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# AN APPLICATION OF THE METHOD OF FUNDAMENTAL SOLUTIONS FOR THE ELASTODYNAMIC PROBLEM

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A method of fundamental solutions (MFS) is employed for the numerical solution of the initial boundary value problem for the elastodynamic equation in annular planar domains. Using the Laguerre transformation in the time, the non-stationary problem is semi discretized to a sequence of stationary Dirichlet problems with an inhomogeneous equation, for which the sequence of fundamental solutions is known. The solutions of stationary problems are found by the MFS, when the unknown functions are approximated by a linear combination of narrowing of the elements from the fundamental sequence, and the source points are placed uniformly on artificial boundaries, located at fixed distances from the boundaries of the domain. The unknown coefficients in the MFS-approximations are found using the collocation method, taking into account the Dirichlet conditions on the boundaries of the domain. As a result, we obtain a sequence of recurrent SLAEs with the same matrix and recurrent right-side parts, that depend on the solutions from previous iterations. In general, for the numerical solution of a problem with an inhomogeneous equation by the method of fundamental solutions, it is necessary to find a partial solution of the inhomogeneous equation, for example, by the method of radial basis functions, however, according to our approach, this is not necessary. A step-by-step algorithm for the numerical solution of the given problem is described and the algorithm for the distribution of collocation points and source points is shown. The results of numerical experiments for different domain configurations are presented, which confirm the applicability and effectiveness of the proposed approach.

*Key words*: Dirichlet problem, elastodynamic equation, method of fundamental solutions, Laguerre transformation.

## 1. Introduction

The method of fundamental solutions was introduced by Kupradze and Aleksidze [19] for solving some homogeneous partial differential equations. The main idea of the method is to represent the solution of the problem by a linear combination of the narrowing of the fundamental solutions, followed by the search for unknown coefficients by the collocation method. The primary advantage over other methods is, that there is no need to discretize the domain of the problem, which makes it meshless, even more the method is simple to implement. For the case of inhomogeneous equations the method was extended by additionally applying the radial basis functions (RBF) technique first for the stationary problems, see [12] and later for the non-stationary problems [3, 13].

Recently, in [4,5] was introduced the approach that dispenses with the use of the RBF method and we are going to derive it for the time dependent boundary value problem for the hyperbolic elasticity equation. There are a few studies for the elastodynamic problems that are based on the MFS-RBF, see [14, 15, 17] or on the boundary integral equations method [8, 10]. This is due to the important application of the problem, for example, in structural engineering, in seismology [20] or the problem appears as a subproblem when solving some ill-posed problems [6].

Let's formulate the problem to be studied. We consider a planer bounded domain $D$, bounded by two simple closed curves $\Gamma_1$ (inner) and $\Gamma_2$ (outer), which are of class $C^2$. We denote by $\nu$ the outward unit normal to the boundaries $\Gamma_1$ and $\Gamma_2$.

We consider the Dirichlet problem for the hyperbolic elastic equation, which consists in finding an unknown vector function $\vec{u} : D \times (0, \infty) \to \mathbb{R}^2$ such that:

$$\begin{cases} \dfrac{\partial^2 \vec{u}}{\partial t^2} = \Delta^* \vec{u} & \text{in } D \times (0,\infty), \\ \vec{u} = \vec{f_\ell} & \text{on } \Gamma_\ell \times (0,\infty),\ \ell = 1,2, \\ \dfrac{\partial \vec{u}}{\partial t}(\cdot, 0) = \vec{u}(\cdot, 0) = 0 & \text{in } D, \end{cases} \tag{1}$$

where $\Delta^*$ is a Lamé operator, defined by

$$\Delta^* = c_s^2 \Delta + (c_d^2 - c_s^2)\,\mathrm{grad}\,\mathrm{div}$$

with

$$c_s = \sqrt{\frac{\mu}{\rho}}, \quad c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}}.$$

Here, $\rho$ is the density, $\lambda, \mu$ are the Lamé constants, and $\vec{f_\ell} : \Gamma_\ell \times (0, \infty) \to \mathbb{R}^2$, $\ell = 1, 2$ are given sufficiently smooth vector functions. The formulated problem (1) is well-posed, for uniqueness and existence we refer to [18]. For the simplicity we consider the Dirichlet boundary conditions, but other types of conditions can be handled similarly.

Following [4] the solution of the problem (1) is represented as a partial sum of the Fourier-Laguerre series, where the unknown coefficients are obtained from the recurrent sequence of the stationary problems for the hyperbolic equation. At the end, the recurrent system is solved using the MFS.

For the outline of the work, in the section 2, using the Laguerre transformation, we reduce the problem (1) to the sequence of the stationary problems. The application of the MFS to the obtained recurrent sequence is shown in the section 3. Section 4 presents results of some numerical experiments.

## 2. Time discretization

Following [6, 10] the Laguerre transformation is used to semi-discretize the problem (1).

**Definition 1.** *The Laguerre transformation with respect to the time-variable of the function $\vec{u}$ has the following representation:*

$$\vec{u}(x,t) = \kappa \sum_{p=0}^{\infty} \vec{u}_p(x) L_p(\kappa t), \tag{2}$$

*where $L_p(t) = \sum_{k=0}^{p} \binom{p}{k} \frac{(-t)^k}{k!}$ is the Laguerre polynomial of order $p$, $\kappa > 0$ is a given scaling constant and the Fourier-Laguerre coefficients $\vec{u}_p$ are defined as:*

$$\vec{u}_p(x) = \int_0^\infty e^{-\kappa t} L_p(\kappa t) \vec{u}(x,t)\,dt, \quad p = 0, 1, 2, \ldots. \tag{3}$$

Using properties of the Laguerre polynomials [1] the unknown coefficients $\vec{u}_p$ can be obtained from the recurrent sequence of the stationary Dirichlet problems, see [9, 10]. The result is summarized in the following theorem.

**Theorem 1.** *Function $\vec{u}$, defined by (2), is a solution of the non-stationary Dirichlet problem (1) if the Fourier-Laguerre coefficients $\vec{u}_p$, $p = 0, 1, 2, \dots$ are a solution of the sequence of stationary Dirichlet problems:*

$$\begin{cases} \Delta^* \vec{u}_p - \kappa^2 \vec{u}_p = \sum\limits_{m=0}^{p-1} \beta_{p-m} \vec{u}_m & \text{in } D, \\ \vec{u}_p = \vec{f}_{\ell,p} & \text{on } \Gamma_2, \text{ for } \ell = 1, 2, \end{cases} \tag{4}$$

*where the Fourier-Laguerre coefficients $\vec{f}_{\ell,p}$, $\ell = 1, 2$ are computed by*

$$\vec{f}_{\ell,p}(x) = \int\limits_0^\infty e^{-\kappa t} L_p(\kappa t) \vec{f}_\ell(x, t)\, dt, \quad p = 0, 1, 2, \dots,$$

*with the coefficients $\beta_p = \kappa^2(p + 1)$, $p = 0, 1, 2, \dots$.*

We refer to [16], for methods of numerical computing of the Fourier-Laguerre coefficients$\vec{f}_{\ell,p}$.

The approximation with respect to the time variable of the exact solution $\vec{u}$ is obtained as a partial sum of the representation (2) for chosen integer $N > 0$

$$\vec{u}(x, t) \approx \kappa \sum_{p=0}^{N} \vec{u}_p(x) L_p(\kappa t), \quad (x, t) \in D \times (0, \infty). \tag{5}$$

Therefore, in the next section, we focus on solving of (4) by the method of fundamental solutions.

Note that, instead of the Laguerre transformation in time, we can use some of the finite differences methods, for example, method of Rothe [7] or method of Houbolt [5].

## 3. Application of the MFS to the stationary problems (4)

### 3.1. Fundamental sequence

For the sequence of the stationary problems (4) it is possible to find a sequence of fundamental solutions $E_p$. More precisely:

**Definition 2.** *The sequence of $2 \times 2$ matrices $\{E_p\}_{p=0}^N$ is a fundamental sequence for the equations in (4) when*

$$\Delta^* E_p(x, y) - \kappa^2 E_p(x, y) - \sum_{m=0}^{p-1} \beta_{p-m} E_m = \delta(x - y) I,$$

*where $\delta$ the Dirac delta function and $I$ the $2 \times 2$ identity matrix.*

It is possible to find the expression of the elements $E_p$, see [7, 8]. Let's recall the result.

**Theorem 2.** *The functions $E_p$ with*

$$E_p(x, y) = \Phi_{1,p}(|x - y|) I + \Phi_{2,p}(|x - y|) J(x - y), \quad x \neq y, \tag{6}$$

*for $p = 0, 1, 2, \ldots, N$, constitute a fundamental sequence of the elliptic equations (4) in sense of definition 2.*

In the representation (6), $I$ is the identity matrix,

$$J(x) = \frac{xx^\top}{|x|^2},\ x \in \mathbb{R}^2 \backslash \{0\},$$

and the scalar functions $\Phi_{1,p}$ and $\Phi_{2,p}$ have the following expressions

$$\Phi_{\ell,p}(r) = \frac{(-\ell)^{\ell-1}}{\kappa^2 r^2} \sum_{k=-2}^{2} \chi_{k,p} \left\{ \Phi_{p+k}\left(\frac{\kappa}{c_s}, r\right) - \Phi_{p+k}\left(\frac{\kappa}{c_d}, r\right) \right\} + \frac{(-1)^{\ell-1}}{c_d^2} \Phi_p\left(\frac{\kappa}{c_d}, r\right) + \frac{\ell-1}{c_s^2} \Phi_p\left(\frac{\kappa}{c_s}, r\right), \text{ for } \ell = 1, 2,$$

$\chi_{-2,p} = p(p-1)$, $\chi_{-1,p} = -4p^2$, $\chi_{0,p} = 2(3p^2 + 3p + 1)$, $\chi_{1,p} = -4(p+1)^2$, $\chi_{2,p} = (p+1)(p+2)$, $c_s$ and $c_d$ are the constants from (1), and

$$\Phi_p(\gamma, r) = K_0(\gamma r) v_p(\gamma, r) + K_1(\gamma r) w_p(\gamma, r),$$

where the functions $K_0$ and $K_1$ are the modified Bessel functions (properties of the $K_0, K_1$ can be found in [1]). The polynomials $v_p$ and $w_p$ for $p = 0, 1, \ldots, N$, are given by:

$$v_p(\gamma, r) = \sum_{m=0}^{[\frac{p}{2}]} a_{p,2m}(\gamma) r^{2m} \quad \text{and} \quad w_p(\gamma, r) = \sum_{m=0}^{[\frac{p-1}{2}]} a_{p,2m+1}(\gamma) r^{2m+1},\ w_0(\gamma, r) = 0,$$

with $[q]$ being the largest integer not greater than $q$. The coefficients $a_p$ for $p = 0, 1, \ldots, N$, are obtained from the recurrence relations:

$$a_{p,0}(\gamma) = 1;$$

$$a_{p,p}(\gamma) = -\frac{\gamma}{p} a_{p-1,p-1}(\gamma);$$

$$a_{p,k}(\gamma) = \frac{1}{2\gamma k} \left\{ 4 \left[\frac{k+1}{2}\right]^2 a_{p,k+1}(\gamma) - \gamma^2 \sum_{m=k-1}^{p-1} (p - m + 1) a_{m,k-1}(\gamma) \right\}, \quad k = p-1, \ldots, 1.$$

In the last relation the coefficients $a_{p,k}$ are calculated from the last $a_{p,p-1}$ to the first $a_{p,1}$, for the fixed $p$.

### 3.2. Application of the MFS

Having a fundamental sequence (6), we can apply the MFS to discretize the stationary problems (4). According to the classical strategy for the scalar problems (see [4,19]), the unknown Fourier-Laguerre functions $\vec{u}_p$ are approximated by a linear combination of the fundamental sequence

$$\vec{u}_p(x) \approx \vec{u}_{p,n}(x) = \sum_{m=0}^{p} \sum_{k=1}^{n} E_{p-m}(x, y_k) \vec{\alpha}_{mk}, \quad x \in D, \tag{7}$$

where $n > 0$ – selected parameter, matrices $E_p$ given by (6), $y_k \notin \overline{D}$, $k = 1, 2, \dots, n$ – selected source points and $\vec{\alpha}_{mk} \in \mathbb{R}^2$, $m = 0, 1, \dots, p$, $k = 1, 2, \dots, n$ are the unknown coefficients.

There is no single strategy for choosing source points $y_k$. For the doubly-connected domains, according to [2], the source points should be placed in the unbounded exterior region of $D$ and in the bounded region enclosed by $\Gamma_1$. In each of these two regions we generate one artificial boundary and place $n/2$ evenly distributed source points $y_k$ on it. Thus, we assume that $n$ is an even integer. The algorithm of the points distribution depends on $\Gamma_1$ and $\Gamma_2$ representation, thus it is presented in the numerical examples section. For more information on the distribution of source points we refer to [11, 14].

By straightforward calculations can be checked that approximations (7) satisfy the equations in (4). The coefficients $\vec{\alpha}_{mk}$ in (7) are determined by the collocation method, from the Dirichlet boundary conditions from (4). As a result, we receive recurrent system for $p = 0, 1, \dots, N$

$$\sum_{k=1}^{n} E_0(x_{\ell j}, y_k)\vec{\alpha}_{pk} = \vec{f}_{\ell,p}(x_{\ell j}) - \sum_{m=0}^{p-1}\sum_{k=1}^{n} E_{p-m}(x_{\ell j}, y_k)\vec{\alpha}_{mk}, \tag{8}$$

for $j = 1, \dots, n/2$, $\ell = 1, 2$, where $x_{\ell j} \in \Gamma_\ell$ are selected collocation points, the collocation point distribution rule is given in next section.

Note that, the systems (8) consists of the same $2n \times 2n$ matrix and recurrent right side vectors of length $2n$.

Having obtained the coefficients $\vec{\alpha}_{pk}$ from the (8) we can construct the approximation to the solution of the time-dependent Dirichlet problem (1). Taking into account (5) and (7), we obtain the approximation

$$\vec{u}(x,t) \approx \vec{u}_{N,n}(x,t) = \kappa \sum_{p=0}^{N}\sum_{m=0}^{p}\sum_{k=1}^{n} E_{p-m}(x, y_k)\vec{\alpha}_{mk} L_p(\kappa t), \quad (x,t) \in D \times (0,\infty). \tag{9}$$

### 3.3. Main steps of the proposed method

Let's summarize the steps of the proposed method of the numerical solution of the Dirichlet problem for the hyperbolic elastodynamic equation (1).

- Choose the scaling constant $\kappa > 0$ and discretization parameters: $N > 0$ in (5), an even $n > 0$ in (7).
- Choose the source points $y_k$, $k = 1, \dots, n$, and collocation points $x_{\ell j}$, $\ell = 1, 2$, $j = 1, \dots, n/2$.
- Calculate the Fourier-Laguerre coefficients $\vec{f}_{\ell,p}$, $\ell = 1, 2$, $p = 0, 1, \dots, N$ in (4).
- Calculate the matrix of the recursive systems (8), where elements $E_0(x_{\ell j}, y_k)$, $\ell = 1, 2$, $k = 1, \dots, n$ are provided in (6).
- For $p = 0, 1, \dots, N$:
  - Generate the right-hand side vector of the system (8), where the matrices $E_p(x_{\ell j}, y_k)$ are calculated by (6). For $p > 0$ the coefficients $\vec{\alpha}_{mk}$, $m = 0, 1, \dots, p-1$, $k = 1, \dots, n$, obtained from the previous iterations unleashing (8).
  - Obtain the solution $\vec{\alpha}_{pk}$, by solving the system (8) for the current $p$.
- Calculate the numerical approximation of the solution of the problem (1) by (9), using the obtained coefficients $\vec{\alpha}_{pk}$.

Results of some numerical experiments together with the distribution of the source and collocation points are given in the next section.

## 4. Numerical examples

Let's consider the results of numerical approximation of the solution of the non-stationary Dirichlet problem (1) for two problems. Assume that the boundaries $\Gamma_\ell$, $\ell = 1, 2$ have the following representation:

$$\Gamma_\ell = \{x_\ell(s) = (x_{\ell 1}, x_{\ell 2}),\ s \in [0, 2\pi]\}\,,\ \ell = 1, 2.$$

We consider such a boundary case only for the simplicity, and the application of the MFS is not limited to it. Artificial boundaries for the source points are generated as $2x_2(s)$ and $0.5x_1(s)$. Therefore, the source points $y_k$ are distributed on these artificial boundaries according to the rule

$$y_k = (1.5\xi_k - 1)x_{\xi_k}(s_k),\ \ \xi_k = ((k+1) \bmod 2) + 1,\ s_k = \frac{2\pi}{n}k, \quad k = 1, \dots, n. \tag{10}$$

The collocation points $x_{\ell k}$ are evenly distributed on the both boundaries $\Gamma_\ell$, $\ell = 1, 2$ by the rule

$$x_{\ell j} = x_\ell(\widetilde{s}_j),\ \ \widetilde{s}_j = \frac{4\pi}{n+1}j, \quad j = 1, \dots, n/2,\ \ell = 1, 2. \tag{11}$$

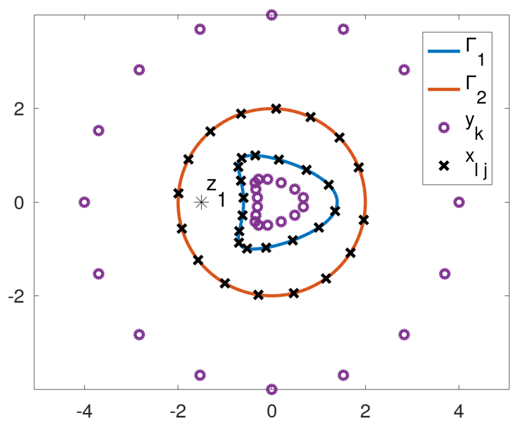


Fig. 1. The domain $D$, test point $z_1$, distribution of the source and collocation points, used in the example 1

### 4.1. Example 1

Let's consider the configuration of the solution domain $D$. The outer boundary $\Gamma_2$ has the following representation (see fig. 1)

$$\Gamma_2 = \{x_2(s) = 2(\cos s, \sin s),\ s \in [0, 2\pi]\}$$

and the inner boundary $\Gamma_1$ is chosen to be

$$\Gamma_1 = \{x_1(s) = (\cos s + 0.4\cos 2s, \sin s),\ s \in [0, 2\pi]\}\,.$$

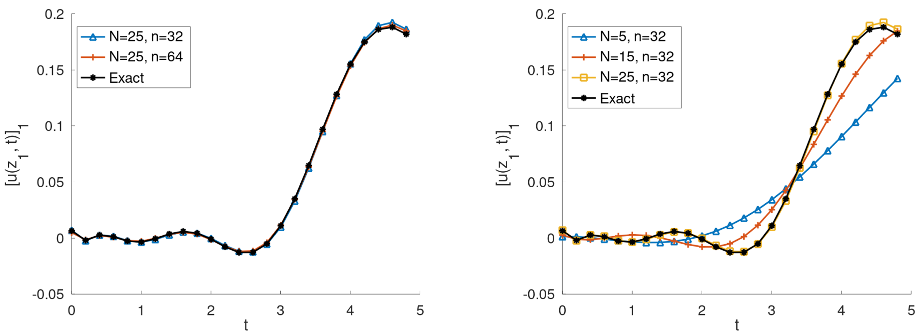


Fig. 2. Values of the first component of the exact and approximated solutions at $(z_1, t)$, $t \in (0, 5]$ for the example 1

*Table* 1

The errors of the approximated solution at $(z_1, t)$, $t \in (0, 5]$ for the example 1

| $N$ | $\| [\vec{u}_{N,32}(z_1, \cdot)]_1 - [\vec{u}_{ex}(z_1, \cdot)]_1 \|_2$ |
|---|---|
| 5 | $1.6E-1$ |
| 15 | $6.4E-2$ |
| 25 | $8.5E-3$ |

| $n$ | $\| [\vec{u}_{25,n}(z_1, \cdot)]_1 - [\vec{u}_{ex}(z_1, \cdot)]_1 \|_2$ |
|---|---|
| 16 | $9.4E-1$ |
| 32 | $8.5E-3$ |
| 64 | $5.6E-3$ |

Source points are generated by the rule (10) and collocation points by (11). The distribution of the source and collocation points for $n = 32$ is presented in the fig. 1.

The Lamé constants are selected as $\lambda = 2, \mu = 1$ and the density $\rho = 1$. The scaling constant $\kappa$ is chosen equal to 1. As the exact solution, we use the first component of a truncated series of the narrowing of fundamental solutions at the source point $z = (0, 6)^\top$

$$\vec{u}_{ex}(x, t) = \kappa \sum_{p=0}^{26} [E_p(x, z)]_1 \, L_p(\kappa t), \quad (x, t) \in D \times (0, \infty).$$

Thus, the Fourier-Laguerre coefficients $\vec{f}_{\ell,p}$ are defined exactly and are:

$$\vec{f}_{\ell,p}(x) = [E_p(x, z)]_1 \,, \quad x \in \Gamma_\ell, \, \ell = 1, 2, \, p = 0, 1, \ldots, N.$$

Let's consider the test point $z_1 = (-1.5, 0)^\top$. In the fig. 2 presented values of the first component of the exact $\vec{u}_{ex}$ and approximated $\vec{u}_{N,n}$ solutions at $(z_1, t)$, $t \in (0, 5]$ for different values of parameters $N$ and $n$. The absolute errors $\|\vec{u}_{N,n}(z_1, \cdot) - \vec{u}_{ex}(z_1, \cdot)\|_2$ of the approximated solution are given in table 1. Similar results are obtained for other test points in the domain $D$ or for the second component of the solution.

### 4.2. Example 2

Due to the fact that it is difficult to find an analytical representation of the exact solution of the problem (1), in the second example we consider the same exact solution

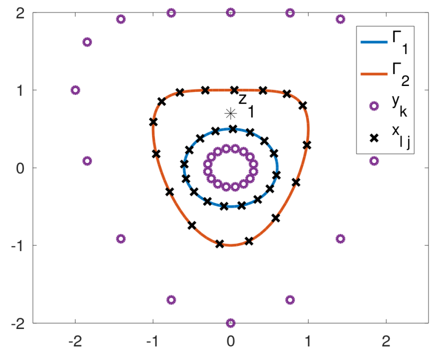

Fig. 3. The domain $D$, test point $z_1$, distribution of the source and collocation points, used in the example 2

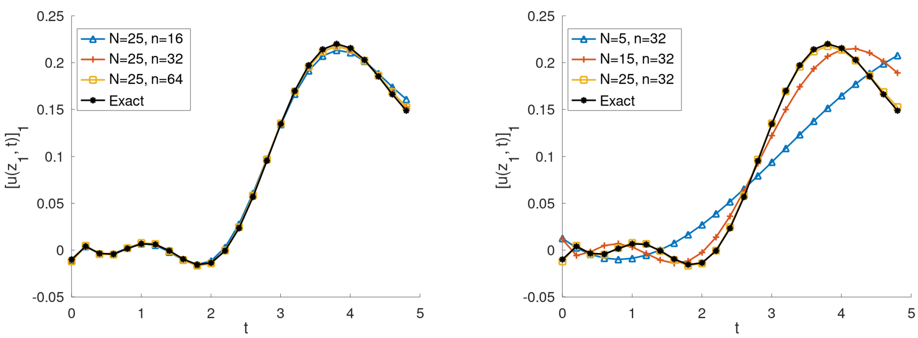

Fig. 4. Values of the first component of the exact and approximated solutions at $(z_1, t)$, $t \in (0, 5]$ for the example 2

as in the example 1, but for a different domain configuration. The boundary curves of the domain $D$ have following representation (see fig. 3)

$$\Gamma_2 = \left\{x_2(s) = (\cos s, \sin s - 0.5\sin^2 s + 0.5),\ s \in [0, 2\pi]\right\}$$

and

$$\Gamma_1 = \{x_1(s) = (0.6\cos s, 0.5\sin s),\ s \in [0, 2\pi]\}\,.$$

Let's consider the test point $z_1 = (0, 0.7)^\top$. In the fig. 4 presented values of the first component of the exact $\vec{u}_{ex}$ and approximated $\vec{u}_{N,n}$ solutions at $(z_1, t)$, $t \in (0, 5]$ for different values of parameters $N$ and $n$. The absolute errors $\|\vec{u}_{N,n}(z_1, \cdot) - \vec{u}_{ex}(z_1, \cdot)\|_2$ of the approximated solution are given in table 2.

In general, time of the program execution is about 45 seconds for $N = 25$ and $n = 32$, the program was executed on a conventional workstation with a 2.60 GHz Intel(R) Core(TM) i7 CPU. Results of both numerical examples confirms the applicability of the MFS for the numerical solution of the non-stationary Dirichlet problem for the elastodynamic equation.

*Table* 2

The errors of the approximated solution at $(z_1, t)$, $t \in (0, 5]$ for the example 2

| $N$ | $\| [\vec{u}_{N,32}(z_1, \cdot)]_1 - [\vec{u}_{ex}(z_1, \cdot)]_1 \|_2$ | $n$ | $\| [\vec{u}_{25,n}(z_1, \cdot)]_1 - [\vec{u}_{ex}(z_1, \cdot)]_1 \|_2$ |
|---|---|---|---|
| 5 | $1.8E-1$ | 16 | $2.1E-2$ |
| 15 | $8.2E-2$ | 32 | $7.4E-3$ |
| 25 | $7.4E-3$ | 64 | $7.2E-3$ |

## 5. Conclusions

The method of fundamental solutions is proposed for numerical solution of the well-posed Dirichlet problem for the hyperbolic elastodynamic equation. Using the Laguerre transformation in time, the problem is reduced to the sequence of the stationary Dirichlet problems for which the fundamental sequence is known. This makes it possible to develop the MFS for the sequence of stationary problems without using the RBF method. The source points for the MFS are distributed on the generated artificial boundaries and by the collocation method, the recurrent linear systems are obtained for the calculation of the unknown coefficients. Results of some numerical experiments are presented, which confirm the applicability of the proposed method.

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## ЗАСТОСУВАННЯ МЕТОДУ ФУНДАМЕНТАЛЬНИХ РОЗВ'ЯЗКІВ ДЛЯ ЗАДАЧІ ЕЛАСТОДИНАМІКИ

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Розглянуто застосування методу фундаментальних розв'язків (МФР) для чисельного розв'язування початково-крайової задачі еластодинаміки в плоских двозв'язних областях. За допомогою перетворення Лагерра по часовій змінній, нестаціонарна задача частково дискретизована до послідовності стаціонарних задач Діріхле з неоднорідним рівнянням, для якої відома послідовність фундаментальних розв'язків. Розв'язки стаціонарних задач знаходимо за МФР, коли невідомі функції апроксимуються лінійною комбінацією звужень елементів з фундаментальної послідовності, а точки джерела розміщуємо рівномірно на штучних границях, розташованих на фіксованих відстанях від границь області. Невідомі коефіцієнти у МФР-апроксимаціях знаходимо, використовуючи метод колокації з врахуванням умов Діріхле на границях області. В результаті, одержуємо послідовність рекурентних СЛАР з однаковою матрицею та рекурентними правими частинами, залежними від розв'язків із попередніх ітерацій. Зазвичай, для чисельного розв'язування задачі з неоднорідним рівнянням за методом фундаментальних розв'язків, потрібно знайти частковий розв'язок неоднорідного рівняння, наприклад, за методом радіальних базисних функцій, проте, за нашим підходом, цього не потрібно. Описано покроковий алгоритм для чисельного розв'язування поставленої задачі та показано алгоритм розподілу точок колокації та точок джерела. Наведено результати чисельних експериментів для різних конфігурацій областей, які підтверджують застосовність та ефективність даного підходу.

*Ключові слова*: задача Діріхле, рівняння еластодинаміки, метод фундаментальних розв'язків, перетворення Лагерра.