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HOLDER ESTIMATES FOR THE SOLUTIONS OF DEGENERATE NONLINEAR ELLIPTIC EQUATIONS OF NON-DIVERGENCE TYPE

T. Gadjiev, S. Aliev, T. Maharramova

Institute of Mathematics and Mechanicsof Azerbaijan NAS,
soltanaliyev@yahoo.com, tgadjiev@mail.az

We establish apriori estimate for the solutions of a degenerate non-divergence nonlinear elliptic equation.

Key words: nonlinear elliptic equations, degenerate estimates, weighted Holder estimate.

1. INTRODUCTION

Let us consider in some ball $B_{2r} \subset R^n$ with radius $2r, r \geq 1$, a solution $u(x)$ in $C(\overline{B}_{2r}) \cap W^{2,n}_{loc}(B_{2r})$ of the non-divergence type nonlinear elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}(x, u(x), Du(x)) D^2 u(x) + f(x, u, Du(x)) = 0, \tag{1}$$

for a.e $x \in B_{2r}$. Here $a_{ij} = a_{ji}$, i.d. $A(x, y, p)$ set of symmetric matrices of size $n \times n$ and $\forall y \in R$, $\forall x, p, \xi \in R^n$ coefficients satisfying

$$\begin{cases} \Lambda^{-1}\lambda(p)\omega(x)|\xi|^2 \leq & (\xi, A(x,y,p)\xi) \leq \Lambda\lambda(p)\omega(x)|\xi|^2 \\ f(x,y,p) \leq & \dfrac{1}{k}\Lambda(1+\lambda(p))(1+|p|) \end{cases} \tag{2}$$

for some $\Lambda \geq 1, k > 1$ and some continuous mapping $\lambda : R^n \to R_+$ for which there exist λ_0 and $M > 0$ such that $\lambda(z) \geq \lambda_0$ for $|z| \geq M$. $\omega(x)$ is Muckenhoupt weight function (see [1]). Let $u : \overline{B}_{2r} \to R$ be a bounded and continuous solution of (1).

The first results of Holder estimates for solution of divergence form equation were obtained by De Giorgi and Nash (see [2, 7]). The case of non-divergence equations is considered by Krylov and Sofonov in [4, 5]. The case of divergence type quasilinear elliptic equations was investigated by Serrin [8] and Ladyzhenskaya, Uraltseva [6].

The goal of this paper is to prove a similar result for degenerate quasilinear elliptic equations of non-divergence form.

Our results are new and can be expanded to the following non-divergence equation

$$\|Du\|^{p-2}\omega(x) \cdot \sum_{i,j=1}^{n} [I_n + (p-2)(|Du|^{-2} \cdot Du) D^2 u],$$

where I_n is the unit matrix of size n.

2. MAIN RESULTS

Theorem 1. Suppose that the aforementioned assumption for equation (1) is satisfied. Then $u(x)$ is Holder continuous on $\overline{B}_r$. Moreover, there exist consts $\beta, C,$ only depend on n, λ, λ_0 and M, such that

$$|\,\omega(x)u(x)-\omega(y)u(y)\,|\le C\,|\,x-y\,|^{\beta}\left(1+\sup_{\overline{B}_{2r}}(|\,u\,|)\right)$$

for any $x, y \in \overline{B}_r$.

The proof relies on a probabilistic interpretation of the quasilinear equations. For linear equation in case A and f are in dependent of y and p. The original proof consists of introducing a diffusion process X, solution to the Stochastic Differential Equation(SDE)

$$dX_t = \sigma(X_t)dW_t, t \ge 0$$

where W is a n- dimensional Wiener process and σ a continuous version of the square root of the matrical mapping $2A.$ The basic idea that the generator of diffusion process has some smoothing property in the surrounding space with a nontrivial probability. Let f vanish and u is smooth, $u(X_t)_{t\le 0}$ is martingale. In this case $u(x)$ may be expressed as the expectation $E[u(X_\tau^x)]$ for any well-controlled stopping time τ and the exponent x indicates the initial position of the diffusion process. As a consequence, $u(x)$ may be understood as a mean over the values of u in a neighborhood of x, since X visits the surrounding space around $x,$ almost all the values of u in the neighborhood of x. In this case the probabilistic representation formula has the form

$$u(x) = E[u(X_\tau^x) + \int_0^\tau f(X_s^x)ds]. \qquad (3)$$

The probability that the diffusion process X hits a Borel subset of non-zero Lebesgue measure included in B_{2r}. In our non-linear case, when $A(x, y, p)$ and $u(x)$ are smooth, we can define X similarly by setting

$$dX_t = \sigma(X_t, u(X_t), Du(X_t))dW_t, t \ge 0 \qquad (4)$$

$(x, y, p) \to \sigma(x, y, p)$ being a smooth version of the square root of $2A(x, y, p)$.

We show this can force the stochastic system on the areas of degeneracy by an additional drift to push it towards the desired Borel subset.

Theorem 2. Suppose that $\sigma : R^n \to R^{n\times n}$ be a Lipschitz continuous mapping such that the function $a(x) = \sigma\cdot\sigma^*(x)$ satisfies the following estimates: $\forall x, \xi \in R^n$

$$\Lambda^{-1}\overline{\lambda}(x)\omega(x)\,|\,\xi\,|^2 \le (\xi, a(x)\xi) \le \Lambda\overline{\lambda}(x)\omega(x)\,|\,\xi\,|^2, \qquad (5)$$

where $\Lambda \ge 1$ and $\overline{\lambda} : R^n \to [0,1].$ Also suppose that $(\Omega, \mathrm{F}, (\mathrm{F}_t)_{t\ge 0}, P)$ be a filtered probability space satisfying the usual conditions endowed with an $(\mathcal{F}_t)_{t\ge 0}$ Brownian motion $(W_t)_{t\ge 0}$, α be a positive real number and Q_1 be some hypercube of R^n of radius r. Then, for any μ in $(0,1)$ there exist some positive constants $\varepsilon(\mu), R(\mu)$ and $(\Gamma_p(\mu))_{1\le p<2}$ only depending on d, α, Λ and $\mu,$ such that for any $\rho \in (0,1)$ and $x_0 \in Q_{\rho/8},$ we can find an integrable n-dimensional $(\mathcal{F}_t)_{t\le 0}$ progressively measurable process $b_t)_{t\le 0}$ such that both $b_t)_{t\le 0}$ and the process X, solution to the SDE

$$X_t = x_0 + \int_0^t b_s ds + \int_0^t \sigma(X_t) X_s dB_s, t \ge 0$$

satisfy the following:

$$\forall t \ge 0,\ \ \overline{\lambda}(X_t) \ge \alpha \Rightarrow b_t = 0 \ \ \forall p \in [1,2),\ \ E\int_0^{+\infty} | b_t |^p \, dt \le \Gamma_p(\mu)\rho^{p-2},$$

and for any Borel subset $V \subset Q_s$ we have that

$$| Q_\rho \setminus V | \le \mu | Q_\rho | \Rightarrow P\{T_V < (R(\mu)\rho^2) \cap S_{Q_\rho}\} \ge \varepsilon(\mu),$$

where T_V is the first hitting time of V and S_{Q_ρ} the first exit time from Q_ρ by X and $|\cdot|$ is a Lebesgue measure.

The connection with Theorem 1 may be understood as follows when $u(x)$ is a strong solution of the (1), then we choose $a(x)$ in the statement of Theorem (2) as $2A(x,u(x),Du(x)).$ The term $\lambda(Du(x))$ in Theorem 1 plays the role of $\overline{\lambda}(x)$ in Theorem 2. By choosing α in the Theorem 2 equal to λ_0 given in Theorem 1, we deduce that

$$| Du(X_t) | \ge M \Rightarrow \lambda(Du(X_t)) \ge \lambda_0 \Rightarrow \overline{\lambda}(X_t) \ge \alpha \Rightarrow b_t = 0.$$

In other words the resulting drift $b_t)_{t\le 0}$ just acts whem the gradient is small, i.e. is bounded by M.

Later we can show how to deduce Theorem 1 from Theorem 2. We prove Theorem 2 when the proportion of V inside Q_ρ is large enough. Compared with the original argument given by Krylov and Safanov, the main difference in the application of the probabilistic estimate follows from the interpretation of the underlying PDE. In the paper of Krylov and Safonov the PDE is understood in the strong sense, i.e. the solution $u(x)$ is assumed to be in $C(\overline{B}_{2r}) \cap W^{2,n}_{loc}(B_{2r})$. To complete the proof of Theorem 1 we also used to Gilbarg and Trudinger [[3], lem. 8.23]. In a future paper we will give strong proof these facts.

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ОЦІНКА ГЕЛЬДЕРА ДЛЯ РОЗВ'ЯЗКУ ВИРОДЖЕНИХ НЕЛІНІЙНИХ ЕЛІПТИЧНИХ РІВНЯНЬ НЕДИВЕРГЕНТНОГО ТИПУ

Т. Гадєєв, С. Алієв, Т. Магеррамова

Інститут математики та механіки НАН Азербайджану,
soltanaliyev@yahoo.com, tgadjiev@mail.az

Знайдено апріорну оцінку для розв'язання виродженого не дивергентного нелінійного рівняння.

Ключові слова: нелінійні еліптичні рівняння, вироджена оцінка, вагова оцінка Гельдера.