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# PURE STRATEGY SADDLE POINT IN PROGRESSIVE DISCRETE SILENT DUEL WITH QUADRATIC ACCURACY FUNCTIONS OF THE PLAYERS

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A zero-sum game defined on a finite subset of the unit square is considered. The game is a progressive discrete silent duel, in which the kernel is skew-symmetric. As the duel starts, time moments of possible shooting become denser by a geometric progression. Apart from the duel beginning and end moments, every following moment is the partial sum of the respective geometric series. Due to the skew-symmetry, both the duelists have the same optimal strategies and the game optimal value is 0. It is proved that the solution of a progressive discrete silent duel, with identical accuracy functions of the duelists, is a pure strategy saddle point. For nontrivial games, where the duelist possesses more than just one moment of possible shooting between the duel beginning and end moments, the saddle point is single. Moreover, the solution renders the game into an invariant decision. For the linear accuracy, whose value is not less than its time moment, the optimal strategy is the middle of the duel time span. For the quadratic accuracy the optimal strategy is the middle of the second half of the duel time span. If the linear accuracy value is less than its time moment, the middle of the duel time span is never optimal.

*Key words*: game theory, silent duel, accuracy function, matrix game, pure strategy saddle point, quadratic accuracy.

## 1. Introduction

In game theory, duels are used to model timing interactions. In particular, they can model competitive auctions between two bidders [3, 4], struggling for market control, product placement, retailing, advertising, and many other economic and social competitions between two sides [6, 7]. Duels are represented as games whose players personify participants of such competitions. The solutions to such games allow players holding at the most reasonable strategies and develop rationalized processes of sharing resources for which the players compete [15, 1, 2].

Games of timing include several distinct types of duels [4, 8, 13]. Silent duels constitute a wide class of these games [8, 15, 1]. In a typical silent duel, which alternatively may be called noiseless, each of two players has exactly one bullet, and it is unknown to them whether a bullet was fired or not until the end of the duel time span [11, 3]. The player is also featured with an accuracy function that must be a nondecreasing function of time [2, 4].

Usually the players are allowed to shoot at any moment during the duel time span. The solutions in this case, even if the accuracy functions are linear, are non-continuous probability distributions as mixed strategies with uncountably infinite supports whose measure is less than the duel time span length [5, 15]. Therefore, practical realization or implementation of such solutions cannot be complete due to naturally existing limits for any sequence of actions [12].

As a subclass of silent duels, discrete silent duels can be considered [9, 11]. In a discrete silent duel, both the players can shoot only at specified time moments whose number is finite. The moments of the duel beginning and end are included in this number [10]. If a discrete silent duel is symmetric, the respective game becomes a matrix game in which the optimal strategies of the players are identical [2, 3, 15].

The discrete silent duel becomes progressive if the importance of the pure strategy of the duelist with approaching to the conflict confrontation period completion grows in geometrical progression. In this case, the density of pure strategies of the duelist grows in geometrical progression as the duelist approaches to the duel end [9, 10, 14]. The solutions of progressive discrete silent duels with skew-symmetric payoff matrices were partly studied for identical linear accuracy functions of the players [11]. In particular, it was shown that the progressive discrete silent duel has a saddle point in pure strategies, and this saddle point is single when the player possesses a set of four or more pure strategies. Moreover, the saddle point is the same, whichever the number of pure strategies is (if the player has no fewer than four strategies).

The case with nonlinear accuracy functions was tackled also [10], but there is no thorough investigation. Meanwhile, the most common nonlinearity is quadratic. Therefore, the goal is to study solutions of the progressive discrete silent duel with quadratic accuracy functions.

The paper proceeds as follows. The preliminaries of the progressive discrete silent duel are given in Section 2. Section 3 summarizes the solution results for the case of linear accuracy functions. The case of quadratic accuracy functions is considered in Section 4. The study is discussed and concluded in Section 5.

## 2. Preliminaries

Consider a zero-sum game

$$\langle X,\ Y,\ K(x,\ y)\rangle \tag{1}$$

whose kernel is defined on unit square

$$X \times Y = [0;\ 1] \times [0;\ 1] \tag{2}$$

and

$$K(x,\ y) = x - y + xy\,\mathrm{sign}\,(y - x)\,. \tag{3}$$

Game (1) by (2) and (3) is a silent duel with linear accuracy functions of the players. As kernel (3) is skew-symmetric, i. e.

$$K(x,\ y) = -K(y,\ x)$$

due to

$$K(y,\ x) = y - x + yx\,\mathrm{sign}\,(x - y) = -K(x,\ y)\,,$$

both the players have the same optimal strategies and the game optimal value is 0 [15, 7, 8].

The duel starts at moment $x = y = 0$ and ends at moment $x = y = 1$. The duel becomes discrete if

$$X = \{x_i\}_{i=1}^{N} = Y = \{y_j\}_{j=1}^{N} = T = \{t_q\}_{q=1}^{N} \subset [0;\ 1]$$

$$\forall\, q = \overline{1,\ N-1} \ \text{ by } \ t_q < t_{q+1} \ \ \forall\, q = \overline{1,\ N-1} \ \text{ and } \ t_1 = 0,\ \ t_N = 1 \tag{4}$$

for $N \in \mathbb{N}\backslash\{1\}$. In this case, no specific locations of moments $\{t_q\}_{q=1}^{N}$ of possible shooting are given. However, in practical applications, as the duelist approaches to moment $t_N = 1$, at which the duel ends, the space between consecutive moments $t_q$ and $t_{q+1}$, $q = \overline{1,\ N-1}$, may shorten. In particular, if

$$t_q = \sum_{l=1}^{q-1} 2^{-l} \text{ for } q = \overline{2,\ N-1} \tag{5}$$

then the duel is progressive [10], where, apart from the duel beginning and end moments, every following moment is the partial sum of the respective geometric series. This duel is a matrix game whose payoff matrix

$$\mathbf{K}_N = [k_{ij}]_{N\times N} \text{ by } k_{ij} = K(x_i,\ y_j) = x_i - y_j + x_i y_j \operatorname{sign}(y_j - x_i). \tag{6}$$

Therefore, matrix game

$$\left\langle \{x_i\}_{i=1}^{N},\ \{y_j\}_{j=1}^{N},\ \mathbf{K}_N \right\rangle \tag{7}$$

by (4) – (6) is the progressive discrete silent duel with linear accuracy functions, where the accuracy is exactly equal to the time moment at which the bullet is fired.

A silent duel with quadratic accuracy functions has kernel

$$K(x,\ y) = x^2 - y^2 + x^2 y^2 \operatorname{sign}(y - x) \tag{8}$$

defined on (2), where the accuracy is the square of the time moment at which the bullet is fired. Kernel (8) is also skew-symmetric as

$$K(y,\ x) = y^2 - x^2 + y^2 x^2 \operatorname{sign}(x - y) = -K(x,\ y).$$

So, just like with kernel (3), in game (1) by (2) and (8) both the players have the same optimal strategies and the game optimal value is 0. Then the progressive discrete silent duel with quadratic accuracy functions is matrix game (7) by (4), (5), and

$$\mathbf{K}_N = [k_{ij}]_{N\times N} \text{ by } k_{ij} = K(x_i,\ y_j) = x_i^2 - y_j^2 + x_i^2 y_j^2 \operatorname{sign}(y_j - x_i). \tag{9}$$

Herein, the quadratic accuracy implies that the duelist is a worse shooter than the duelist with the linear accuracy. However, as the duelist approaches to the duel end, the difference between the linear and quadratic accuracies decreases.

## 3. Linear accuracy functions

First of all, it is useful to reminisce about the discrete duel with linear accuracy functions by (4) – (6) whose peculiarities can be found in [9, 11].

**Theorem 1.** *In a progressive discrete silent duel (7) by (4) – (6), situation*

$$\{x_2,\ y_2\} = \left\{\frac{1}{2},\ \frac{1}{2}\right\} \tag{10}$$

*is optimal and it is single for every $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$.*

*Proof.* In the case of $N = 2$ the shooting is allowed only at moments $t_1 = 0$, $t_2 = 1$. The respective payoff matrix is

$$\mathbf{K}_2 = [k_{ij}]_{2\times 2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{11}$$

and situation

$$\{x_2,\ y_2\} = \{1,\ 1\} \tag{12}$$

is the single saddle point in this trivial case. In the case of $N = 3$ the shooting is also allowed, apart from the very beginning and end moments $t_1 = 0$, $t_3 = 1$, at moment $t_2 = \frac{1}{2}$. The respective payoff matrix is

$$\mathbf{K}_3 = [k_{ij}]_{3\times 3} = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

This matrix game has four saddle points: situation (10), situation

$$\{x_3,\ y_3\} = \{1,\ 1\} \tag{13}$$

and non-symmetric situations

$$\{x_2,\ y_3\} = \left\{\frac{1}{2},\ 1\right\},$$

$$\{x_3,\ y_2\} = \left\{1,\ \frac{1}{2}\right\}.$$

It follows from (3) that $K(0,\ 0) = 0$ and $K(0,\ 1) = -1$, so $k_{11} = 0$ and $k_{1N} = -1$ for any $N \in \mathbb{N}\backslash\{1\}$. So, the game optimal value $v_{\text{opt}} = 0$ cannot be reached in the first row of matrix (6). Besides, $K(1,\ 0) = 1$, so it cannot be reached in the first column of matrix (6). Therefore, neither the first row nor the first column contains a saddle point. For $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$ consider the second row of matrix (6). Here, $x_2 = \frac{1}{2}$ and

$$K(x_2,\ y_1) = K\left(\frac{1}{2},\ 0\right) = \frac{1}{2}, \tag{14}$$

$$K(x_2,\ y_2) = K\left(\frac{1}{2},\ \frac{1}{2}\right) = 0. \tag{15}$$

For $x_2 = \frac{1}{2} < y < 1$

$$K(x_2,\ y) = K\left(\frac{1}{2},\ y\right) = \frac{1}{2} - y + \frac{1}{2}y = \frac{1-y}{2} > 0 \tag{16}$$

and

$$K(x_2,\ y_N) = K\left(\frac{1}{2},\ 1\right) = \frac{1}{2} - 1 + \frac{1}{2} = 0. \tag{17}$$

Thus, the second row of matrix (6) contains positive entries except for the second column and $N$-th column whose entries are 0. As $\mathbf{K}_N = -\mathbf{K}_N^{\mathrm{T}}$, i. e.

$$k_{ij} = -k_{ji}\ \ \forall\, i = \overline{1,\ N}\ \text{ and }\ \forall\, j = \overline{1,\ N}, \tag{18}$$

then the second column of matrix (6) contains negative entries except for the second row and $N$-th row whose entries are 0. Hence, entry $k_{22} = K(x_2,\ y_2) = 0$ is minimal in the second row and is maximal in the second column. So, it is saddle point (10).

To prove that for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$ saddle point (10) is single, consider the other rows of matrix (6), without the first, second, and $N$-th rows. This is to show that the rows of this matrix, starting from the third one and finishing at the $(N-1)$-th one, contain only negative entries under the main diagonal, except for the first column. Here, $i = \overline{3,\ N-1}$ and $j = \overline{2,\ i-1}$ that is $\frac{1}{2} \leqslant y < x < 1$, whence

$$K(x,\ y) = x - y - xy = x(1-y) - y.$$

Inasmuch as

$$-y \leqslant -\frac{1}{2}, \tag{19}$$

$$1 - y \leqslant \frac{1}{2},$$

then

$$x(1-y) < \frac{1}{2} \ \text{ by } \ x \in \left(\frac{1}{2};\ 1\right). \tag{20}$$

The sum of inequalities (19) and (20) is inequality

$$x(1-y) - y < 0 \ \text{ by } \ \frac{1}{2} \leqslant y < x < 1. \tag{21}$$

Inequality (21) implies that matrix (6) contains negative entries under its main diagonal in every $i$-th row for $i = \overline{3,\ N-1}$, except for the first column. So, these rows do not contain saddle points. In the last row,

$$K(1,\ y) = 1 - y - y = 1 - 2y < 0 \ \text{ by } \ y \in \left(\frac{1}{2};\ 1\right). \tag{22}$$

Inequality (22) implies that the last row of matrix (6) contains negative entries starting from the third column and finishing at the $(N-1)$-th column, and this row does not contain saddle points. Inequality (22) also implies that the $N$-th column does not contain saddle points that completes the proof of the singleness of saddle point (10). ☐

In fact, Theorem 1 renders any non-trivial ($2 \times 2$ and $3 \times 3$ games are apparently trivial) progressive discrete silent duel (7) by (4) – (6) into an invariant decision. This decision is saddle point (10). And, as any matrix game pure strategy solution, it factually annihilates the game. Nevertheless, this has been just the case when the accuracy is exactly equal to the time moment at which the bullet is fired. In other versions of linear accuracy functions, when the accuracy is linearly proportional to the time moment, the solution depends on the proportionality factor. Let this factor be denoted by $a$, where, obviously, $a > 0$. The case $a = 1$ is covered by Theorem 1, and it remains to consider the cases $a > 1$ and $0 < a < 1$.

**Theorem 2.** *In a progressive discrete silent duel (7) by (4), (5), and*

$$\mathbf{K}_N = [k_{ij}]_{N \times N} \ \ \textit{by}$$

$$k_{ij} = K(x_i,\ y_j) = ax_i - ay_j + a^2 x_i y_j \operatorname{sign}(y_j - x_i) \ \ \textit{for} \ \ a > 1, \tag{23}$$

*situation (10) is optimal and it is single for every $N \in \mathbb{N} \backslash \{1,\ 2\}$.*

*Proof.* In the trivial case of $N = 2$ the respective payoff matrix is

$$\mathbf{K}_2 = [k_{ij}]_{2\times 2} = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \tag{24}$$

and situation (12) is the single saddle point. In the case of $N = 3$ the respective payoff matrix is

$$\mathbf{K}_3 = [k_{ij}]_{3\times 3} = \begin{bmatrix} 0 & -\frac{a}{2} & -a \\ \frac{a}{2} & 0 & \frac{a}{2}(a-1) \\ a & -\frac{a}{2}(a-1) & 0 \end{bmatrix}. \tag{25}$$

When $a > 1$, the second row of matrix (25) is nonnegative and so situation (10) is optimal. Each of the first and third rows contains negative entries, so saddle point (10) is single for matrix (25).

As $K(0,\ 0) = 0$, $K(0,\ 1) = -a$, $K(1,\ 0) = a$, then $k_{11} = 0$, $k_{1N} = -a$, $k_{N1} = a$ for any $N \in \mathbb{N} \backslash \{1\}$. Due to (18), the game optimal value $v_{\text{opt}} = 0$ cannot be reached in the first row of matrix (23), nor can it be reached in the first column of matrix (23). Therefore, neither the first row nor the first column contains a saddle point. For $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$ consider the second row of matrix (23). Here,

$$K(x_2,\ y_1) = K\left(\frac{1}{2},\ 0\right) = \frac{a}{2} \tag{26}$$

and (15). For $x_2 = \frac{1}{2} < y < 1$

$$K(x_2,\ y) = K\left(\frac{1}{2},\ y\right) = \frac{a}{2} - ay + \frac{a^2}{2}y = \frac{a}{2}(1 - 2y + ay) \tag{27}$$

and

$$K(x_2,\ y_N) = K\left(\frac{1}{2},\ 1\right) = \frac{a}{2} - a + \frac{a^2}{2} = \frac{a}{2}(a-1). \tag{28}$$

Entry (27) is positive if

$$1 - 2y + ay > 0,$$

whence

$$a > \frac{2y-1}{y} \ \text{ by } \ y \in \left(\frac{1}{2};\ 1\right). \tag{29}$$

As, by $\frac{1}{2} < y < 1$,

$$(1-y)^2 > 0,$$
$$1 - 2y + y^2 > 0,$$
$$2y - 1 < y^2,$$
$$\frac{2y-1}{y} < y,$$

then

$$\frac{2y-1}{y} < y < 1 < a$$

and so condition (29) holds. Entry (28) is positive by $a > 1$. Thus, the second row of matrix (23) by $a > 1$ contains positive entries except for the second column whose entry is 0. Due to (18), entry $k_{22} = K(x_2, y_2) = 0$ is minimal in the second row and is maximal in the second column. So, it is saddle point (10), and there are no other saddle points in the second row.

To prove that for $N \in \mathbb{N} \backslash \{1, 2, 3\}$ saddle point (10) is single, consider the other rows of matrix (23), without the first, second, and $N$-th rows. Here, $i = \overline{3, N-1}$ and $j = \overline{2, i-1}$ that is $\frac{1}{2} \leqslant y < x < 1$, whence

$$K(x, y) = ax - ay - a^2xy = a[x(1-ay) - y].$$

Inasmuch as (19) holds,

$$1 - ay \leqslant 1 - \frac{a}{2}.$$

But $a > 1$, so

$$-a < -1,$$

$$-\frac{a}{2} < -\frac{1}{2},$$

$$1 - \frac{a}{2} < \frac{1}{2},$$

that is

$$1 - ay < \frac{1}{2}.$$

Then

$$x(1-ay) < \frac{1}{2} \text{ by } x \in \left(\frac{1}{2}; 1\right). \tag{30}$$

The sum of inequalities (19) and (30) is inequality

$$x(1-ay) - y < 0 \text{ by } \frac{1}{2} \leqslant y < x < 1. \tag{31}$$

Inequality (31) implies that matrix (23) contains negative entries under its main diagonal in every $i$-th row for $i = \overline{3, N-1}$, except for the first column. So, these rows do not contain saddle points. In the last row,

$$K(1, y) = a - ay - a^2y = a[1 - y(1+a)].$$

Owing to (19),

$$-y(1+a) < -\frac{1+a}{2},$$

$$1 - y(1+a) < 1 - \frac{1+a}{2} = \frac{1-a}{2} < 0 \text{ by } a > 1,$$

whence

$$K(1, y) = a[1 - y(1+a)] < 0 \text{ by } y \in \left(\frac{1}{2}; 1\right). \tag{32}$$

Inequality (32) implies that the last row of matrix (23) contains negative entries starting from the third column and finishing at the $(N-1)$-th column, and this row does not contain saddle points. Inequality (32) also implies that the $N$-th column does not contain saddle points that completes the proof of the singleness of saddle point (10). □

By the way, entry $k_{N2} < 0$ of matrix (23), unlike entry $k_{N2} = 0$ of matrix (6). So, generally speaking, the last row of matrix (23) contains negative entries starting from the second column and finishing at the $(N-1)$-th column. This is why Theorem 2 is not a generalization of Theorem 1. In addition, it does not relate to the case when $0 < a < 1$. It is easy to get convinced that situation (10) is not optimal for $0 < a < 1$.

**Theorem 3.** *Situation (10) is never optimal in a progressive discrete silent duel (7) by (4), (5), and*

$$\mathbf{K}_N = [k_{ij}]_{N \times N} \quad \text{by}$$

$$k_{ij} = K(x_i,\ y_j) = ax_i - ay_j + a^2 x_i y_j \operatorname{sign}(y_j - x_i) \quad \text{for} \ \ 0 < a < 1. \tag{33}$$

*Proof.* In the trivial case of $N = 2$ the respective payoff matrix is (24) and situation (12) is the single saddle point. For $N \in \mathbb{N} \backslash \{1,\ 2\}$ consider the second row of matrix (33). There are entries (26) and (28). Entry (26) is positive, and entry (28) is negative due to $0 < a < 1$. Therefore, the second row does not contain a saddle point. □

In the $3 \times 3$ trivial case of progressive discrete silent duel (7) by (4), (5), (33), the payoff matrix is still (25). The first row of matrix (25) contains negative entries and thus it does not contain a saddle point. Nor does the second row by, e. g., referring to Theorem 3. The third row contains positive entries except for $k_{33} = 0$. Consequently, situation (13) is the single saddle point in this trivial case.

## 4. Quadratic accuracy functions

It is interesting whether the case of quadratic accuracy functions bears an invariant decision similar to that in the subcase of linear accuracy functions by Theorem 2 along with Theorem 1. This question is answered by the following assertion.

**Theorem 4.** *In a progressive discrete silent duel (7) by (4), (5), (9), situation*

$$\{x_3,\ y_3\} = \left\{\frac{3}{4},\ \frac{3}{4}\right\} \tag{34}$$

*is optimal and it is single for every $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$.*

*Proof.* In the case of $N = 2$ the respective payoff matrix is (11) and situation (12) is the single saddle point in this trivial case. In the case of $N = 3$ the respective payoff matrix is

$$\mathbf{K}_3 = [k_{ij}]_{3 \times 3} = \begin{bmatrix} 0 & -\frac{1}{4} & -1 \\ \frac{1}{4} & 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$$

and situation (13) is the single saddle point.

As $K(0,\ 0) = 0$, $K(0,\ 1) = -1$, $K(1,\ 0) = 1$, then $k_{11} = 0$, $k_{1N} = -1$, $k_{N1} = 1$ for any $N \in \mathbb{N} \backslash \{1\}$. Due to (18), the game optimal value $v_{\text{opt}} = 0$ cannot be reached in the first row of matrix (9), nor can it be reached in the first column of matrix (9). Therefore, neither the first row nor the first column contains a saddle point. Analogously to that, inasmuch as

$$k_{21} = K\left(\frac{1}{2},\ 0\right) = \frac{1}{4}$$

and

$$k_{2N} = K\left(\frac{1}{2},\ 1\right) = -\frac{1}{2},$$

then the second row and second column do not contain a saddle point. For $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$ consider the third row of matrix (9). Here, $x_3 = \frac{3}{4}$ and

$$K(x_3,\ y_1) = K\left(\frac{3}{4},\ 0\right) = \frac{9}{16}, \tag{35}$$

$$K(x_3,\ y_2) = K\left(\frac{3}{4},\ \frac{1}{2}\right) = \frac{11}{64}. \tag{36}$$

For $x_3 = \frac{3}{4} < y \leqslant 1$

$$K(x_3,\ y) = K\left(\frac{3}{4},\ y\right) = \frac{9}{16} - y^2 + \frac{9}{16}y^2 = \frac{9}{16} - \frac{7}{16}y^2 > 0 \tag{37}$$

due to

$$\frac{9}{16} - \frac{7}{16}y^2 > 0\ \ \forall\, y \in [0;\ 1]\,.$$

Thus, the third row of matrix (9) contains positive entries except for the third column whose entry is 0. Due to (18) the third column of matrix (9) contains negative entries except for the third row whose entry is 0. Hence, entry $k_{33} = K(x_3,\ y_3) = 0$ is minimal in the third row and is maximal in the third column. So, it is saddle point (34).

To prove that for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$ saddle point (34) is single, consider the other rows of matrix (9), without the first, second, third, and $N$-th rows. This is to show that the rows of this matrix, starting from the fourth one and finishing at the $(N-1)$-th one, contain only negative entries under the main diagonal, except for the first and second columns. Here, $i = \overline{4,\ N-1}$ and $j = \overline{3,\ i-1}$ that is $\frac{3}{4} \leqslant y < x < 1$, whence

$$K(x,\ y) = x^2 - y^2 - x^2y^2 = x^2\left(1 - y^2\right) - y^2.$$

Inasmuch as

$$\begin{aligned} y^2 &\geqslant \frac{9}{16}, \\ -y^2 &\leqslant -\frac{9}{16}, \\ 1 - y^2 &\leqslant \frac{7}{16}, \end{aligned} \tag{38}$$

then

$$x^2\left(1 - y^2\right) < \frac{7}{16}\ \text{ by }\ x \in \left(\frac{3}{4};\ 1\right). \tag{39}$$

The sum of inequalities (38) and (39) is inequality

$$x^2\left(1 - y^2\right) - y^2 < \frac{7}{16} - \frac{9}{16} = -\frac{1}{8}\ \text{ by }\ \frac{3}{4} \leqslant y < x < 1. \tag{40}$$

Inequality (40) implies that matrix (9) contains negative entries under its main diagonal in every $i$-th row for $i = \overline{4,\ N-1}$, except for the first and second columns. So, these rows do not contain saddle points. In the last row,

$$K(1,\ y) = 1 - y^2 - y^2 = 1 - 2y^2.$$

Owing to (38),

$$-2y^2 \leqslant -\frac{9}{8},$$

$$1 - 2y^2 \leqslant -\frac{1}{8},$$

i. e.

$$K(1,\ y) = 1 - 2y^2 < 0 \ \text{ by } \ y \in \left[\frac{3}{4};\ 1\right). \tag{41}$$

Inequality (41) implies that the last row of matrix (9) contains negative entries starting from the third column and finishing at the $(N-1)$-th column, and this row does not contain saddle points. Inequality (41) also implies that the $N$-th column does not contain saddle points that completes the proof of the singleness of saddle point (34). □

Although the approach to proving Theorem 4 resembles that one to proving Theorems 1 and 2, the invariant decisions (that annihilate the respective games) differ. The patterns of the respective payoff matrices slightly differ as well. For instance, the payoff matrix of the linear accuracy duel by $a = 1$ is such that the second row, containing the saddle point by $N \in \mathbb{N} \backslash \{1,\ 2\}$, contains another zero entry in the last column of this row. This follows from (14), (16) and (17). In other words, the saddle point row, apart from the saddle point, is not completely positive. To some contrary, the payoff matrix of the quadratic accuracy duel is such that the saddle point row (which is third) by $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$, apart from the saddle point, is completely positive due to (35) – (37).

## 5. Discussion and conclusion

The quadratic accuracy implying the worse-shooting duelist seems a more practicable version as there are almost no linearly developing real-time processes. At least, such processes (or objects) are quite rare. In addition, specifying locations of moments (4) of possible shooting as (5) is natural as well. As the duelist approaches to the duel end, not only the difference between the linear and quadratic accuracies decreases, but the tension builds up. This justifies the pattern by which moments of possible shooting are made progressively denser.

The proved assertions contribute a discrete silent duel specificity to the games of timing. The specificity consists in that the solution of a progressive discrete silent duel, with identical accuracy functions of the players, is a pure strategy saddle point. For nontrivial games, where the duelist possesses more than just one moment of possible shooting between the duel beginning and end moments, the saddle point is single. Moreover, the saddle point is the same: for the linear accuracy, whose value is not less than its time moment, the optimal strategy is the middle of the duel time span (Theorems 1 and 2); for the quadratic accuracy the optimal strategy is the middle of the second half of the duel time span (Theorem 4). If the linear accuracy value is less than its time moment, the middle of the duel time span is never optimal (Theorem 3).

Progressive discrete silent duels can be further studied for other nonlinearities in the accuracy function. For instance, it can be the cubic accuracy and, as a case of the better-shooting duelist, the square-root accuracy. Besides, some peculiar solutions for the low-accurate duelist (that is the case of Theorem 3) are still a matter of interest.

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# СІДЛОВА ТОЧКА У ЧИСТИХ СТРАТЕГІЯХ ПРОГРЕСУЮЧОЇ ДИСКРЕТНОЇ БЕЗШУМНОЇ ДУЕЛІ З КВАДРАТИЧНИМИ ФУНКЦІЯМИ ВЛУЧНОСТІ ГРАВЦІВ

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Розглянуто гру з нульовою сумою, яка визначена на скінченній підмножині одиничного квадрата. Ця гра є прогресуючою дискретною безшумною дуеллю, в якій ядро кососиметричне. Як тільки дуель розпочинається, моменти часу можливих пострілів стають щільнішими за геометричною прогресією. Не враховуючи моментів початку і закінчення дуелі, кожний наступний момент є частинною сумою відповідного геометричного ряду. Внаслідок кососиметричності обидва дуелянти мають ті самі оптимальні стратегії, а оптимальне значення гри дорівнює 0. Доведено, що розв'язок прогресуючої дискретної безшумної дуелі за однакових функцій влучності дуелянтів є сідловою точкою у чистих стратегіях. Для нетривіальних ігор, де дуелянт володіє більш ніж лише одним моментом можливого пострілу між початком та закінченням дуелі, сідлова точка є єдиною. Ба більше, розв'язок зводить таку гру до інваріантного рішення. У випадку лінійної влучності, чиє значення є не меншим за відповідний момент часу, оптимальною стратегією є середина інтервалу часу тривання дуелі. Для квадратичної влучності оптимальною стратегією є середина другої половини часу тривання дуелі. Якщо значення лінійної влучності є меншим за відповідний момент часу, то середина інтервалу часу тривання дуелі ніколи не є оптимальною.

*Ключові слова*: теорія ігор, безшумна дуель, функція влучності, матрична гра, сідлова точка у чистих стратегіях, квадратична влучність.