_T, \\ u\big|_{t=0} = \varphi \in V, \end{cases} \tag{3}$$

where $\varphi = \varphi(x) \in V$ denotes initial conditions, $f = f(x,t) \in C$ represents internal sources and K denotes the differential operator of a problem.

According to [3, 4] we introduce substitution $u(x,t) = v(x,t) + \varphi(x)$ into (3) which leads to considering the auxiliary Cauchy problem with homogeneous initial condition

$$\begin{cases} \text{find function } v = v(x,t) \in V \text{ such, that} \\ v_t = Kv + K\varphi + f, \; \forall (x,t) \in Q_T, \\ v\big|_{t=0} = 0. \end{cases} \tag{4}$$

The idea of such substitution is to reduce the computation effort with preserved accuracy which was demonstrated in the case of boundary value problem for elliptic equation [2].

The solution of problem (4) we seek in the subspace of such functions which are homogeneous at the initial moment of time: $B = \{v \in V : v\big|_{t=0} = 0\}$.

Adriana Kindybaliuk, Arkadii Kindybaliuk, M. Prytula

70   ISSN 2078–5097. Вісн. Львів. ун-ту. Сер. прикл. матем. та інф. 2017. Вип. 25

Denoting the structure elements in (4) by

,

we obtain a problem for operator equation:

$$\begin{cases} \text{for given operator } A : B \to C \text{ and element } \tilde{f} \in C, \\ \text{find element } v \in B \text{ such, that } Av = \tilde{f}. \end{cases}$$

The Cauchy problem was reduced into the problem for the operator equation. This operator equation was solved by means of the generalized method of Lie algebraic discrete approximations and there is proven its convergence for some particular cases [3, 18, 19]. We focus on application of the scheme to Cauchy problem for backward heat equation in this paper.

2. PROBLEM FORMULATION

Considering a bounded domain $\Omega := (a, b) \subset R$, time limit $T < +\infty$, cylinder $Q_T = \Omega \times (0, T]$ we take the Banach spaces $V = C_{x,t}^{2,1}(Q_T) \cap C(\overline{Q_T})$, $C = C(Q_T)$ and formulate the Cauchy problem

$$\begin{cases} \text{find function } u = u(x,t) \in V \text{ such, that} \\ u_t = -au_{xx} + f, \ \forall (x,t) \in Q_T, \\ u\big|_{t=0} = \varphi \in V, \end{cases} \quad (5)$$

where $a > 0$ denotes heat transition parameter, $\varphi = \varphi(x) \in V$ denotes initial conditions and $f = f(x,t) \in C$ represents internal sources. This problem is equivalent to

$$\begin{cases} \text{find function } u = u(x,t) \in V \text{ such, that} \\ u_t = au_{xx} + f, \ \forall (x,t) \in Q_T, \\ u\big|_{t=T} = \varphi \in V. \end{cases}$$

According to [3, 4] we introduce substitution $u(x,t) = v(x,t) + \varphi(x)$ into (5) which leads to considering the auxiliary Cauchy problem with homogeneous initial condition

$$\begin{cases} \text{find function } v = v(x,t) \in V \text{ such, that} \\ v_t = -av_{xx} - a\varphi'' + f, \ \forall (x,t) \in Q_T, \\ v\big|_{t=0} = 0. \end{cases} \quad (6)$$

The solution of problem (6) we seek in the subspace of such functions which are homogeneous at the initial moment of time: $B = \left\{ v \in V : v\big|_{t=0} = 0 \right\}$.

Denoting the structure elements in (6) by

$$A := \partial / \partial t + a \ \partial^2 / \partial x^2, \ \tilde{f} = -a\varphi'' + f \in C(Q_T),$$

we obtain a problem for operator equation:

$$\begin{cases} \text{for given operator } A : B \to C \text{ and element } \tilde{f} \in C \\ \text{find element } v \in B \text{ such, that } Av = \tilde{f}. \end{cases} \quad (7)$$

This operator equation is in the form which allows us using the generalized Lie algebraic approach. First of all we construct numerical scheme in third chapter, approximation properties are investigated in fourth chapter and finally in fifth chapter there

is given a proof of the factorial convergence of the built numerical scheme. Multidimensional generalization of the numerical scheme is covered in sixth and seventh chapters. Numerical example is provided in eighth chapter.

3. NUMERICAL SCHEME AND UNIQUENESS OF DISCRETE SOLUTION

Let n_x denotes the count of nodes in domain Ω and n_t denotes count of nodes in interval $[0,T]$. Set of nodes we denote $Q_{T,h}$.

For every variable we construct a set of Lagrange polynomials, which satisfy property $l_j(x_i) = \delta_{ij}$ and $l_j(t_i) = \delta_{ij}$, where δ_{ij} denotes Kronecker symbol.

According to the Weierstrass approximation theorem the set of polynomials with real-valued coefficients is dense set in the space of continuous real-valued functions. Choosing $l(t) = \left\{ l_j(t) \right\}_{j=2}^{n_t}$ we obtain system of polynomials without polynomial associated with initial moment of time. Its easy verifies that $\forall j = \overline{2,n_t}$, $l_j(t)\big|_{t=0} = 0$ and $l(t) \in B$, moreover basis functions $l(t) \otimes l(x) \in B$ are linearly independent, hence system of these functions create basis for approximation spaces $B_h \subset B$, $C_h \subset C$. Thus, we seek the solution as a Lagrange interpolation in the following form

$$v_h(x,t) = \sum_{j_t=2}^{n_t} \sum_{j_x=1}^{n_x} v_j l_{j_x}(x) l_{j_t}(t) = \bar{v}\big(l(t) \otimes l(x) \big),\tag{8}$$

where h denotes the discretization parameter, j_t and j_x are indexes of nodes by corresponding variables, j denotes the unique number of the node $j = (j_t - 1)n_x + j_x$ and $\bar{v}$ denotes the set of values $\bar{v} = \left\{ v_j \right\}_{j=n_x+1}^{n_x, n_t}$.

Substitution (8) into equation (7) leads to $Av_h = \tilde{f}$ and further using of calculations yields

$$\big(l'(t) \otimes l(x) - l(t) \otimes l''(x) \big) v_h = \tilde{f}_h.\tag{9}$$

Taking $i_x = \overline{1,n_x}$ and $i_t = \overline{2,n_t}$ in (9) we obtain a system of linear algebraic equations

$$\big(Z_t \otimes I_x - I_t \otimes Z_x^2 \big) \bar{v} = \tilde{f}(x_{i_x}, t_{i_t}),\ i_x = \overline{1,n_x},\ i_t = \overline{2,n_t}.$$

Denoting $A_h = Z_t \otimes I_x - I_t \otimes Z_x^2$, and $\tilde{f}_h = \tilde{f}(x_{i_x}, t_{i_t})$, $i_x = \overline{1,n_x}$, $i_t = \overline{2,n_t}$ we obtain discrete formulation of operator problem

$$\begin{cases} \text{for given operator } A_h : B_h \to C_h \text{ and element } \tilde{f}_h \in C_h \\ \text{find element } v_h \in B_h \text{ such, that } A_h v_h = \tilde{f}_h, \end{cases}\tag{10}$$

where matrices of corresponding finite dimensional quasi representations have been built upon these rules

$$Z_{t,ij} = l'_j(t_i), Z_{x,ij}^2 = (l''_j)(x_i), I_{t,ij} = l_j(t_i), I_{x,ij} = l_j(x_i).$$

According to theorem determining the rank of finite dimensional quasi representations [13] we obtain

$$rank\big(Z_t \big) = n_t - 1, rank\big(I_t \big) = n_t - 1, rank\big(Z_x^2 \big) = n_x - 2,\ rank\big(I_x \big) = n_x.$$

Using property of tensor product we verify that

$$rank(Z_t \otimes I_x) = n_x(n_t - 1), rank(I_t \otimes K_x) = (n_t - 1)(n_x - 2).$$

The rank of whole matrix A_h remains an open question, and further lemmas give an answer to this question.

Lemma 3.1. Matrix $(Z_t^{-1} \otimes Z_x^2)$ is nilpotent.

Proof. Since finite dimensional quasi representation of operator d^2/dx^2 has the form $Z_x^2 = Z_x Z_x$ and matrix Z_x^2 is nilpotent, hence $\exists m \in N, \forall n \geq m : (Z_x^2)^n = 0$.

The property of tensor product [5, 13] $\forall n \in N \; (A \otimes B)^n = A^n \otimes B^n$ yields

$$\exists m \in N, \forall n \geq m : (Z_t^{-1} \otimes Z_x^2)^n = (Z_t^{-1})^n \otimes (Z_x^2)^n = (Z_t^{-1})^n \otimes (Z_x)^{2n} = 0,$$

hence matrix $(Z_t^{-1} \otimes Z_x^2)$ is nilpotent.

Lemma 3.2. Matrix $(I_t \otimes I_x + a Z_t^{-1} \otimes Z_x^2)$ has an inverse matrix and its rank is $(n_t - 1)n_x$ i.e. has full rank.

Proof. Let us rewrite matrix $(I_t \otimes I_x + a Z_t^{-1} \otimes Z_x^2)^{-1}$ as a formal series:

$$(I_t \otimes I_x + a Z_t^{-1} \otimes Z_x^2)^{-1} = a \sum_{n=0}^{+\infty} (-1)^n (Z_t^{-1} \otimes Z_x^2)^n = a \sum_{n=0}^{+\infty} (-1)^n (Z_t^{-1})^n \otimes (Z_x^2)^n.$$

Since matrices $Z_x^2, (Z_t^{-1} \otimes Z_x^2)$ are nilpotent (Lemma 3.1) we obtain that the inverse matrix exists because of the existence of finite expansion

$$(I_t \otimes I_x + a Z_t^{-1} \otimes Z_x^2)^{-1} = a \sum_{n=0}^{m-1} (-1)^n (Z_t^{-1} \otimes Z_x^2)^n = a \sum_{n=0}^{m-1} (-1)^n (Z_t^{-1})^n \otimes (Z_x^2)^n.$$

However matrix has $n_x(n_t - 1)$ rows and columns and has an inverse matrix therefore it has full rank: $rank(I_t \otimes I_x - Z_t^{-1} \otimes K_x) = n_x(n_t - 1)$.

Using these lemmas we can prove the next theorem.

Theorem 3.1. The rank of finite dimensional quasi representation A_h of operator A has full rank and its rank is $n_x(n_t - 1)$ and there exists a unique solution of discrete problem (10).

Proof. Let us rewrite

$$A_h = Z_t \otimes I_x + a I_t \otimes Z_x^2 = (Z_t \otimes I_x)(I_t \otimes I_x + a Z_t^{-1} \otimes Z_x^2).$$

However $rank(Z_t \otimes I_x) = n_x(n_t - 1)$ and due to lemma 3.2

$$rank(I_t \otimes I_x + a Z_t^{-1} \otimes Z_x^2) = n_x(n_t - 1)$$

using property for two square matrices A, B: $rank(AB) = \min\{rank(A), rank(B)\}$, yields

$$rank(Z_t \otimes I_x + a I_t \otimes Z_x^2) = n_x(n_t - 1).$$

Since matrix has full rank then a unique solution of the problem (10) exists there.

4. APPROXIMATION PROPERTIES OF NUMERICAL SCHEME

According to construction of finite dimensional quasi representation of the operator it can be verified that $(Av - A_h v)_i = (Av(M) - Av_I(M))|_{M=M_i}$, where v_I denotes Lagrange interpolant and $M_i = (x_{i_x}, t_{i_t})$ denotes node from $Q_{T,h}$.

Let the dimension of finite dimensional subspaces B_h, C_h be $\dim B_h = \dim C_h = N_h$. In these spaces we can define the norm in a similar way as it was proposed in [5] namely

$$\|v\|_{B_h} = \|v\|_{C_h} = \sqrt{\frac{1}{N_h}\sum_{j=1}^{N_h} v_j^2} \; .$$

Assume that $v \in W^{n_x n_t, \infty} = \left\{ v : Q_T \to R : D^a v \in L^\infty(Q_T), \forall |a| \le n_x n_t \right\}$ which means that all possible derivatives till order $n_x n_t$ are bounded.

The residual of Lagrange interpolation polynomial can be written in the following form:

$$v(x,t) - v_I(x,t) = \frac{\omega_{n_x}(x)}{(n_x)!}\frac{\partial^{n_x} v(\xi,t)}{\partial x^{n_x}} + \frac{\omega_{n_t}(t)}{(n_t)!}\frac{\partial^{n_t} v(x,\eta)}{\partial t^{n_t}} - \frac{\omega_{n_x}(x)}{(n_x)!}\frac{\omega_{n_t}(y)}{(n_t)!}\frac{\partial^{n_x+n_t} v(\xi_1,\eta_1)}{\partial x^{n_x}\partial t^{n_t}} \; .$$

where

$$\omega_{n_x}(x) = \prod_{j=1}^{n_x}(x - x_i), \; \omega_{n_y}(y) = \prod_{j=1}^{n_y}(y - y_i) \text{ and } \xi, \xi_1 \in \Omega, \; \eta, \eta_1 \in (0,T) \; .$$

Theorem 4.1. Finite dimensional quasi representation A_h of the operator A approximates the operator A on element $v \in B$ and error estimation of approximation has the following form

$$\|Av - A_h v\|_{C_h} \le \ln(n_t)\left(\frac{1}{n_t - 1}\right)^{n_t - 1}\left\|\frac{\partial^{n_t} v}{\partial t^{n_t}}\right\|_\infty + a\left(\ln(n_x)\ln(n_x - 1)\right)\left(\frac{1}{n_x - 1}\right)^{n_x - 2}\left\|\frac{\partial^{n_x} v}{\partial x^{n_x}}\right\|_\infty \; . \quad (11)$$

Proof. Since the norm of space C_h is vector norm then according to the construction of finite quasi representation A_h of operator A it can verified that

$$(Av - A_h v)_i = (Av(M) - Av_I(M))\big|_{M=M_i} \; ,$$

where v_I denotes Lagrange interpolant and $M_i = (x_{i_x}, t_{i_t})$ denotes node from $Q_{T,h}$. Using the definition of the norm in space C_h after calculations we obtain

$$\|Av - A_h v\|_{C_h} \le \|Av - Av_I\|_\infty = \|A(v - v_I)\|_\infty \; .$$

Acting with operator A on residual of Lagrange polynomial we obtain

$$A(v(x,t) - v_I(x,t)) \approx a\frac{\omega_{n_x}^2(x)}{(n_x)!}\frac{\partial^{n_x} v}{\partial x^{n_x}} + \frac{\omega_{n_t}'(t)}{(n_t)!}\frac{\partial^{n_t} v}{\partial t^{n_t}} \; .$$

Estimation

$$\left|\omega_{n_t}'(t)\right| \le (n_t)!\ln(n_t)\left(\frac{1}{n_t - 1}\right)^{n_t - 1}, \left|\omega_{n_x - 2}(x)\right| \le (n_x)!\left(\ln(n_x)\ln(n_x - 1)\right)\left(\frac{1}{n_x - 1}\right)^{n_x - 2}$$

and $v \in W^{n_x n_t, \infty}(Q_t) \cap B$ yields

$$\|Av - Av_I\|_\infty \le \ln(n_t)\left(\frac{1}{n_t - 1}\right)^{n_t - 1}\left\|\frac{\partial^{n_t} v}{\partial t^{n_t}}\right\|_\infty + a\left(\ln(n_x)\ln(n_x - 1)\right)\left(\frac{1}{n_x - 1}\right)^{n_x - 2}\left\|\frac{\partial^{n_x} v}{\partial x^{n_x}}\right\|_\infty \; ,$$

and finally (11) can be obtained.

5. CONVERGENCE AND ERROR ESTIMATIONS

According to the Kantorovich convergence theorem [8] of abstract approximation scheme $\lim_{h \to 0}\|v - v_h\|_B = 0$ holds if

Adriana Kindybaliuk, Arkadii Kindybaliuk, M. Prytula

74   ISSN 2078–5097. Вісн. Львів. ун-ту. Сер. прикл. матем. та інф. 2017. Вип. 25

1. there exists a unique solution of equation $Av = \tilde{f}$,

2. for all operators approximating A_h operator A exist inverse bounded operators,

3. operator A_h approximates operator A on element $v \in B$: $\lim\limits_{h \to 0} \|Av - A_h v\|_{C_h} = 0$.

The first requirement can be easily verified and the third requirement has been already satisfied in theorem 2, thus we should prove the second requirement.

Theorem 5.1. If finite dimensional quasi representation A_h of operator A has a full rank and is the same as finite dimensional subspace B_h then the bounded inverse operator exists i.e.:

$$\forall A_h, \exists M > 0, \exists A_h^{-1} : \left\| A_h^{-1} \right\| \le M < +\infty . \tag{12}$$

Proof. Although the norm satisfies the axiom of positivity, we obtain $\|A_h v\|_{C_h} \ge 0,\ \forall v \in D(A_h)$ and $\|A_h v\|_{C_h} = 0 \Leftrightarrow A_h v = 0_{C_h}$.

Let $\|A_h v\|_{C_h} = 0$, then $A_h v = 0_{C_h}$. For $v \ne 0_{B_h}$ it is possible if $\det A_h = 0$ and $rank A_h < \dim B_h$. However A_h : $rank A_h = \dim B_h$ then $A_h v = 0_{C_h}$ is possible if $v = 0_{B_h}$ only. Thus $\forall v \in D(A_h) \setminus \{0_{B_h}\} : \|A_h v\|_{C_h} > 0$. Since values $\|A_h v\|_{C_h} > 0$ and $\|v\|_{B_h} > 0$ are strictly positive for $\forall v \in D(A_h) \setminus \{0_{B_h}\}$ then there exists such constant $\mu > 0$ such that $\|A_h v\|_{C_h} \ge \mu \|v\|_{B_h}$.

According to the theorem of existence of bounded inverse operator [5] setting $M = \dfrac{1}{\mu} > 0$ we obtain (12).

Theorem 5.2. If $\lim\limits_{h \to 0} \left\| \tilde{f} - \tilde{f}_h \right\|_{C_h} = 0$ holds and conditions of theorems 3.1, 4.1, 5.1 are satisfied then $\lim\limits_{h \to 0} \|u - u_h\|_{B_h} = 0$ i.e. numerical scheme (10) is convergent and

$$\|u - u_h\|_{B_h} \le M\left(\ln(n_t) \left(\frac{1}{n_t - 1} \right)^{n_t - 1} \left\| \frac{\partial^{n_t} v}{\partial t^{n_t}} \right\|_\infty + |a_\alpha| \left(\prod_{k=0}^{\alpha-1} (\ln(n_x - k)) \right) \left(\frac{1}{n_x - 1} \right)^{n_x - \alpha} \left\| \frac{\partial^{n_x} v}{\partial x^{n_x}} \right\|_\infty \right) \tag{13}$$

Proof. Let us consider $\|v - v_h\|_{B_h}$:

$$\|v - v_h\|_{B_h} = \left\| A_h^{-1} A_h (v - v_h) \right\|_{B_h} \le \left\| A_h^{-1} \right\| \left\| A_h (v - v_h) \right\|_{C_h} .$$

Since the inverse operator is bounded (12) then

$$\left\| A_h^{-1} \right\| \left\| A_h (v - v_h) \right\|_{C_h} \le M \left\| A_h (v - v_h) \right\|_{C_h} .$$

Let us estimate $\left\| A_h (v - v_h) \right\|_{C_h}$:

$$\left\| A_h (v - v_h) \right\|_{C_h} = \left\| A_h v - Av + Av - A_h v_h \right\|_{C_h} \le \left\| A_h v - Av \right\|_{C_h} + \left\| Av - A_h v_h \right\|_{C_h} .$$

Since $Av = \tilde{f}$ and $A_h v_h = \tilde{f}_h$, we obtain

$$\left\| A_h (v - v_h) \right\|_{C_h} \le \left\| A_h v - Av \right\|_{C_h} + \left\| \tilde{f} - \tilde{f}_h \right\|_{C_h} .$$

Finally we obtain

$$\left\| v - v_h \right\|_{B_h} \le M \left(\left\| A_h v - A v \right\|_{C_h} + \left\| \tilde{f} - \tilde{f}_h \right\|_{C_h} \right)$$

and

$$\lim_{h \to 0} \left\| v - v_h \right\|_{B_h} \le M \left(\lim_{h \to 0} \left\| A_h v - A v \right\|_{C_h} + \lim_{h \to 0} \left\| \tilde{f} - \tilde{f}_h \right\|_{C_h} \right) = 0,$$

thus $\lim_{h \to 0} \left\| v - v_h \right\|_{B_h} = 0$.

Substitution leading to homogeneous initial condition $u = v + \varphi$ implies $u_h = v_h + \varphi$ which yields the estimation $\left\| u - u_h \right\| = \left\| v + \varphi - v_h - \varphi \right\| = \left\| v - v_h \right\|$. However $\tilde{f}_h$ is Lagrange approximation of $\tilde{f}$, error estimation of $\left\| \tilde{f} - \tilde{f}_h \right\|_{B_h}$ has the following form:

$$\left\| \tilde{f} - \tilde{f}_h \right\|_{B_h} \le \left(\frac{1}{n_t - 1} \right)^{n_t} \left\| \frac{\partial^{n_t} \tilde{f}}{\partial t^{n_t}} \right\|_\infty + \left(\frac{1}{n_x - 1} \right)^{n_x} \left\| \frac{\partial^{n_x} \tilde{f}}{\partial x^{n_x}} \right\|_\infty.$$

Since (11) and $\left(\dfrac{1}{n_t - 1} \right)^{n_t}$ tends to zero faster than $\left(\dfrac{1}{n_t - 1} \right)^{n_t - 1}$ when $n_t \to \infty$, then

neglecting terms with $\left(\dfrac{1}{n_t - 1} \right)^{n_t}$ and $\left(\dfrac{1}{n_x - 1} \right)^{n_x}$ yields to error estimation (13).

6. THE CAUCHY PROBLEM FOR BACKWARD HEAT EQUATION IN SEVERAL DIMENSIONS

The results obtained in previous sections can be generalized in natural way for a multidimensional case. Let us consider q-dimensional bounded domain $\Omega := (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_q, b_q) \subset R^q$, time limit $T < +\infty$ and cylinder $Q_T = \Omega \times (0, T]$. Let $diam \Omega_i$ denote the length of the range (a_i, b_i). We assume, that linear differential operator K is formal polynomial of elements from Lie algebra $\bigoplus_{i=1}^{q} \{ x_j, \partial / \partial x_j, 1 \}$ [3, 10]. Let a_i denotes coefficient standing by second derivative by variable x_i.

Let operator K has the following representation

$$K = -\sum_{i=1}^{q} \left(a_i \frac{\partial^2}{\partial x_i} \right), \; a_i > 0, \, i = \overline{1, q}$$

and let us consider Banach spaces $V = C_{x_1, \ldots x_d, t}^{2, \ldots 2, 1}(Q_T) \cap C(\overline{Q}_T)$, $C = C(Q_T)$ and formulation of the Cauchy problem with linear backward heat equation [20] is given below

$$\begin{cases} \text{find function } u = u(x_1, \ldots, x_q, t) \in V \text{ such, that} \\ u_t = Ku + f, \; \forall (x_1, \ldots, x_q, t) \in Q_T, \\ u \big|_{t=0} = \varphi \in V, \end{cases} \tag{14}$$

where $\varphi = \varphi(x_1, \ldots, x_d) \in V$ denotes initial conditions and $f = f(x_1, \ldots, x_q, t) \in C$ represents internal sources and

According to [3, 19] we introduce substitution

$$u(x_1,\ldots,x_q,t) = v(x_1,\ldots,x_q,t) + \varphi(x_1,\ldots,x_q)$$

into (14) which leads to considering an auxiliary Cauchy problem with homogeneous initial condition:

$$\begin{cases} \text{find function } v = v(x_1,\ldots,x_q,t) \in V \text{ such, that} \\ v_t = Kv + K\varphi + f, \ \forall (x_1,\ldots,x_q,t) \in Q_T, \\ v\big|_{t=0} = 0. \end{cases} \qquad (15)$$

The solution of problem (15) we seek in the subspace of such functions which are homogeneous at initial moment of time: $B = \left\{ v \in V : v\big|_{t=0} = 0 \right\}$.

Denoting structure elements in (15) by

$$A := \partial/\partial t - K, \ \tilde{f} = K\varphi + f \in C(Q_T)$$

we obtain the problem for operator equation:

$$\begin{cases} \text{for given operator } A : B \to C \text{ and element } \tilde{f} \in C \\ \text{find element } v \in B \text{ such, that } Av = \tilde{f}. \end{cases} \qquad (16)$$

The Cauchy problem has been reduced into problem for operator equation. This operator equation we solve by means of Generalized Method of Lie algebraic discrete approximations.

7. APPROXIMATION PROPERTIES AND CONVERGENCE IN MULTIDIMENSIONAL CASE

The numerical scheme for problem (16) is built using dimensional Lagrange interpolation. Let n_{x_i} denote the count of nodes by variable x_i.

Discrete problem is formulated below

$$\begin{cases} \text{for given operator } A_h : B_h \to C_h \text{ and element } \tilde{f}_h \in C_h \\ \text{find element } v_h \in B_h \text{ such, that } A_h v_h = \tilde{f}_h. \end{cases} \qquad (17)$$

Theorem 7.1. The rank of finite dimensional quasi representation A_h of the operator A has a full rank and its rank is $(n_t - 1)\prod_{i=1}^{q} n_{x_i}$ and there exists a unique solution of the discrete problem (17).

Proof. Using property that finite dimensional quasi representation K_h of operator K is nilpotent matrix. Similarly, as in proof of theorem 3.1, we obtain that finite dimensional quasi representation has a full rank and hence a unique solution of discrete problem (17) exists there.

Theorem 7.2. Finite dimensional quasi representation A_h approximates operator A on element and error estimation of approximation has the following form

$$\left\| Av - A_h v \right\|_{C_h} \le \ln n_t \left(\frac{1}{n_t - 1} \right)^{n_t - 1} \left\| \frac{\partial^{n_t} v}{\partial t^{n_t}} \right\|_{\infty} +$$

$$+ \sum_{i=1}^{q} \left(|a_i| \left(\ln(n_{x_i}) \ln(n_{x_i} - 1) \right) \left(\frac{1}{n_{x_i} - 1} \right)^{n_{x_i} - 2} \left\| \frac{\partial^{n_{x_i}} v}{\partial x^{n_{x_i}}} \right\|_{\infty} \right). \qquad (18)$$

Proof. Using formula of dimensional Lagrange interpolation and acting by operator A on residual yields (18).

Theorem 7.3. If $\lim\limits_{h\to 0}\left\|\tilde{f} - \tilde{f}_h\right\|_{C_h} = 0$ holds and conditions of theorems 5.1, 7.1, 7.2 are satisfied then $\lim\limits_{h\to 0}\left\|u - u_h\right\|_{B_h} = 0$ and

$$
\left\|u - u_h\right\|_{B_h} \le M\left(\ln n_t \left(\frac{1}{n_t - 1}\right)^{n_t - 1} \left\|\frac{\partial^{n_t} v}{\partial t^{n_t}}\right\|_\infty + \right.
$$
$$
\left. + \sum_{i=1}^q \left(|a_i| \big(\ln(n_{x_i})\ln(n_{x_i} - 1)\big) \left(\frac{1}{n_{x_i} - 1}\right)^{n_{x_i} - 2} \left\|\frac{\partial^{n_{x_i}} v}{\partial x^{n_{x_i}}}\right\|_\infty \right) \right)
\tag{19}
$$

Proof. According to the Kantorovich convergence theorem of abstract approximation scheme all requirements are satisfied, thus similarly to theorem 5.2 using inequality

$$
\left\|u - u_h\right\|_{B_h} \le \left\|v - v_h\right\|_{B_h} \le M\left\|Av - A_h v\right\|_{C_h}
$$

and neglecting terms $\left(\dfrac{1}{n_t - 1}\right)^{n_t}$ and $\left\{\left(\dfrac{1}{n_{x_i} - 1}\right)^{n_{x_i}}\right\}_{i=1}^q$ we obtain (19).

8. NUMERICAL EXAMPLE

We consider model problem with backward heat equation which is similar to model problem in [4, 12]

$$
\begin{cases}
\text{find function } u = u(x,t) \text{ such, that :} \\
\dfrac{\partial u}{\partial t} = -\dfrac{\partial^2 u}{\partial x^2}, \forall (x,t) \in Q_T, \\
u\mid_{t=0} = \sin x,
\end{cases}
\tag{8.1}
$$

having the exact solution $u(x,t) = e^t \sin x$.

If the exact solution is known, we use the following rule for evaluating the rate of convergence:

$$
p_h = \log_2\left(\frac{\left\|u - u_h\right\|}{\left\|u - u_{h/2}\right\|}\right).
$$

If we get value $\left\|u - u_h\right\| = 0$ and $\left\|u - u_{h/2}\right\| = 0$, thus the value $0/0$ is shown as NaN (*not a number*).

The model problem is investigated by means Lax–Wendroff scheme of finite differences method (FDM), method of Lie-algabraic discrete approximations (MLADA) and generalized method of Lie-algebraic discrete approximations (GMLADA). The solution of Cauchy problem with the system of differential equations was performed using Mathematica.

Let us denote $\Delta x = 1/(n_x - 1)$ – the step of discretization by space variable, $\Delta t = 1/(n_t - 1)$ – the step of discretization by time variable. If discretization steps by both variables are equal then we use $h = \Delta x = \Delta t$ for FDM and GMLADA. Nevertheless h

denotes the step of discretization by space variable, because time step is chosen automatically while solving the Cauchy problem with the system of differential equation by means of Mathematica software.

Table 1

Error estimations in $L^2(Q_T)$ space

Step h	FDM	MLADA	GMLADA
$h = 1/2$	0.129611	0.256518	0.0844799
$h = 1/4$	0.0713553	0.0808159	0.020411
$h = 1/8$	0.0380837	0.00404986	0.000689873
$h = 1/16$	–	–	$1.43891 \cdot 10^{-7}$

Table 2

Error estimations in $L^\infty(Q_{T,h})$ space

Step h	FDM	MLADA	GMLADA
$h = 1/2$	0.420034	0.976364	0.36965
$h = 1/4$	0.241512	0.356581	0.099300
$h = 1/8$	0.130843	0.0212378	0.0038209
$h = 1/16$	–	–	$9.1009 \cdot 10^{-7}$

Table 3

Error estimations in $W^{1,2}(Q_T)$ space

Step h	FDM	MLADA	GMLADA
$h = 1/2$	0.413269	0.984519	0.374173
$h = 1/4$	0.232347	0.360836	0.104918
$h = 1/8$	0.124427	0.0239794	0.00464834
$h = 1/16$	–	–	$1.49632 \cdot 10^{-6}$

Table 4

Rates of convergence in $L^2(Q_T)$ space

Step h	FDM	MLADA	GMLADA
$h = 1/2$	0.861099	1.66635	2.04926
$h = 1/4$	0.905849	4.3187	4.88687
$h = 1/8$	–	–	12.2271

Table 5

Rates of convergence in $L^{\infty}(Q_T)$ space

Step h	FDM	MLADA	GMLADA
$h = 1/2$	0.798413	1.45319	1.89631
$h = 1/4$	0.884252	4.06952	4.69979
$h = 1/8$	–	–	12.0356

Table 6

Rates of convergence in $W^{1,2}(Q_T)$ space

Step h	FDM	MLADA	GMLADA
$h = 1/2$	0.830802	1.44808	1.83445
$h = 1/4$	0.900985	3.91147	4.4964
$h = 1/8$	–	–	11.6011

9. CONCLUSIONS

We present the application of the generalized method of Lie algebraic discrete approximations for solving the Cauchy problem for backward heat equation in this paper. The key finding of this research is the opportunity to provide a factorial rate of convergence by all variables in the equation, including time variable. The Cauchy problem for backward heat equation has been reduced to a system of linear algebraic equations.

There were compared different numerical schemes (finite difference method, classical method of Lie-algebraic discrete approximations and generalized method of Lie-algebraic discrete approximations) for solving the Cauchy problem for backward heat equation.

Substitution allows the rapid solving of the problem when initial data or internal sources are mutable but coefficients of differential operator in the problem remained constant. That was available by keeping in memory the inverse matrix and multiplying it on the vector which represents initial data and/or internal sources. Such substitution encapsulates the idea which in some way separates the data of the problem and the internal structure of the problem.

REFERENCES

1. *Бігун О.* Метод Лі-алгебричних апроксимацій у теорії динамічних систем / О. Бігун, М. Притула // Математичний вісник НТШ. – Т. 1. – 2004. – С. 24–31.
2. *Кіндибалюк А. А.* Модифікований метод Лі-алгебричних апроксимацій для еліптичних рівнянь з неоднорідними крайовими умовами / А. А. Кіндибалюк, М. М. Притула // Вісн. Львів. ун-ту. Серія прикладна математика та інформатика. – 2013. – Вип. 19. – С. 56–67.
3. *Кіндибалюк А. А.* Застосування узагальненого методу Лі-алгебричних дискретних апроксимацій до розв'язування задачі Коші для двовимірного рівняння адвекції / А. А. Кіндибалюк // Вісн. Львів. ун-ту. Серія механіко-математична. – 2014. – Вип. 79. – С. 48–68.

Adriana Kindybaliuk, Arkadii Kindybaliuk, M. Prytula

80   ISSN 2078–5097. Вісн. Львів. ун-ту. Сер. прикл. матем. та інф. 2017. Вип. 25

4. *Кіндибалюк А. А.* Застосування узагальненого методу Лі-алгебраїчних дискретних апроксимацій до розв'язування задачі Коші для рівняння теплопровідності / А. А. Аркадій // Математичні студії. – 2014. – Т. 42, № 2. – С. 181–194.

5. *Люстерник Л. А.* Элементы функционального анализа / Л. А. Люстерник, В. И. Соболев. – Москва: Наука, 1965. – 519 с.

6. *Люстик М.* Функціонально-операторний аналіз проблеми збіжності для методу дискретних апроксимацій Ф. Калоджеро в банахових просторах / М. Люстик, А. Прикарпатський, М. Притула, М. Вовк // Математичний вісник НТШ. – Т. 9. – 2012. – С. 168–179.

7. *Митропольский А.* Алгебраическая схема дискретных аппроксимаций линейных и нелинейных динамических систем математической физики / А. Митропольский, А. К. Прикарпатский, В. Гр. Самойленко // Укр. мат. журн. – 1988. – Т. 40. – С. 453–458.

8. *Рихтмайер Р.* Разностные методы решения краевых задач / Р. Рихтмайер, К. Моротон. – Москва: Мир, 1972. – 418 с.

9. *Самойленко В. Гр.* Алгебраическая схема дискретных аппроксимаций динамических систем математической физики и оценки её точности / В. Гр. Самойленко // Асимптотические методы в задачах мат. физики. – Киев: Ин-т математики АН УССР. – 1988. – С. 144–151.

10. *Bihun O. H.* Approximation properties of the Lie-algebraic scheme / O. H. Bihun // Matematychni Studii. – 2003. – Vol. 20, № 1. – P. 85–91.

11. *Bihun O. H.* Modification of the Lie-algebraic scheme and approximation error estimations / O. H. Bihun // Matematychni Studii. – 2003. – Vol. 20, № 2. – P. 179–184.

12. *Bihun O. H.* Numerical tests and theoretical estimations for a Lie-algebraic scheme of discrete approximations / O. H. Bihun, M. S. Lustyk // Visnyk of the Lviv University. Series Applied Mathematics and Computer Science. – 2003. – Vol. 6. – P. 3–10.

13. *Bihun O.* The rank of projection-algebraic representations of some differential operators / O. Bihun, M. Prytula // Matematychni Studii. – 2011. – Vol. 35, Is 1. – P. 9–21.

14. *Calogero F.* Interpolation, differentiation and solution of eigen value problems in more than one dimension / F. Calogero // Lett. Nuovo Cimento. – 1983. –Vol. 38, № 13. – P. 453–459.

15. *Calogero F.* Numerical tests of a novel technique to compute the eigen values of differential operators / F. Calogero, E. Franko // Il Nuovo Cins. –1985. – Vol. 89, № 2. – P. 161–208.

16. *Casas F.* Solution of linear partial differential equations by Lie algebraic method / F. Casas // J. of Comp. and Appl. Math. – 1996. – Vol. 76. – P. 159–170.

17. *Kindybaliuk Arkadii.* Generalized method of Lie-algebraic discrete approximations for solving Cauchy problems with evolution equation / Arkadii Kindybaliuk // Journal of Applied Mathematics and Computational Mechanics. – 2014. – Vol. 12, № 2. – P. 51–62.

18. *Kindybaliuk Arkadii.* Application of the generalized method of Lie-algebraic discrete approximations to the solution of the Cauchy problem with the advection equation / Arkadii Kindybaliuk, M. Prytula // Journal of Mathematical Sciences. – 2015. – Vol. 204, Is. 3. – P. 280–297.

19. *Kindybaliuk Arkadii.* Generalized Lie-algebraic discrete approximation for heat equation. / Arkadii Kindybaliuk, M. Prytula // XXV International Conference.

Problems o decision making under uncertainties (PDMU-2015). – Skhidnytsia, Ukraine, 11–15 May, 2015. – Abstracts. – P. 44–45.

20. *Prykarpatsky A. K.* The Lie-algebraic discrete approximations in computing analysis / A. K. Prykarpatsky, M. M. Prytula, O. O. Yerchenko // Volyn Mathematical Bulletin. – 1996. – Vol. 3. – P. 113–116.

21. *Wolf F.* Lie algebraic solutions of linear Foker-Plank equations / F. Wolf // J. Math. Phys. –1988. – Vol. 29. – P. 305–307.

РОЗВ'ЯЗУВАННЯ ОБЕРНЕНОГО РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ ЛІ-АЛГЕБРИЧНИМИ ДИСКРЕТНИМИ АПРОКСИМАЦІЯМИ

Адріана Кіндибалюк[1], Аркадій Кіндибалюк[2], М. Притула[3]

[1]Київський національний універврситет будівництва і архітектури,
Повітрофлотський проспект, 31, Київ, 03680,
[2,3]Shape, Польова 21, Київ, 03056, e-mail: kindybaluk.arkadii@outlook.com
[3]Львівський національний університет імені Івана Франка,
вул. Університетська, 1, Львів, 79000, e-mail: mykola.prytula@gmail.com

Розглянуто обчислювальну схему для наближеного розв'язування задачі Коші для оберненого рівняння теплопровідності з використанням узагальненого методу Лі-алгебричних дискретних апроксимацій. Дискретизацію рівняння виконано за усіма змінними, що уможливлює факторіальну збіжність за усіма змінними, які входять до рівняння у випадку побудови квазізображень диференціального оператора з використанням інтерполяції Лагранжа. Визначено ранг скінченновимірного оператора, а також з'ясовано його апроксимаційні властивості. Наведено оцінки похибок, доведено факторіальну швидкість збіжності.

Key words: узагальнений метод Лі-алгебричних дискретних апроксимацій, обернене рівняння теплопровідності, скінченновимірне квазізображення, поліном Лагранжа, факторіальна збіжність.