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FINITE ELEMENT ANALYSIS OF GREEN-LINDSAY THERMOPIEZOELECTRICITY TIME-DEPENDENT PROBLEM

R. Drebotiy, V. Stelmashchuk, H. Shynkarenko

Ivan Franko National University of Lviv,
1, Universytetska str., 79000, Lviv, Ukraine
e-mail: roman.drebotiy@lnu.edu.ua, vitalii.stelmashchuk@lnu.edu.ua,
heorhiy.shynkarenko@lnu.edu.ua

On the basis of Green-Lindsay thermopiezoelectricity model with so-called "relaxation time"-parameters, which influence on the way of interaction between mechanical, electrical and thermal fields in pyroelectric materials, we formulate initial boundary value problem and the corresponding variational problem in terms of vector of elastic displacements, electrical potential, and temperature increment. We derive the respective energy balance law and perform energy estimates of the solutions of the variational problem. Using the latter the well-posedness of the variational problem is proved. Based on finite element method and Newmark scheme the numerical scheme is developed for approximate solution of this problem. The unconditional stability of the constructed time integration scheme is proved. Finally we show the results of the numerical experiment which demonstrates the influence of the values of "time relaxation" parameters of Green-Lindsay model on the obtained solution.

Key words: pyroelectric effect, thermopiezoelectricity, Green-Lindsay model, initial boundary value problem, variational problem, well-posedness of the variational problem, Galerkin semi-discretization, finite element method, Newmark scheme, stability of time integration numerical scheme.

1. Introduction

Nowadays various mathematical models of behaviour of pyrolectric and piezoelectric materials are in place. The classic one is the linear thermopiezoelectricity model proposed by Mindlin [10] and then comprehensively studied by Nowacki [11]. However, that theory has a significant drawback because it assumes infinite speed of heat propagation wave in the material. To overcome it, Green and Lindsay [7] proposed a modified theory for thermoelasticity problem where heat conduction equation is considered to be hyperbolic by introduction of two so-called "relaxation time"-parameters. Chandrasekharaiah [6] extended that approach to the linear thermopiezoelectricity model.

Apart from Green-Lindsay theory of generalized thermoelasticity, there exists a set of other theories, namely Lord-Shulman, Chandrasekharaiah-Tzou, Green-Naghdi, etc. The effect of the "relaxation time" parameter of Lord-Shulman theory in piezoelectric materials has been studied in works of Ashida&Tauchert [2], [3]. Bassiouny&Youssef [4] have studied a two-temperature generalized thermopiezoelasticity of finite rod using Laplace transform method. Kuang [9] considered the application of variational principles for generalized dynamical theory of thermopiezoelectricity. More recent works of Zenkour et al. [17], Sharifi [12] are dedicated to problems of Green-Lindsay thermoelasticity. Shivay&Mukhopadhyay [13] show the application of finite element method for Green-Lindsay thermoelasticity problem.

In Stelmashchuk&Shynkarenko paper [15] Lord-Shulman theory of thermopiezoelectricity for dynamical problems was investigated. The Green-Lindsay model of thermoelasticity have been extensively studied in work of Chyr&Shynkarenko [1]. Then in article of Stelmashchuk&Shynkarenko [14] forced vibrations of pyroelectric materials under Green-Lindsay theory were studied. In the current article we extend the investigation of Green-Lindsay thermopiezoelectricity problem done in [14] by considering an initial boundary value problem for Green-Lindsay model. Similar techniques as in the work [15] are applied in this paper.

In section 2 the initial boundary value problem of Green-Lindsay thermopiezoelectricity is formulated. Section 3 is dedicated to construction of the corresponding variational problem. In section 4 we prove the well-posedness of the constructed variational problem. Section 5 describes how Galerkin semi-discretization allows us to build a Cauchy problem for solving this variational problem. Section 6 is dedicated to time integration scheme. Section 7 contains a proof of stability of the time integration scheme. Section 8 shows the results of the numerical experiments. Finally, in section 9 the conclusions are made.

2. Problem statement

We will set up the initial boundary value problem of Green-Lindsay thermopiezoelectricity just like it was done in the article [14].

Consider a bounded connected domain Ω of points $x = (x_1, ..., x_d) \in \mathbb{R}^d$ with Lipschitz-continuous boundary $\partial\Omega = \Gamma$ that defines the pyroelectric specimen. Let $n = (n_1, ..., n_d)$ be the unit outer normal vector, $n_i = cos(n, x_i)$. We also consider a time interval $[0, T], 0 < T < +\infty$. Just like in classic thermopiezoelectricity problem, our goal is to find a vector of elastic displacements $u = \{u_i(x,t)\}_{i=1}^d$, electric potential $p = p(x,t)$ and temperature increment $\theta = \theta(x,t)$ which satisfy the following system of partial differential equations in $\Omega \times (0, T]$ (here and everywhere below the ordinary summation by repetitive indices is expected):

$$\rho u_i'' - \sigma_{ij,j} = \rho f_i, \tag{1}$$

$$D'_{k,k} + J_{k,k} = 0, \tag{2}$$

$$\rho(T_0 S' - w) + q_{i,i} = 0. \tag{3}$$

In fundamental equations (1)–(3) by prime symbol $'$ we denote a partial differentiation by a time variable. The aforementioned expressions (1)-(3) are equation of motion, differentiated Maxwell's equation and heat conduction equation respectively. The notation σ_{ij} is used for a stress tensor which is defined by the following constitutive equation:

$$\sigma_{ij} = c_{ijkm}[\varepsilon_{km} - \alpha_{km}(\theta + t_1\theta')] - e_{kij}E_k + a_{ijkm}\varepsilon'_{km}. \tag{4}$$

Constitutive equation for electric displacement D_k:

$$D_k = e_{kij}\varepsilon_{ij} + \chi_{km}E_m + \pi_k(\theta + t_1\theta'). \tag{5}$$

And the entropy density S is defined by:

$$\rho S = c_{ijkm}\alpha_{km}\varepsilon_{ij} + \pi_k E_k + \tfrac{\rho c_v}{T_0}(\theta + t_0\theta'). \tag{6}$$

Here in the constitutive equations (4)–(6) the parameters $t_1 \geq t_0 > 0$ are of time dimension and were firstly introduced by Green and Lindsay in [7] for the classic heat conduction problem to resolve the problem of infinite speed of heat propagation wave. These parameters are also often referred in the literature as so-called "relaxation times" or "relaxation time"-parameters. The values of these parameters are always taken very small (less than $10^{-10}s$). If we put $t_1 = t_0 = 0$ we come to the constitutive equations of the classic thermopiezoelectricity model.

Vector $q = \{q_i\}_{i=1}^d$ is a heat flux in different directions and it is assumed to satisfy the classic Fourier's law:

$$q_i = -\lambda_{ij}\theta_{,j}.$$

Vector $J = \{J_k\}_{k=1}^d$ defines the electrical current density. We also assume that the electric current that runs through the pyroelectric specimen satisfies standard Ohm's law:

$$J_k = z_{km}E_m(p).$$

Strain tensor $\varepsilon = \{\varepsilon_{km}\}_{k,m=1}^d$ and electrical field vector $E = \{E_k\}_{k=1}^d$ satisfy the equations:

$$\varepsilon_{km} = \varepsilon_{km}(u) = \tfrac{1}{2}(u_{k,m} + u_{m,k}),$$
$$E_k = E_k(p) = -p_{,k}.$$

Here and elsewhere in the problem statement a comma in the subscript means a partial derivative by a spatial variable, i.e. $g_{,k} = \partial g/\partial x_k$.

Notation ρ is a mass density of pyroelectric material, c_v is its specific heat and T_0 is a fixed uniform reference temperature of the specimen. Vector $f = \{f_i\}_{i=1}^d$ defines mechanical volume forces and w represents volume heat forces. Tensors $\{a_{ijkm}\}$ and $\{c_{ijkm}\}$ describe the viscosity and elasticity properties of a pyroelectric material and satisfy common conditions of symmetry and ellipticity. Notation $\{e_{ij}\}$ is a piezoelectricity coefficients tensor with symmetric properties:

$$e_{kij} = e_{kji}.$$

Coefficients z_{km}, χ_{km}, λ_{ij}, α_{km} determine the symmetrical and elliptical electric conductivity, dielectric susceptibility, heat conductivity and thermal expansion coefficients correspondingly. Notation π_k defines a vector of pyroelectricity coefficients that satisfies the inequality [11], [14]:

$$\chi_{km}y_k y_m + 2\pi_k y_k \xi + \rho c_v \xi^2 \geq 0 \quad \forall \xi, y_k \in \mathbb{R}. \tag{7}$$

The boundary conditions for mechanical field are [14]:

$$\begin{cases} u_i = 0 \quad \text{on} \quad \Gamma_u \times [0,T],\ \Gamma_u \subset \Gamma, mes(\Gamma_u) > 0, \\ \sigma_{ij}n_j = \overline{\sigma}_i \quad \text{on} \quad \Gamma_\sigma \times [0,T], \Gamma_\sigma := \Gamma \setminus \Gamma_u. \end{cases}$$

The boundary conditions for electrical field are [14]:

$$\begin{cases} p = 0 \quad \text{on} \quad \Gamma_p \times [0,T],\ \Gamma_p \subset \Gamma, mes(\Gamma_p) > 0, \\ [D'_k + J_k]\, n_k = 0 \quad \text{on} \quad \Gamma_d \times [0,T],\ \Gamma_d \subset \Gamma, \quad \Gamma_p \cap \Gamma_d = \oslash, \\ \int\limits_{\Gamma_e} [D'_k + J_k]\, n_k d\gamma = I \quad \text{on} \quad \Gamma_e \times [0,T], \quad \Gamma_e = \Gamma \setminus (\Gamma_d \cup \Gamma_p), \\ E_k(p) - n_k E_m(p) n_m = 0 \quad \text{on} \quad \Gamma_e \times [0,T]. \end{cases}$$

The boundary conditions for heat field are [14]:

$$\begin{cases} \theta = 0 \quad \text{on} \quad \Gamma_\theta \times [0,T],\ \Gamma_\theta \subset \Gamma, mes(\Gamma_\theta) > 0, \\ q_i n_i = \overline{q} \quad \text{on} \quad \Gamma_q \times [0,T], \Gamma_q := \Gamma \setminus \Gamma_\theta. \end{cases} \tag{8}$$

To finalize the formulation of the initial boundary value problem of Green-Lindsay thermopiezoelectricity all the aforementioned equations are complemented by the initial conditions:

$$u|_{t=0} = u_0, \quad u'|_{t=0} = v_0, p|_{t=0} = p_0, \quad \theta|_{t=0} = \theta_0 \quad \text{in} \quad \Omega.$$

3. Variational problem statement

Let us introduce the spaces of admissible elastic displacements, electric potentials and temperature increments

$$\begin{aligned} &V = \left\{ v \in [H^1(\Omega)]^d |\ v = 0 \ \text{ on } \ \Gamma_u \right\}, \\ &Q = \left\{ p \in H^1(\Omega) |\ p = 0 \ \text{ on } \ \Gamma_p, \quad p = const \ \text{ on } \ \Gamma_e \right\}, \\ &Z = \left\{ \xi \in [H^1(\Omega)] |\ \xi = 0 \ \text{ on } \ \Gamma_\theta \right\}. \end{aligned}$$

Here $H^m(\Omega)$ is a standard Sobolev space.

We will also denote $\Phi = V \times Q \times Z$ and its dual space $\Phi' = V' \times Q' \times Z'$. Besides, we will use the Lebesgue spaces $L^2(0,T;\Phi)$ and $L^2(0,T;\Phi')$. The space $L^2(0,T;\overline{\Phi})$, where $\overline{\Phi} = \Phi$ or $\overline{\Phi} = \Phi'$, means that a function of two variables (x,t) to be the element of that space has to be square integrable by variable t via Lebesque integral over the time interval $[0,T]$ and regarding the integration by space variable x the function must satisfy the restrictions of the corresponding space $\overline{\Phi}$. Then the initial boundary value problem of Green-Lindsay thermopiezoelectricity defined in the previous section can be rewritten in the following variational formulation:

$$\begin{cases} \text{given } \psi_0 = (u_0, p_0, \theta_0) \in \Phi,\ v_0 \in [L^2(\Omega)]^d, \theta_{10} \in L^2(\Omega) \\ \text{and } (l, r, \mu) \in L^2(0,T;\Phi'); \\ \text{find } \psi = \{u(x,t), p(x,t), \theta(x,t)\} \in L^2(0,T;\Phi) \ \text{ such that} \\ m(u''(t), v) + a(u'(t), v) + c(u(t), v) - e(p(t), v) - \\ -\gamma(\theta(t) + t_1\theta'(t), v) =< l(t), v >, \\ g(p'(t), q) + e(q, u'(t)) + z(p(t), q) + \pi(\theta'(t) + t_1\theta''(t), q) =< r(t), q >, \\ s(\theta'(t) + t_0\theta''(t), \xi) + k(\theta(t), \xi) + \pi(\xi, p'(t)) + \\ +\gamma(\xi, u'(t)) =< \mu(t), \xi > \quad \forall\ t \in (0,T], \\ m(u'(0) - v_0, v) = 0, \qquad c(u(0) - u_0, v) = 0 \qquad \forall v \in V, \\ g(p(0) - p_0, q) = 0 \qquad \forall q \in Q, \\ s(\theta(0) - \theta_0, \xi) = 0, \quad s(\theta'(0) - \theta_{10}, \xi) = 0 \quad \forall \xi \in Z. \end{cases} \tag{9}$$

In the aforementioned variational problem statement (9) the following bilinear and linear forms are used:

$$\begin{aligned}
&m(u,v) := \int_\Omega \rho u_i v_i dx = \int_\Omega \rho u \cdot v dx,\\
&a(u,v) := \int_\Omega a_{ijkm}\varepsilon_{ij}(u)\varepsilon_{km}(v)dx,\\
&c(u,v) := \int_\Omega c_{ijkm}\varepsilon_{ij}(u)\varepsilon_{km}(v)dx,\\
&< l, v > := \int_\Omega \rho f_i v_i dx + \int_{\Gamma_\sigma} \overline{\sigma}_i v_i d\gamma,\\
&\gamma(\xi,v) := \int_\Omega \xi c_{ijkm}\alpha_{km}\varepsilon_{ij}(v)dx,\\
&e(q,v) := \int_\Omega e_{kij}E_k(q)\varepsilon_{ij}(v)dx \quad \forall u,v\in V,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&g(p,q) := \int_\Omega \chi_{km}E_k(p)E_m(q)dx,\\
&z(p,q) := \int_\Omega z_{km}E_k(p)E_m(q)dx,\\
&< r, q > := Iq|_{\Gamma_e},\\
&\pi(\xi,q) = \int_\Omega \xi\pi_k E_k(q)dx \ \forall p,q\in Q,\\
&s(\theta,\xi) = \int_\Omega c_v T_0^{-1}\theta\xi dx,\\
&k(\theta,\xi) = \int_\Omega T_0^{-1}\lambda_{ij}\theta_{,i}\xi_{,j}dx,\\
&< \mu, \xi > := \int_\Omega T_0^{-1} w\xi dx + \int_{\Gamma_q} T_0^{-1}\overline{q}\xi d\gamma \ \forall \xi,\theta\in Z.
\end{aligned} \tag{11}$$

4. Energy balance equation

To simplify things instead of (8) we will use the following uniform boundary conditions for heat field:

$$\begin{cases} \theta = 0 \quad \text{on} \quad \Gamma_\theta\times[0,T],\ \Gamma_\theta\subset\Gamma, mes(\Gamma_\theta)>0,\\ q_i n_i = 0 \quad \text{on} \quad \Gamma_q\times[0,T], \Gamma_q := \Gamma\setminus\Gamma_\theta. \end{cases}$$

Thus the linear form $< \mu, \xi >$ used in variational problem statement (9) will be simplified to:

$$< \mu, \xi > := \int_\Omega T_0^{-1} w\xi dx.$$

This assumption allows us to estimate the linear form $< \mu, \xi >$ using Cauchy-Schwartz inequality using the norm of $L^2(\Omega)$ space, that is:

$$| < \mu, \xi > | \le ||\mu||_{**}\cdot||\xi||_{L^2(\Omega)}.$$

Here $||\mu||_{**}$ is a norm of linear form μ in the space dual to $L^2(\Omega)$.

Also, in our further analysis we will assume a strict inequality $t_1 > t_0$ instead of $t_1 \ge t_0$ as it was in problem statement.

Note that bilinear forms in (10)–(11) admit a clear physical interpretation and continuity and ellipticity properties of some of them allow us to introduce the following energy norms:

$$\begin{aligned}
&||v||_m^2 = m(v,v),\ ||v||_c^2 = c(v,v),\ ||v||_a^2 = a(v,v) \ \ \forall v\in V,\\
&||q||_g^2 = g(q,q),\ \ ||q||_z^2 = z(q,q) \ \ \forall q\in Q,\\
&||\zeta||_k^2 = k(\zeta,\zeta) \ \ \forall\zeta\in Z, ||\zeta||_s^2 = s(\zeta,\zeta) \ \ \forall\zeta\in L^2(\Omega).
\end{aligned}$$

Let us assume (9) admits a solution $\psi = (u, p, \theta)$. Then after substitution $\varphi = (v, q, \xi) = (u', p, \theta + t_1\theta')$ into equations of (9) and after addding them up we receive the following integral identity:

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left[||u'(t)||_m^2 + ||u(t)||_c^2 + ||p(t)||_g^2 + 2\pi(\theta(t) + t_1\theta'(t), p(t)) + t_1||\theta(t)||_k^2\right]\\&+\left[||u'(t)||_a^2 + ||p(t)||_z^2 + ||\theta(t)||_k^2 + s(\theta'(t) + t_0\theta''(t), \theta(t) + t_1\theta'(t))\right]\\&=< l, u'(t) > + < r, p(t) > + < \mu, \theta(t) + t_1\theta'(t) > \quad \forall t \in (0, T].\end{aligned} \tag{12}$$

Let us perform the following transformations with the term $s(\theta'(t)+t_0\theta''(t), \theta(t)+t_1\theta'(t))$:

$$\begin{aligned}&s(\theta' + t_0\theta'', \theta + t_1\theta') = \tfrac{1}{2}\tfrac{d}{dt}||\theta + t_0\theta'||_s^2 + s(\theta' + t_0\theta'', (t_1 - t_0)\theta') =\\&= \tfrac{1}{2}\tfrac{d}{dt}||\theta + t_0\theta'||_s^2 + (t_1 - t_0)||\theta'||_s^2 + \tfrac{1}{2}\tfrac{d}{dt}(t_1 - t_0)t_0||\theta'||_s^2 =\\&= \tfrac{1}{2}\tfrac{d}{dt}\left[||\theta + t_0\theta'||_s^2 + (t_1 - t_0)t_0||\theta'||_s^2\right] + (t_1 - t_0)||\theta'||_s^2.\end{aligned} \tag{13}$$

Then substitute (13) into (12) and receive:

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left[||u'(t)||_m^2 + ||u(t)||_c^2 + ||p(t)||_g^2 + t_1||\theta(t)||_k^2 + ||\theta(t) + t_0\theta'(t)||_s^2 +\right.\\&\left.+(t_1 - t_0)t_0||\theta'(t)||_s^2 + 2\pi(\theta(t) + t_1\theta'(t), p(t))\right] +\\&+\left[||u'(t)||_a^2 + ||p(t)||_z^2 + ||\theta(t)||_k^2 + (t_1 - t_0)||\theta'||_s^2\right] =\\&=< l, u'(t) > + < r, p(t) > + < \mu, \theta(t) + t_1\theta'(t) > \quad \forall t \in (0, T].\end{aligned} \tag{14}$$

After integration of (14) over any time interval $[0, t]$ and utilizing the initial conditions of the problem we obtain:

$$\begin{aligned}&\tfrac{1}{2}\left[||u'(t)||_m^2 + ||u(t)||_c^2 + ||p(t)||_g^2 + t_1||\theta(t)||_k^2 + ||\theta(t) + t_0\theta'(t)||_s^2 +\right.\\&\left.+(t_1 - t_0)t_0||\theta'(t)||_s^2 + 2\pi(\theta(t) + t_1\theta'(t), p(t))\right] +\\&+\int_0^t\left[||u'(t)||_a^2 + ||p(t)||_z^2 + ||\theta(t)||_k^2 + (t_1 - t_0)||\theta'||_s^2\right]d\tau =\\&= \tfrac{1}{2}\left[||v_0||_m^2 + ||u_0||_c^2 + ||p_0||_g^2 + t_1||\theta_0||_k^2 + ||\theta_0 + t_0\theta_{10}||_s^2 +\right.\\&\left.+(t_1 - t_0)t_0||\theta_{10}||_s^2 + 2\pi(\theta_0 + t_1\theta_{10}, p_0))\right] +\\&+\int_0^t\left[< l, u'(\tau) > + < r, p(\tau) > + < \mu, \theta(\tau) + t_1\theta'(\tau) >\right]d\tau \quad \forall t \in [0, T].\end{aligned} \tag{15}$$

The term $2\pi(\theta(t) + t_1\theta'(t), p(t)) > 0$ because of the inequality (7). Therefore we can introduce the notations:

$$\begin{aligned}&||\psi(t)||^2 = ||u'(t)||_m^2 + ||u(t)||_c^2 + ||p(t)||_g^2 + t_1||\theta(t)||_k^2 + ||\theta(t) + t_0\theta'(t)||_s^2 +\\&+(t_1 - t_0)t_0||\theta'(t)||_s^2 + 2\pi(\theta(t) + t_1\theta'(t), p(t)),\end{aligned} \tag{16}$$

$$|||\psi(t)|||^2 = ||u'(t)||_a^2 + ||p(t)||_z^2 + ||\theta(t)||_k^2 + (t_1 - t_0)||\theta'(t)||_s^2. \tag{17}$$

Then the obtained energy balance equation for Green-Lindsay thermopiezoelectricity (15) can be represented as:

$$\begin{aligned}&\tfrac{1}{2}||\psi(t)||^2 + \int_0^t |||\psi(\tau)|||^2 d\tau = \tfrac{1}{2}||\psi(0)||^2 +\\&+\int_0^t\left[< l, u'(\tau) > + < r, p(\tau) > + < \mu, \theta(\tau) + t_1\theta'(\tau) >\right]d\tau \quad \forall t \in [0, T].\end{aligned} \tag{18}$$

Here the term $||\psi(t)||^2$ defines a total energy of the pyroelectric specimen. The energy dissipation is determined by the term $\int\limits_0^t |||\psi(\tau)|||^2 d\tau$.

Now we will use an inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \forall a, b \in R, \quad \forall \varepsilon > 0$$

and Cauchy-Schwartz inequality to estimate the terms in the right part in the energy balance law (18). Therefore, the following inequalities are held:

$$\int\limits_0^t < l, u'(\tau) > d\tau \leq \frac{1}{2} \int\limits_0^t \left[||l||_*^2 + ||u'(\tau)||_a^2 \right] d\tau, \tag{19}$$

$$\int\limits_0^t < r, p(\tau) > d\tau \leq \frac{1}{2} \int\limits_0^t \left[||r||_*^2 + ||p(\tau)||_z^2 \right] d\tau, \tag{20}$$

$$\begin{aligned} &\int\limits_0^t < \mu, \theta(\tau) + t_1 \theta'(\tau) > d\tau \leq \int\limits_0^t \left[||\mu||_* ||\theta(\tau)||_k + t_1 ||\mu||_{**} ||\theta'(\tau)||_s \right] d\tau \leq \\ &\leq \frac{1}{2} \int\limits_0^t \left[||\mu||_*^2 + ||\theta(\tau)||_k^2 \right] d\tau + \frac{1}{2} \int\limits_0^t \left[\frac{t_1^2}{t_1 - t_0} ||\mu||_{**}^2 + (t_1 - t_0) ||\theta'(\tau)||_s^2 \right] d\tau. \end{aligned} \tag{21}$$

Here and everywhere by symbol $||\cdot||_*$ we denote the norms in dual spaces V', Q' and Z' respectively. Substituting (19)–(21) into energy balance law (18) we receive:

$$\begin{aligned} &\tfrac{1}{2} ||\psi(t)||^2 + \int\limits_0^t |||\psi(\tau)|||^2 d\tau \leq \tfrac{1}{2} ||\psi(0)||^2 + \\ &+ \tfrac{1}{2} \int\limits_0^t \left[||l||_*^2 + ||u'(\tau)||_a^2 \right] d\tau + \tfrac{1}{2} \int\limits_0^t \left[||r||_*^2 + ||p(\tau)||_z^2 \right] d\tau + \\ &+ \tfrac{1}{2} \int\limits_0^t \left[||\mu||_*^2 + ||\theta(\tau)||_k^2 \right] d\tau + \tfrac{1}{2} \int\limits_0^t \left[\frac{t_1^2}{t_1 - t_0} ||\mu||_{**}^2 + (t_1 - t_0) ||\theta'(\tau)||_s^2 \right] d\tau \\ &\forall t \in [0, T]. \end{aligned}$$

Taking into account the definition $|||\psi(\tau)|||^2$, after transformations we obtain:

$$\begin{aligned} &\tfrac{1}{2} ||\psi(t)||^2 + \tfrac{1}{2} \int\limits_0^t \left[||u'(t)||_a^2 + ||p(t)||_z^2 + ||\theta(t)||_k^2 + (t_1 - t_0) ||\theta'(t)||_s^2 \right] d\tau \leq \\ &\leq \tfrac{1}{2} ||\psi(0)||^2 + \tfrac{1}{2} \int\limits_0^t \left[||l||_*^2 + ||r||_*^2 + ||\mu||_*^2 + \frac{t_1^2}{t_1 - t_0} ||\mu||_{**}^2 \right] d\tau \quad \forall t \in [0, T]. \end{aligned}$$

Having divided the latter by $\frac{1}{2}$ we receive:

$$\begin{aligned} &||\psi(t)||^2 + \int\limits_0^t |||\psi(\tau)|||^2 d\tau \leq \\ &\leq ||\psi(0)||^2 + \int\limits_0^t \left[||l||_*^2 + ||r||_*^2 + ||\mu||_*^2 + \frac{t_1^2}{t_1 - t_0} ||\mu||_{**}^2 \right] d\tau \quad \forall t \in [0, T]. \end{aligned} \tag{22}$$

The estimation (22) shows that solution continuously changes depending on the input data of the variational problem. Let us denote

$$|||\psi(t)|||_{\Phi}^2 = ||\psi(t)||^2 + \int\limits_0^t |||\psi(\tau)|||^2 d\tau \quad \forall t \in [0,T].$$

Then the expression $|||\psi(t)|||_{\Phi}^2$ defines so-called energy norm of the solution of the variational problem.

Proposition 1. *The estimation (22) is valid if the input data of the variational problem (9) satisfy the following conditions:*

$$\begin{cases} v_0 \in [L^2(\Omega)]^d, u_0 \in [H^1(\Omega)]^d, \\ p_0 \in H^1(\Omega), \theta_0 \in L^2(\Omega), \theta_{10} \in L^2(\Omega), \\ (l,r,\mu) \in L^2(0,T;\Phi'). \end{cases} \tag{23}$$

Moreover, a solution $\psi = (u,p,\theta)$ of the problem (9), if one exists, is characterized by the following regularity inclusions:

$$\begin{cases} u' \in L^\infty(0,T;[L^2(\Omega)]^d) \cap L^2(0,T;V), \;\; u \in L^\infty(0,T;V) \cap L^2(0,T;V), \\ p \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;Q), \\ \theta' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;Z), \;\; \theta \in L^\infty(0,T;Z) \cap L^2(0,T;Z) \end{cases} \tag{24}$$

and stability inequality

$$\begin{aligned} &|||\psi(t)|||_{\Phi}^2 \le ||\psi(0)||^2 + \int\limits_0^t \left[||l||_*^2 + ||r||_*^2 + ||\mu||_*^2 + \frac{t_1^2}{t_1 - t_0} ||\mu||_{**}^2 \right] d\tau \\ &\forall t \in [0,T]. \end{aligned} \tag{25}$$

Proof. The conditions (23) are defined in the variational problem statement (9) and are necessary for the boundedness of the term $\frac{1}{2}||\psi(0)||^2$ of the energy balance equation (18). The regularity inclusions (24) provide us the boundedness of all other terms in the energy balance equation (18). Finally, the stability inequality (25) is proved via the above mentioned procedures. □

Proposition 2. *A solution $\psi = (u,p,\theta)$ of the problem (9), if one exists, is unique.*

Proof. By contradiction. Suppose there exist two different solutions $\psi_1(t)$ and $\psi_2(t)$ of the problem (9). Then their difference $\psi(t) = \psi_1(t) - \psi_2(t) \neq 0$ satisfies the homogeneous equations of the problem (9). Therefore, from the inequality (25) we have:

$$|||\psi(t)|||_{\Phi}^2 \le 0 \qquad \forall t \in [0,T],$$

which contradicts with our assumption that $\psi(t) \neq 0$. □

5. Finite element semi-discretization

Let us define in the space $\Phi = V \times Q \times Z$ a sequence of finite-dimensional subspaces $\Phi_h = V_h \times Q_h \times Z_h$, such that $\dim \Phi_h \to \infty$ when $h \to 0$ with the following density properties:

$$\begin{cases} \forall \phi \in \Phi \cap [H^{k+1}(\Omega)]^{d+2}, \;\; k \ge 1, \\ \exists \phi_h \in \Phi_h \;\; \text{and} \;\; C = const > 0 \text{ such that} \\ \|\phi - \phi_h\|_{m,\Omega} \le C h^{k+1-m} \|\phi\|_{k+1,\Omega}, 0 \le m \le k. \end{cases} \tag{26}$$

Here k is the greatest degree of the polynomial that approximates the unknown solution (it will be precisely defined by the base functions of Φ_h). The norm $\|\cdot\|_{m,\Omega}$ is a norm in a standard Sobolev space $H^m(\Omega)$. Also, here and everywhere below the symbol C defines different positive constants, values of which do not depend on the solutions of our problem. For each fixed $h>0$ a solution $\psi_h=(u_h,p_h,\theta_h)$ of the problem

$$\left\{\begin{array}{l} \text{given } \psi_0=(u_0,p_0,\theta_0)\in\Phi_h,\ v_0\in V_h, \theta_{10}\in Z_h \ \text{ and } (l,r,\mu)\in L^2(0,T;\Phi_h');\\ \text{find } \ \psi=\{u_h(x,t),p_h(x,t),\theta_h(x,t)\}\in L^2(0,T;\Phi_h) \text{ such that}\\ m(u_h''(t),v)+a(u_h'(t),v)+c(u_h(t),v)-e(p_h(t),v)-\\ -\gamma(\theta_h(t)+t_1\theta_h'(t),v)=<l(t),v>,\\ g(p_h'(t),q)+e(q,u_h'(t))+z(p_h(t),q)+\pi(\theta_h'(t)+t_1\theta_h''(t),q)=<r(t),q>,\\ s(\theta_h'(t)+t_0\theta_h''(t),\xi)+k(\theta_h(t),\xi)+\pi(\xi,p_h'(t))+\\ +\gamma(\xi,u_h'(t))=<\mu(t),\xi> \quad \forall\ t\in(0,T],\\ m(u_h'(0)-v_0,v)=0, \qquad c(u_h(0)-u_0,v)=0 \qquad \forall v\in V_h,\\ g(p_h(0)-p_0,q)=0 \qquad \forall q\in Q_h,\\ s(\theta_h(0)-\theta_0,\xi)=0, \quad s(\theta_h'(0)-\theta_{10},\xi)=0 \quad \forall\xi\in Z_h \end{array}\right. \tag{27}$$

we will call a semi-discrete Galerkin approximation of the solution $\psi=(u,p,\theta)$ of the variational problem (9). The constant h will be called a space discretization parameter of the problem (9).

Let us fix some bases $\{v_i\}$, $\{q_i\}$, $\{\xi_i\}$ in the approximation subspaces V_h, Q_h and Z_h respectively. Those bases we will select by means of finite element method. Then our unknown solutions might be represented as follows:

$$\begin{aligned} u_h(x,t)&=\sum_{i=1}^{dimV_h}U_i(t)v_i(x),\\ p_h(x,t)&=\sum_{i=1}^{dimQ_h}P_i(t)q_i(x),\\ \theta_h(x,t)&=\sum_{i=1}^{dimZ_h}\Theta_i(t)\xi_i(x). \end{aligned} \tag{28}$$

Then by the Galerkin procedure we obtain a Cachy problem for determining the unknown coefficients $U(t)=\{U_i(t)\}$, $P(t)=\{P_i(t)\}$ and $\Theta(t)=\{\Theta_i(t)\}$ of approximations u_h, p_h and θ_h:

$$\left\{\begin{array}{c} MU^{''}(t)+AU^{'}(t)+CU(t)-E^TP(t)-Y^T(\Theta(t)+t_1\Theta^{'}(t))=L(t),\\ EU^{'}(t)+GP^{'}(t)+\Pi^T(\Theta^{'}(t)+t_1\Theta^{''}(t))+ZP(t)=R(t),\\ YU^{'}(t)+\Pi P^{'}(t)+S^T(\Theta^{'}(t)+t_1\Theta^{''}(t))+K\Theta(t)=F(t),\\ MU^{'}(0)=V^0, \quad CU(0)=U^0,\\ GP(0)=P^0, \quad S(\Theta(0)+t_0\Theta^{'}(0))=\Theta^0+t_0\Theta^{10}. \end{array}\right. \tag{29}$$

Here the coefficients of matrices M,A,C,E,Y,G,Π,Z,S,K in the system above are calculated by the corresponding bilinear forms (10)-(11) applied to the base functions of the finite-dimensional subspaces V_h, Q_h and Z_h, for example, $M=\{m_{ij}\}=m(v_i,v_j)$, $Y=\{\gamma_{ij}\}=\gamma(\xi_i,v_j)$, $\Pi=\{\pi_{ij}\}=\pi(\xi_i,q_j)$, $E=\{e_{ij}\}=e(q_i,v_j)$ (in general, a matrix denoted by some capital letter is formed by a bilinear form denoted by the respective lowercase letter). The vectors $L(t)$, $R(t)$, $F(t)$ are formed using the expression below:

$$L(t) = \{< l(t), v_i >\}, \quad R(t) = \{< r(t), q_i >\}, \quad F(t) = \{< \mu(t), \xi_i >\}$$

and vectors that represent the initial conditions of the problem are calculated via expressions:

$$V^0 = \{m(v_0, v_i)\}, \quad U^0 = \{c(u_0, v_i)\}, \quad P^0 = \{g(p_0, q_i)\},$$
$$\Theta^0 = \{s(\theta_0, \xi_i)\}, \quad \Theta^{10} = \{s(\theta_{10}, \xi_i)\}.$$

Since matrices M, A, C, G, Z, S, K are positively defined for each $h > 0$, the Cauchy problem (29) has a unique solution which in its turn defines a semi-discrete Galerkin approximation $\psi_h = (u_h, p_h, \theta_h)$ in a form(28). Moreover, taking into account Proposition 1 from the previous section we can formulate the following theorem.

Theorem 3 (*about well-posedness of semi-discretized variational problem of Green-Lindsay thermopiezoelectricity*)**.**

Let the input data of the variational problem (9) satisfy the conditions (23). Then for each $h > 0$ there exists a unique solution $\psi_h = (u_h, p_h, \theta_h)$ of the problem (27), such that

$$|||\psi_h(t)|||^2_\Phi \le ||\psi_h(0)||^2 + \int_0^t \left[||l||^2_* + ||r||^2_* + ||\mu||^2_* + \frac{t_1^2}{t_1 - t_0} ||\mu||^2_{**} \right] d\tau$$
$$\forall t \in (0, T].$$

Proof. The existence implies from the aforementioned procedure. The uniqueness is proved by Proposition 2 of the previous section. ☐ The above mentioned results let us to formulate the following theorem.

Theorem 4 (*about well-posedness of variational problem of Green-Lindsay thermopiezoelectricity*)**.** *Let the input data of the variational problem (9) satisfy the conditions (23). Then the variational problem (9) has a unique and stable solution $\psi = (u, p, \theta)$ that is characterized by regularity conditions (24) and stability conditions (25).*

Proof. Implies from the density properties of the finite-dimensional subspaces (26). ☐

6. Discretization in time

To complete the discretization of the Cauchy problem (29) we will consider a uniform partition of the time interval $[0, T]$ by nodes $t_j = j\Delta t, j = 0, 1, ..., N_T$, where N_T is some fixed number and $T = N_T \Delta t$. Then we will apply a standard Newmark scheme [5], [8] for the hyperbolic equations of motion and heat conduction with parameters γ, β and a generalized trapezoidal rule [8] with parameter $\alpha = \gamma$ for the parabolic differentiated Maxwell's equation.

The nodal approximations of elastic displacement U^{j+1} and its velocity $\dot{U}^{j+1}$ according to the Newmark scheme are defined by:

$$\begin{aligned} U^{j+1} &= U^j + \Delta t \dot{U}^j + \tfrac{\Delta t^2}{2} \left[(1 - 2\beta)\ddot{U}^j + 2\beta \ddot{U}^{j+1} \right] \\ \dot{U}^{j+1} &= \dot{U}^j + \Delta t \left[(1 - \gamma)\ddot{U}^j + \gamma \ddot{U}^{j+1} \right], \end{aligned} \tag{30}$$

where $\ddot{U}^j$ is an approximation of the mechanical acceleration at node t_j.

The nodal approximations of electric potential P^{j+1} according to the generalized trapezoidal rule are calculated as:

$$P^{j+1} = P^j + \Delta t \left[(1-\gamma)\dot{P}^j + \gamma \dot{P}^{j+1} \right], \tag{31}$$

where $\dot{P}^j$ is an approximation of the velocity of electric potential at node t_j.

The nodal approximations of temperature change Θ^{j+1} and its velocity $\dot{\Theta}^{j+1}$ according to the Newmark scheme are defined by:

$$\begin{aligned} \Theta^{j+1} &= \Theta^j + \Delta t \dot{\Theta}^j + \tfrac{\Delta t^2}{2} \left[(1-2\beta)\ddot{\Theta}^j + 2\beta \ddot{\Theta}^{j+1} \right] \\ \dot{\Theta}^{j+1} &= \dot{\Theta}^j + \Delta t \left[(1-\gamma)\ddot{\Theta}^j + \gamma \ddot{\Theta}^{j+1} \right], \end{aligned} \tag{32}$$

where $\ddot{\Theta}^j$ is an approximation of the acceleration of temperature increment at node t_j.

Combining those approximations (30)–(32) with the Cauchy problem (29) we come to the following numerical scheme:

$$\left\{ \begin{array}{l} \text{given } \Delta t > 0, t_1 \ge t_0 > 0,\ 1 \ge \gamma \ge 0,\ \tfrac{1}{2} \ge \beta \ge 0,\ (\dot{U}^j, U^j, P^j, \dot{\Theta}^j, \Theta^j); \\ \text{find } \ (\ddot{U}^{j+1}, \dot{P}^{j+1}, \ddot{\Theta}^{j+1}) \ \text{ such that} \\ \begin{pmatrix} M + \Delta t\gamma A + \Delta t^2 \beta C & -\Delta t \gamma E^T & -(\Delta t \gamma t_1 + \Delta t^2 \beta) Y^T \\ \Delta t \gamma E & G + \Delta t \gamma Z & (t_1 + \Delta t \gamma)\Pi^T \\ \Delta t \gamma Y & \Pi & (t_0 + \Delta t\gamma) S + \Delta t^2 \beta K \end{pmatrix} \times \\ \times \begin{pmatrix} \ddot{U}^{j+1} \\ \dot{P}^{j+1} \\ \ddot{\Theta}^{j+1} \end{pmatrix} = \begin{pmatrix} L_{j+1} - A\tilde{\dot{U}}^{j+1} - C\tilde{U}^{j+1} + E^T \tilde{P}^{j+1} + Y^T \left[\tilde{\Theta}^{j+1} + t_1 \tilde{\dot{\Theta}}^{j+1} \right] \\ R_{j+1} - E\tilde{\dot{U}}^{j+1} - Z\tilde{P}^{j+1} - \Pi^T \tilde{\dot{\Theta}}^{j+1} \\ F_{j+1} - Y\tilde{\dot{U}}^{j+1} - S\tilde{\dot{\Theta}}^{j+1} - K\tilde{\Theta}^{j+1} \end{pmatrix}, \end{array} \right.$$

where we have used predictors

$$\begin{aligned} \tilde{U}^{j+1} &= U^j + \Delta t \dot{U}^j + \tfrac{\Delta t^2}{2}(1-2\beta)\ddot{U}^j, \\ \tilde{\dot{U}}^{j+1} &= \dot{U}^j + \Delta t (1-\gamma)\ddot{U}^j, \\ \tilde{P}^{j+1} &= P^j + \Delta t(1-\gamma)\dot{P}^j, \\ \tilde{\Theta}^{j+1} &= \Theta^j + \Delta t \dot{\Theta}^j + \tfrac{\Delta t^2}{2}(1-2\beta)\ddot{\Theta}^j, \\ \tilde{\dot{\Theta}}^{j+1} &= \dot{\Theta}^j + \Delta t(1-\gamma)\ddot{\Theta}^j \end{aligned}$$

and then the following correctors to compute the final values at time step $j+1$

$$\begin{aligned} U^{j+1} &= \tilde{U}^{j+1} + \Delta t^2 \beta \ddot{U}^{j+1}, \\ \dot{U}^{j+1} &= \tilde{\dot{U}}^{j+1} + \Delta t \gamma \ddot{U}^{j+1}, \\ P^{j+1} &= \tilde{P}^{j+1} + \Delta t \gamma \dot{P}^{j+1}, \\ \Theta^{j+1} &= \tilde{\Theta}^{j+1} + \Delta t^2 \beta \ddot{\Theta}^{j+1}, \\ \dot{\Theta}^{j+1} &= \tilde{\dot{\Theta}}^{j+1} + \Delta t \gamma \ddot{\Theta}^{j+1}. \end{aligned} \tag{33}$$

7. Proof of time integration scheme stability

For pure hyperbolic problems the Newmark scheme is known to be unconditionally stable [5], [8] if $\gamma \geq \frac{1}{2}$ and the selection of parameters $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ gives the highest rate of convergence (equal to 2). For pure parabolic problems the generalized trapezoidal rule is also unconditionally stable [8] if $\gamma \geq \frac{1}{2}$. Therefore we would like to prove that in our case of the system of coupled PDEs (which is not pure hyperbolic or parabolic) the numerical scheme defined by (30)–(33) is unconditionally stable for $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$. For this selection of parameters γ and β the accelerations of elastic displacement $\ddot{U}^{j+\frac{1}{2}}$ and acceleration of temperature change $\ddot{\Theta}^{j+\frac{1}{2}}$ and velocity of electric potential $\dot{P}^{j+\frac{1}{2}}$ are assumed to be constant within the time interval $[t_j, t_{j+1}]$. Therefore the relations (30)-(32) for approximation of the values of unknowns at time step $j+1$ reduce to the following:

$$\begin{aligned} &U^{j+1} = U^j + \Delta t \dot{U}^{j+\frac{1}{2}}, \\ &\Theta^{j+1} = \Theta^j + \Delta t \dot{\Theta}^{j+\frac{1}{2}}, \\ &P^{j+1} = P^j + \Delta t \dot{P}^{j+\frac{1}{2}}, \\ &\dot{U}^{j+1} = \dot{U}^j + \Delta t \ddot{U}^{j+\frac{1}{2}}, \\ &\dot{\Theta}^{j+1} = \dot{\Theta}^j + \Delta t \ddot{\Theta}^{j+\frac{1}{2}}, \\ &U^{j+1} = U^{j+\frac{1}{2}} + \tfrac{1}{2}\Delta t \dot{U}^j + \tfrac{1}{4}\Delta t^2 \ddot{U}^{j+\frac{1}{2}}. \end{aligned} \tag{34}$$

Since we can do the discretization by space variables and by time variable in any order, for further proof let us consider the variational problem of Green-Lindsay thermopiezoelectricity (9) in the middle of the time interval $[t_j, t_{j+1}]$ and let us do the necessary substitutions instead of admissible functions to obtain the energy relations:

$$\begin{cases} m(\ddot{u}^{j+\frac{1}{2}}, \dot{u}^{j+\frac{1}{2}}) + a(\dot{u}^{j+\frac{1}{2}}, \dot{u}^{j+\frac{1}{2}}) + c(u^{j+\frac{1}{2}}, \dot{u}^{j+\frac{1}{2}}) - e(p^{j+\frac{1}{2}}, \dot{u}^{j+\frac{1}{2}}) - \\ -\gamma(\theta^{j+\frac{1}{2}} + t_1\dot{\theta}^{j+\frac{1}{2}}, \dot{u}^{j+\frac{1}{2}}) = < l^{j+\frac{1}{2}}, \dot{u}^{j+\frac{1}{2}} >, \\ g(\dot{p}^{j+\frac{1}{2}}, p^{j+\frac{1}{2}}) + e(p^{j+\frac{1}{2}}, \dot{u}^{j+\frac{1}{2}}) + z(p^{j+\frac{1}{2}}, p^{j+\frac{1}{2}}) + \pi(\dot{\theta}^{j+\frac{1}{2}} + t_1\ddot{\theta}^{j+\frac{1}{2}}, p^{j+\frac{1}{2}}) = \\ = < r^{j+\frac{1}{2}}, p^{j+\frac{1}{2}} >, \\ s(\dot{\theta}^{j+\frac{1}{2}} + t_0\ddot{\theta}^{j+\frac{1}{2}}, \theta^{j+\frac{1}{2}} + t_1\dot{\theta}^{j+\frac{1}{2}}) + k(\theta^{j+\frac{1}{2}}, \theta^{j+\frac{1}{2}} + t_1\dot{\theta}^{j+\frac{1}{2}}) + \\ +\pi(\theta^{j+\frac{1}{2}} + t_1\dot{\theta}^{j+\frac{1}{2}}, \dot{p}^{j+\frac{1}{2}}) + \gamma(\theta^{j+\frac{1}{2}} + t_1\dot{\theta}^{j+\frac{1}{2}}, \dot{u}^{j+\frac{1}{2}}) = \\ = < \mu^{j+\frac{1}{2}}, \theta^{j+\frac{1}{2}} + t_1\dot{\theta}^{j+\frac{1}{2}} > . \end{cases} \tag{35}$$

Here the notations $\dot{y}$ and $\ddot{y}$ denote the velocities and accelerations of the corresponding variable y (just like they were described in the previous section for semi-discretized solutions). Besides, the superscript $j+\frac{1}{2}$ always means the value of the respective function or linear form at the middle of the time interval $t_{j+\frac{1}{2}} = \frac{t_j + t_{j+1}}{2}$, for example, $u^{j+\frac{1}{2}} = u(t_{j+\frac{1}{2}})$, $l^{j+\frac{1}{2}} = l(t_{j+\frac{1}{2}})$.

It is worth to note that the relations (34) remain valid for non-discretized by space variables u^{j+1}, θ^{j+1}, p^{j+1}, $\dot{u}^{j+1}$, $\dot{\theta}^{j+1}$ as well. Therefore, for every variable y and every bilinear form $b(\cdot,\cdot)$:

$$\begin{aligned} &b(y^{j+\frac{1}{2}}, \dot{y}^{j+\frac{1}{2}}) = b(\tfrac{y^{j+1}+y^j}{2}, \tfrac{y^{j+1}-y^j}{\Delta t}) = \tfrac{1}{2\Delta t} b(y^{j+1}+y^j, y^{j+1}-y^j) = \\ &= \tfrac{1}{2\Delta t}[b(y^{j+1}, y^{j+1}) - b(y^j, y^j)] = \tfrac{1}{2\Delta t}[||y^{j+1}||_b^2 - ||y^j||_b^2]. \end{aligned} \tag{36}$$

In addition notice that

$$\begin{aligned}
&\pi(\dot\theta^{j+\frac12} + t_1\ddot\theta^{j+\frac12}, p^{j+\frac12}) + \pi(\theta^{j+\frac12} + t_1\dot\theta^{j+\frac12}, \dot p^{j+\frac12}) = \\
&= \pi(\tfrac{\theta^{j+1}-\theta^j}{\Delta t} + t_1\tfrac{\dot\theta^{j+1}-\dot\theta^j}{\Delta t}, \tfrac{p^{j+1}+p^j}{2}) + \pi(\tfrac{\theta^{j+1}+\theta^j}{2} + t_1\tfrac{\dot\theta^{j+1}+\dot\theta^j}{2}, \tfrac{p^{j+1}-p^j}{\Delta t}) = \\
&= \tfrac{1}{2\Delta t}[\pi(\theta^{j+1} + t_1\dot\theta^{j+1}, p^{j+1}) - \pi(\theta^j + t_1\dot\theta^j, p^{j+1}) + \pi(\theta^{j+1} + t_1\dot\theta^{j+1}, p^j) - \\
&-\pi(\theta^j + t_1\dot\theta^j, p^j) + \pi(\theta^{j+1} + t_1\dot\theta^{j+1}, p^{j+1}) + \pi(\theta^j + t_1\dot\theta^j, p^{j+1}) - \\
&-\pi(\theta^{j+1} + t_1\dot\theta^{j+1}, p^j) - \pi(\theta^j + t_1\dot\theta^j, p^j)] = \\
&= \tfrac{1}{2\Delta t}[2\pi(\theta^{j+1} + t_1\dot\theta^{j+1}, p^{j+1}) - 2\pi(\theta^j + t_1\dot\theta^j, p^j)]
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&s(\dot\theta^{j+\frac12} + t_0\ddot\theta^{j+\frac12}, \theta^{j+\frac12} + t_1\dot\theta^{j+\frac12}) = \\
&= s(\dot\theta^{j+\frac12} + t_0\ddot\theta^{j+\frac12}, \theta^{j+\frac12} + t_0\dot\theta^{j+\frac12}) + (t_1 - t_0)s(\dot\theta^{j+\frac12} + t_0\ddot\theta^{j+\frac12}, \theta^{j+\frac12}) = \\
&= \tfrac{1}{2\Delta t}[s(\theta^{j+1} + t_0\dot\theta^{j+1}, \theta^{j+1} + t_0\dot\theta^{j+1}) - s(\theta^j + t_0\dot\theta^j, \theta^j + t_0\dot\theta^j)] + \\
&+\tfrac{1}{2\Delta t}t_0(t_1 - t_0)[s(\dot\theta^{j+1}, \dot\theta^{j+1}) - s(\dot\theta^j, \dot\theta^j)] + (t_1 - t_0)s(\dot\theta^{j+\frac12}, \dot\theta^{j+\frac12}).
\end{aligned} \tag{38}$$

Therefore, after summing up the equations from (35) and using the expressions (36)–(38), we obtain:

$$\begin{aligned}
&\tfrac{1}{2\Delta t}[m(\dot u^{j+1}, \dot u^{j+1}) - m(\dot u^j, \dot u^j)] + \tfrac{1}{2\Delta t}[c(u^{j+1}, u^{j+1}) - c(u^j, u^j)] + \\
&+\tfrac{1}{2\Delta t}[g(\dot p^{j+1}, \dot p^{j+1}) - g(\dot p^j, \dot p^j)] + \tfrac{1}{2\Delta t}t_1[k(\theta^{j+1}, \theta^{j+1}) - k(\theta^j, \theta^j)] + \\
&+\tfrac{1}{2\Delta t}[s(\theta^{j+1} + t_0\dot\theta^{j+1}, \theta^{j+1} + t_0\dot\theta^{j+1}) - s(\theta^j + t_0\dot\theta^j, \theta^j + t_0\dot\theta^j)] + \\
&+\tfrac{1}{2\Delta t}t_0(t_1 - t_0)[s(\dot\theta^{j+1}, \dot\theta^{j+1}) - s(\dot\theta^j, \dot\theta^j)] + \\
&+\tfrac{1}{2\Delta t}[2\pi(\theta^{j+1} + t_1\dot\theta^{j+1}, p^{j+1}) - 2\pi(\theta^j + t_1\dot\theta^j, p^j)] + \\
&+a(\dot u^{j+\frac12}, \dot u^{j+\frac12}) + z(p^{j+\frac12}, p^{j+\frac12}) + \\
&+k(\theta^{j+\frac12}, \theta^{j+\frac12} + (t_1 - t_0)s(\dot\theta^{j+\frac12}, \dot\theta^{j+\frac12}) = \\
&=< l^{j+\frac12}, \dot u^{j+\frac12} > + < r^{j+\frac12}, p^{j+\frac12} > + < \mu^{j+\frac12}, \theta^{j+\frac12} + t_1\dot\theta^{j+\frac12} > .
\end{aligned} \tag{39}$$

Using the notations (16)–(17), the identity (39) can be rewritten as follows:

$$\begin{aligned}
&\tfrac{1}{2\Delta t}[||\psi^{j+1}||^2 - ||\psi^j||^2] + |||\psi^{j+\frac12}|||^2 = \\
&=< l^{j+\frac12}, \dot u^{j+\frac12} > + < r^{j+\frac12}, p^{j+\frac12} > + < \mu^{j+\frac12}, \theta^{j+\frac12} + t_1\dot\theta^{j+\frac12} > .
\end{aligned} \tag{40}$$

Using the estimates for the linear functionals like in expressions (19)-(21) we obtain:

$$\begin{aligned}
&< l^{j+\frac12}, \dot u^{j+\frac12} > + < r^{j+\frac12}, p^{j+\frac12} > + < \mu^{j+\frac12}, \theta^{j+\frac12} + t_1\dot\theta^{j+\frac12} > \le \\
&\le \tfrac12\left[||l^{j+\frac12}||_*^2 + ||r^{j+\frac12}||_*^2 + ||\mu^{j+\frac12}||_*^2 + \frac{t_1^2}{t_1 - t_0}||\mu^{j+\frac12}||_{**}^2\right] + \tfrac12|||\psi^{j+\frac12}|||^2.
\end{aligned}$$

Applying the latest inequality to (40) we receive:

$$\begin{aligned}
&\tfrac{1}{2\Delta t}[||\psi^{j+1}||^2 - ||\psi^j||^2] + |||\psi^{j+\frac12}|||^2 \le \\
&\le \tfrac12\left[||l^{j+\frac12}||_*^2 + ||r^{j+\frac12}||_*^2 + ||\mu^{j+\frac12}||_*^2 + \frac{t_1^2}{t_1 - t_0}||\mu^{j+\frac12}||_{**}^2\right] + \tfrac12|||\psi^{j+\frac12}|||^2.
\end{aligned} \tag{41}$$

Let's introduce a notation:

$$||K^{j+\frac{1}{2}}||_*^2 = ||l^{j+\frac{1}{2}}||_*^2 + ||r^{j+\frac{1}{2}}||_*^2 + ||\mu^{j+\frac{1}{2}}||_*^2 + \frac{t_1^2}{t_1 - t_0}||\mu^{j+\frac{1}{2}}||_{**}^2.$$

Then the inequality (41) transforms into:

$$\left[||\psi^{j+1}||^2 - ||\psi^j||^2\right] + \Delta t|||\psi^{j+\frac{1}{2}}|||^2 \le \Delta t||K^{j+\frac{1}{2}}||_*^2. \qquad (42)$$

Now let's add up the inequalities (42) for $j = 0, 1, ..., m$, $\forall m = 0, 1, ..., N_T - 1$:

$$||\psi^{m+1}||^2 + \Delta t \sum_{j=0}^{m} |||\psi^{j+\frac{1}{2}}|||^2 \le ||\psi^0||^2 + \Delta t \sum_{j=0}^{m} ||K^{j+\frac{1}{2}}||_*^2. \qquad (43)$$

The inequality (43) shows that the solution ψ^{m+1} obtained by the time integration scheme for each node $t_{m+1}, m = 0, 1, ..., N_T - 1$ is bounded, so the utilized scheme is stable.

8. Numerical experiments

We consider a piezoelectric bar of length $L = 10^{-8}$ m made of PZT-4 ceramics. The behavior of the bar is examined during a very short time interval $T = 11.2 \cdot 10^{-12}$ s. The boundary conditions for temperature increment θ are the ones that describe a ramp-type heating of one edge of the bar while another one is kept at the initial temperature:

$$\theta(0,t) = \theta_c \begin{cases} \frac{t}{t_p}, 0 \le t \le t_p \\ 1,\ t_p \le t \le T \end{cases},$$

$$\theta(L,t) = 0,\ 0 \le t \le T.$$

where $t_p = 10^{-12}$ s and $\theta_c = 293$ K. The boundary conditions for both mechanical and electric fields are of Neumann type:

$$\sigma = \overline{\sigma} = 0 \text{ N/m}^2 \text{ on } \Gamma_\sigma \times [0,T], \Gamma_\sigma = \{x = 0\} \cup \{x = L\},$$
$$(D' + J) = 0 \text{ A on } \Gamma_d \times [0,T], \Gamma_d = \{x = 0\} \cup \{x = L\}.$$

The initial conditions are taken to be zeros:

$$\begin{matrix} u(x,0) = 0, \\ u'(x,0) = 0, \\ p(x,0) = 0, \\ \theta(x,0) = 0 \end{matrix} \quad \forall x \in [0, L].$$

Thus, our numerical experiment reproduces the one described in the article [16] (where actually Lord-Shulman model of thermopiezoelectricity is considered, but we can compare results when the "relaxation time" parameters are small enough). So, we will take physical coefficients of PZT-4 ceramics just like in [16]:

$$\rho = 7500 \text{ kg/m}^3,$$
$$c_v = 420 \text{ J/kg} \cdot \text{K},$$
$$\lambda = 2.1 \text{ W/m} \cdot \text{K},$$
$$c = 115 \ \times 10^9 \text{ N/m}^2,$$
$$e = 15.1 \text{ C/m}^2,$$
$$\pi = -2.12 \times 10^{-4} \text{ C/K} \cdot \text{m}^2$$
$$\chi = 5.62 \times 10^{-9} \text{ C}^2/\text{N} \cdot \text{m}^2.$$

Also we will take

$$\alpha = 3.13 \times 10^{-5} \text{ K}^{-1},$$
$$z = 5 \times 10^{-12} \ \Omega^{-1} \cdot \text{m}^{-1},$$
$$a = 0 \text{ m}^2 \cdot \text{s}^{-1},$$
$$T_0 = 293 \text{ K}.$$

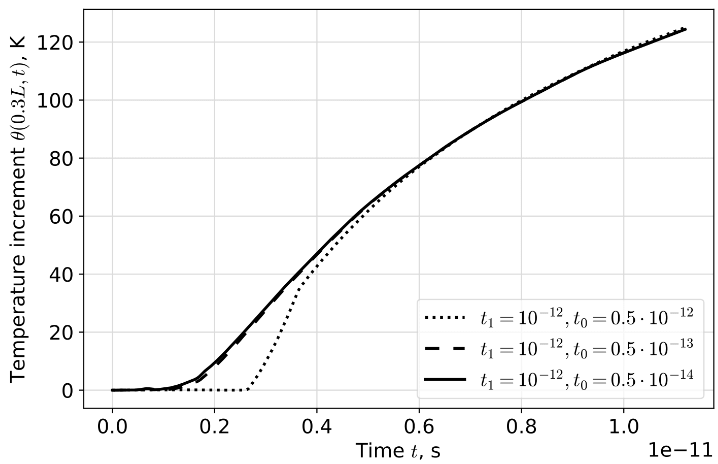

Fig. 1. Temperature increment θ at $x = 0.3L$ depending on t_0 and t_1

However, the exact values of "relaxation time" parameters t_0, t_1 are unknown for majority of materials (including PZT-4 ceramics). But practical experiments show that these values can vary between 10^{-10} s for gases and 10^{-14} s for metals. Also, recalling the condition that $t_1 \geq t_0 \geq 0$, in our experiment we will fix $t_1 = 10^{-12}$ s and show the influence of t_0 parameter of the Green-Lindsay model by sequentially setting its value to $0.5 \cdot 10^{-12}$ s, $0.5 \cdot 10^{-13}$ s and $0.5 \cdot 10^{-14}$ s.

For discretization in space we divide the interval $[0, L]$ into $N = 256$ finite elements with piecewise quadratic approximations. For time discretization we uniformly divide the time interval $[0, T]$ into $N_T = 1200$ subintervals. Besides, we will take the parameters of the time integration scheme $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ (the ones that provide unconditional stability of the numerical scheme).

Fig. 1 shows the temperature increment θ varying in time at point $x = 0.3L$ of the piezoelectric bar when one can obviously see the impact of t_0 parameter on the received solution. Moreover, if $t_0 = 0.5 \cdot 10^{-14}$ s the solution is very close to the one of classical thermopiezoelectricity problem and the one with a very small ($\tau = 10^{-14}$ s) "relaxation time" parameter of Lord-Shulman thermopiezoelectricity model, see [15] and [16].

9. Conclusions

The dynamical behaviour of the pyroelectric specimen have been studied under generalized Green-Lindsay thermopiezoelectricity model. The initial boundary value problem was formulated, transformed into the variational problem. After formulating the energy balance law the regularity conditions of the input data have been defined (quite acceptable for practical applications), which allow us to prove the well-posedness of the variational problem.

We have used similar technique as in the article [15] to construct a numerical scheme for solving the variational problem of Green-Lindsay thermopiezoelectricity. It has been proved that the constructed time integration scheme is unconditionally stable if its parameters are taken $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$.

Finally, we perform the numerical experiment inspired by other researchers' work [16] and the obtained results show us the impact of the "relaxation time" parameters on the solution of the problem.

References

1. Чир І.А. Коректність варіаційної задачі термопружності Гріна-Ліндсея / І.А. Чир, Г.А. Шинкаренко // Математичні методи та фізико-механічні поля. – 2015. – Т. 58, № 3. – С. 15–25.
2. Ashida F. Thermally-induced wave propagation in piezoelectric plate / F. Ashida, T.R. Tauchert // Acta Mechanica. – 2003. – Vol. 161. – P. 1–16.
3. Ashida F. Piezothermoelastic response of a circular plate with thermal relaxation / F. Ashida, T.R. Tauchert // J. Thermal Stresses. – 2004. – Vol. 27. – P. 513–536.
4. Bassiouny E. Two-temperature generalized thermopiezoelasticity of finite rod subjected to different types of thermal loading / E. Bassiouny, H.M. Youssef // Journal of Thermal Stresses. – 2008. – Vol. 31. – P. 233–245.
5. Bathe K.J. Finite Element Procedures. 2nd ed. / K.J. Bathe. – Watertown: K. J. Bathe, 2014. – 1043 p.
6. Chandrasekharaiah D.S. A generalized linear thermoelasticity theory for piezoelectric media / D.J. Chandrasekharaiah // Acta Mechanica. – Springer-Verlag. – 1988. – Vol. 71, No. 1-4. – P. 39–49.
7. Green A.E. Thermoelasticity / A.E. Green, K.A. Lindsay // J. Elasticity. – 1972. – Vol. 2, No. 1. – P. 1–7.
8. Hughes T.J.R. The Finite Element Method. Linear Static and Dynamic Finite Element Analysis / T.J.R. Hughes // Dover Publications. – 2000. – 704 p.
9. Kuang Z.-B. Variational principles for generalized dynamical theory of thermopiezoelectricity / Z.-B. Kuang // Acta Mechanica. – 2009. – Vol. 203. – P. 1–11.
10. Mindlin R.D. On the equations of motion of piezoelectric crystals, problems of continuum / R.D. Mindlin // In: Problems of Continuum Mechanics 70th Birthday Volume (N.I. Muskelishvili, ed.). – SIAM, Philadelpia. – 1961. – P. 282–290.
11. Nowacki W. Some general theorems of thermo-piezoelectricity / W. Nowacki // Journal of Thermal Stresses. – 1978. – Vol. 1. – P. 171–182.
12. Sharifi H. Dynamic response of an orthotropic hollow cylinder under thermal shock based on Green-Lindsay theory / H. Sharifi // Thin-Walled Structures. – 2023. – Vol.182. –110221. – P. 1-16.

13. Shivay O.N. A complete Galerkin's type approach of finite element for the solution of a problem on modified Green-Lindsay thermoelasticity for a functionally graded hollow disk / O.N. Shivay, S. Mukhopadhyay // European Journal of Mechanics – A/Solids. – 2020. – Vol. 80. – 103914. – P. 1-22.
14. Stelmashchuk V. Finite-element analysis of Green-Lindsay thermopiezoelectricity time-harmonic problem / V. Stelmashchuk, H. Shynkarenko // Вісник Львів. ун-ту. Серія прикл. матем. та інформ. – 2017. – Вип. 25. – С. 136–147.
15. Stelmashchuk V. Numerical solution of Lord-Shulman thermopiezoelectricity dynamical problem / V. Stelmashchuk, H. Shynkarenko // AIP Conf. Proc. – 2018. – Vol. 1922. – 040006. – P. 1–10.
16. Sumi N. Solution for thermal and mechanical waves in a piezoelectric plate by the method of characteristics / N. Sumi, F. Ashida // J. Thermal Stresses. – 2003. – Vol. 26. – P. 1113–1123.
17. Zenkour A.M. Refined Green-Lindsay model for the response of skin tissue under a ramp-type heating / A.M. Zenkour, T. Saeed, K.M. Alnefaie // Mathematics. – 2023. – Vol. 11. – 1437. – P. 1–18.

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СКІНЧЕННОЕЛЕМЕНТНИЙ АНАЛІЗ ДИНАМІЧНОЇ ЗАДАЧІ ТЕРМОП'ЄЗОЕЛЕКТРИКИ ГРІНА-ЛІНДСЕЯ

Р. Дреботій, В. Стельмащук, Г. Шинкаренко

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, Львів, 79000
e-mail: roman.drebotiy@lnu.edu.ua, vitalii.stelmashchuk@lnu.edu.ua,
heorhiy.shynkarenko@lnu.edu.ua

На основі моделі термоп'єзоелектрики Гріна-Ліндсея з так званими параметрами релаксації, які впливають на характер взаємодії механічного, електричного та теплового полів у піроелектриках, сформульовано початково-крайову та відповідну їй варіаційну задачу цієї моделі в термінах вектора пружних переміщень, електричного потенціалу, приросту температури. Виведено відповідне рівняння енергетичного балансу та зроблено енергетичні оцінки розв'язків варіаційної задачі. На цій підставі доведено коректність варіаційної задачі. На базі методу скінченних елементів та схеми Ньюмарка розроблено чисельну схему для наближеного розв'язування цієї задачі. Доведено безумовну стійкість побудованої схеми інтегрування в часі. Наведено результати числового експерименту, який демонструє вплив значень параметрів релаксації моделі Гріна-Ліндсея на отриманий розв'язок.

Ключові слова: піроелектричний ефект, термоп'єзоелектрика, модель Гріна-Ліндсея, початково-крайова задача, варіаційна задача, коректність варіаційної задачі, напівдискретизація Гальоркіна, метод скінченних елементів, схема Ньюмарка, стійкість числової схеми інтегрування в часі.