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# DIRECT METHOD OF LIE-ALGEBRAIC DISCRETE APPROXIMATIONS FOR SOLVING BACKWARD HEAT EQUATION

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A direct method of Lie-algebraic discrete approximation for numerical solving the Cauchy problem for the backward heat equation is proposed in this paper. The key idea of direct method of Lie-algebraic discrete approximations is using analytical approaches, in particular the method of small parameter or Taylor series expansion, to construct analytical approximation of the solution for the problem in the form of power series with respect to the time variable.

The conditions for convergence of analytical series are studied in particular. By means of small parameter method the recurrence relation for evaluation of each member of a sequence is provided. This approach enables fast computation and significant reduction of computational cost in compare to Generalized method of Lie-algebraic discrete approximations which performs complete discretization by all variables.

Thereafter, the discrete match of recurrence relation is built using quasi-representations of the Lie-algebra basis elements, which means, that each differential operator is replaced by its analogue matrix which is quasi-representation of differential operator in finite dimensional space. It is proved that computational scheme has a factorial rate of convergence.

The proposed approach is applied to model case and obtained results are compared with finite difference method, classical method of Lie-algebraic discrete approximations and Generalized method of Lie-algebraic discrete approximation. The convergence rates for all of these methods are compared in different functional spaces. In addition, we study the count of arithmetical operations for equal set of nodes. Demonstrated a possibility for reusability of the numerical scheme for heat equation.

*Key words*: direct method of Lie-algebraic discrete approximations, backward heat equation, finite dimensional quasi representation, Lagrange polynomial, small parameter method, factorial convergence.

## 1. Introduction

Backward heat equation has many applications in the diverse scientific fields: signal processing, image processing, eliminating of diffusion [29]. Hence effective numerical solution is an actual problem besides the existing of various approaches [19, 20].

We propose solution via the Direct method of Lie-algebraic discrete approximations that was firstly proposed for advection equation in [24] and has been approbated on conference [27]. This method was extended for nonlinear equation, namely Burger's nonviscous equation and was discussed on conference [5]. This method belongs to wide family of the methods that use Lie-algebraic discrete approximations [1-5, 7, 8, 10, 12-18, 21-28, 30].

Main prerequisite for these methods is that differential operator should be an element of the universe enveloping Heisenberg-Weyl's algebra with basis elements from the Lie algebra $\{1, x, d/dx\}$, i.e. differential operator within the differential equation must be a superposition and/or linear combination of these base elements of Lie algebra. As a next step we introduce the finite dimensional discrete quasi representations of $\{1, x, d/dx\}$ as matrices $\{I, X, Z\}$.

Next, if we reduce partial differential equation to system of ordinary differential equations we get the (classic) Method of Lie-algebraic discrete approximations [1, 8, 10, 14]; if we reduce partial differential equation to the system of algebraic equation (either linear or nonlinear) we get the Generalized Method of Lie-algebraic discrete approximations [21-23].

Let us explain the idea of Direct method of Lie-algebraic discrete approximation on the model problem that is investigated in [24]. Considering a bounded domain $\Omega := (0,1) \subset \mathbb{R}$, time limit $T < +\infty$, cylinder $Q_T = \Omega \times (0,T]$ we take the Banach space $V = W^{\infty,\infty}\left(\overline{Q_T}\right)$ and formulate the Cauchy problem

$$\left\{\begin{array}{l} \text{given advection coefficient } c \in \mathbb{R}, \\ \text{distribution at initial moment of time } \varphi = \varphi(x); \\ \text{find function } u = u(x,t) \in V \text{ such, that:} \\ \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0,\ \forall (x,t) \in Q_T, \\ u|_{t=0} = \varphi, \varphi \in W^{\infty,\infty}\left((-|c|T, |c|T) \cup \overline{Q}\right), \end{array}\right. \tag{1}$$

where space $V = W^{\infty,\infty}\left(\overline{Q_T}\right)$ denotes the functional space in which all functions and its derivatives up to arbitrary order are bounded in the domain $\overline{Q_T}$, i.e.

$$W^{\infty,\infty}\left(\overline{Q_T}\right) = \left\{u : Q_T \to \mathbb{R} : D^\alpha u \in L^\infty(Q_T), \forall \alpha \in \mathbb{N}\right\}.$$

The idea of a direct method of Lie-algebraic discrete approximations consists in the use of analytical approaches, in particular the method of a small parameter, to construct an approximate analytic solution of a problem (1) in the form of a power series

$$u_n(x,t) = \sum_{k=0}^{n}\left(\tilde{u}_k \frac{t^k}{k!}\right) = \varphi - c\varphi' t + c^2\varphi'' \frac{t^2}{2!} + \ldots + (-1)^n c^n \varphi^{(n)} \frac{t^n}{n!}. \tag{2}$$

After this, the corresponding discrete series was constructed for (2) using the finite dimensional quasi-representations of elements of the Lie algebra

$$u_{n,h}(t) = \sum_{k=0}^{n}\left(\tilde{u}_{k,h} \frac{t^k}{k!}\right) = \varphi_h - cZ\varphi_h t + c^2 Z^2 \varphi_h \frac{t^2}{2!} + \ldots + (-1)^n c^n Z^n \varphi_h \frac{t^n}{n!}, \tag{3}$$

where the matrix $Z$ approximates the differential operator $d/dx$. Moreover, the series (3) is finite, since the matrix $Z$ is nilpotent [15].

It was proved in [24] that the computational scheme is convergent with error rate

$$\|u - u_h\|_{V_h} \leq \frac{|c|^{n+1} T^{n+1} + \left(2max\left\{|c|T, diam\Omega\right\}\right)^{n+1}}{(n+1)!} \|\varphi^{(n+1)}\|_\infty.$$

Computational experiments showed that with the same accuracy and convergence indicators that are characteristic for the generalized method of Lie-algebraic discrete approximations, we succeeded in significantly reducing the number of arithmetic operations using approach from [24].

This paper is constructed in the following way: we formulate the model problem to which we apply the proposed numerical scheme in second chapter, analytical foundations for the proposed numerical approach are discussed in the third chapter and its Lie-algebraic discretization of the recurrence relation is investigated in the fourth chapter. Numerical results with arithmetic operations count for the model problem are provided in the fifth chapter.

## 2. Problem formulation

Considering a bounded domain $\Omega = (0,1) \subset \mathbb{R}$, time limit $T < +\infty$, cylinder $Q_T = \Omega \times (0,T]$ we take the Banach space $V = W^{\infty,\infty}\left(\overline{Q_T}\right)$ and formulate the Cauchy problem

$$\begin{cases} \text{given heat conduction coefficient } a \in \mathbb{R}, a > 0, \\ \text{temperature at initial moment of time } \varphi = \varphi(x); \\ \text{find function } u = u(x,t) \in V \text{ such, that} \\ \dfrac{\partial u}{\partial t} = -a\dfrac{\partial^2 u}{\partial x^2},\ (x,t) \in Q_T, \\ u|_{t=0} = \varphi(x),\ \ \varphi(x) \in C_x^{\infty}(\Omega)\,. \end{cases} \tag{4}$$

The solution of problem (4) we seek using iterative approach method via Lie-algebraic discrete approximations, i.e. by means of Direct method of Lie-algebraic discrete approximations.

## 3. Iterative approach and its convergence

The main focus of the direct method of Lie-algebraic discrete approximation is to approximate the solution directly. First of all we make the analytical setup for the proposed approach.

Proofs for the following lemmas can be easy obtained from the proof in [26] for heat equation by changing $a = 1$ to $a = -1$ for the current problem.

Let us denote the derivative of the function as $\varphi^{(k)} = d^k\varphi/dx^k$.

**Lemma 1.** *(The identity of series expansions). The solution expansion*

$$u_I = \sum_{k=0}^{+\infty} \tilde{u}_k \frac{t^k}{k!}, \tag{5}$$

*where $\tilde{u}_k = (-1)^k a^k \varphi^{(2k)}$, can be derived by means of iterative approach and provided here expansion is a Taylor series expansion with respect to time variable.*

**Lemma 2.** *(Convergence of the iterative approach). The sequence $\{u_k(x,t)\}$ defined in (5) converges uniformly to the exact solution, i.e.*

$$\lim_{n\to\infty} u_n(x,t) = u(x,t),$$

*where $u(x,t)$ is the solution of the problem (4).*

**Lemma 3.** *(Recurrence relation for the expansion terms). Terms $\{\tilde{u}_k(x)\}_{k=0}^{n}$ in expression (5) can be computed by means of the following recurrence relation*

$$\begin{cases} \tilde{u}_{k+1} = -a\dfrac{d^2}{dx^2}\left(\tilde{u}_k\right), \\ \tilde{u}_0 = \varphi\,. \end{cases} \tag{6}$$

Approach based on *Small Parameter Method* for obtaining (6) allows fast symbolic computation and it is a good foundation for direct method Lie-algebraic discrete approximations. Next chapter is devoted to constructing the numerical scheme on top of recurrence relation via finite dimensional quasi representations. This is an essence of *Direct method of Lie-algebraic discrete approximations.*

## 4. Numerical scheme

Let $n_x$ denotes the count of nodes in domain $\Omega$ and $n_t$ denotes count of nodes in interval $[0, T]$ and $Q_{T,h}$ denotes the mesh of nodes built upon nodes $\{x_i\}_{i=0}^{n_x}$ and $\{t_i\}_{i=0}^{n_t}$. Lagrange polynomials $l_j(x)$ built at the nodes $\{x_i\}_{i=0}^{n_x}$ form the basis in finite dimensional space $V_h$.

Let us denote the matrix $Z$ as finite dimensional quasi representation of the differential operator $d/dx$. The matrix $Z$ is built upon the rule $Z_{ij} = l'_j(x_i)$ [16]. The key property of this matrix is such, that matrix $Z^k = (Z)^k$ approximates differential operator $d^k/dx^k$ and matrix $Z$ is nilpotent [15], i.e. there is some number $n$ that all further multiplications give nil matrix: $\forall k \geq n : Z^k = \mathbf{0}$.

Let $v_I$ denotes the Lagrange interpolation of function $v(x)$ built at nodes $\{x_i\}_{i=0}^{n}$. Main approximation inequalities are proved in [24] for approximation and its derivatives:

$$\left\| v^{(k)} - v_I^{(k)} \right\|_\infty \leq \frac{(diam\Omega)^{n-k+1}}{(n-k+1)!} \|v^{(n+1)}\|_\infty$$

and approximating operator for arbitrary order of derivative

$$\|v^{(k)} - Z^k v\|_{V_h} \leq \|v^{(k)} - v_I^{(k)}\|_\infty .$$

Having built all required quasi-representations we provide the following lemma as a key finding of this paper, namely the discrete recurrence relation as a Lie-algebraic discrete approximation of the recurrence relation.

**Lemma 4.** *(Finite dimensional recurrence relation for the expansion terms). Terms $\{\tilde{u}_{k,h}\}_{k=0}^{n}$ in expression*

$$u_{n,h} = \sum_{k=0}^{n} \tilde{u}_{k,h} \frac{t^k}{k!}, \tag{7}$$

*can be computed by means of the following recurrence relation:*

$$\begin{cases} \tilde{u}_{k+1,h} = -aZ^2\left(\tilde{u}_{k,h}\right), \\ \tilde{u}_{0,h} = \varphi_h \, . \end{cases} \tag{8}$$

*which is the Lie-algebraic discretization of the recurrence relation (6).*

*Proof.* The proof of current lemma is identical to the proof in [26] by changing $a = 1$ to $a = -1$ for the current problem. □

The key finding of this paper is the proposition of method which has almost the same properties regarding the convergence but has more comprehensive way in the constructing and implementation of the numerical scheme.

Let us recall the following series estimation

$$\sum_{k=0}^{n/2} \frac{1}{(n-2k)!k!} \leq \frac{2^{n-1}}{\left(\frac{n}{2}-1\right)!}$$

proved in [26], lemmas from [24] formulate a theorem regarding convergence of the numerical scheme.

**Theorem 5.** *(Convergence of the direct Lie-algebraic numerical scheme). Let* $u = u(x,t)$ *be the solution of the problem (4),* $u_n = \sum_{k=0}^{n/2}\left((-1)^k a^k \varphi^{(2k)}\frac{t^k}{k!}\right)$ *be the Taylor expansion of the solution and* $u_h = \sum_{j=0}^{n}\left[\left(\sum_{k=0}^{n/2}\left((-1)^k a^k Z^{2k}\varphi_h\frac{t^k}{k!}\right)\right) l_j(x)\right]$ *be the finite dimensional solution. Then built numerical scheme (8) is convergent having the factorial rate of convergence:*

$$\|u-u_h\|_{V_h} \le \frac{T^{n/2+1}}{\left(\frac{n}{2}+1\right)!}\left\|\frac{\partial^{n+1}u}{\partial t^{n+1}}\right\|_\infty + \frac{\left(2max\left\{a, diam\Omega, T\right\}\right)^{n+1}}{4\left(n/2-1\right)!}\left\|\varphi^{(n+1)}\right\|_\infty. \quad (9)$$

*Proof.* Triangle inequality shows the natural way to split the norm $\|u-u_h\|_{V_h}$ in the following way:

$$\|u-u_h\|_{V_h} \le \|u-u_{n/2}\|_{V_h} + \|u_{n/2}-u_h\|_{V_h}$$

where the first norm $\|u-u_{n/2}\|_{V_h}$ represents the accuracy of approximation of the solution by means Taylor expansion and second form represents the error of Taylor series approximation by means of Lie-algebraic finite dimensional quasi representations. Using the property of error estimation of Taylor series we obtain the estimation for the first norm:

$$\|u-u_{n/2}\|_{V_h} \le \|u-u_{n/2}\|_\infty \le \frac{T^{n/2+1}}{\left(\frac{n}{2}+1\right)!}\left\|\frac{\partial^{n+1}u}{\partial t^{n+1}}\right\|_\infty.$$

Decomposition of the $\|u_{n/2}-u_h\|_{V_h}$ implies yields the following calculations:

$$\|u_{n/2}-u_h\|_{V_h} = \left\|\sum_{k=0}^{n/2}(-1)^k a^k\varphi^{(2k)}\frac{t^k}{k!} - \sum_{k=0}^{n/2}(-1)^k a^k Z^{2k}\varphi_h\frac{t^k}{k!}\right\|_{V_h} =$$
$$= \left\|\sum_{k=0}^{n/2}(-1)^k a^k\left(\varphi^{(2k)} - Z^{2k}\varphi_h\right)\frac{t^k}{k!}\right\|_{V_h} \le \sum_{k=0}^{n/2} a^k\left\|\varphi^{(2k)} - Z^{2k}\varphi_h\right\|_{V_h}\frac{t^k}{k!} \le$$
$$\le \sum_{k=0}^{n/2} a^k\left\|\varphi^{(2k)} - \varphi_I^{(2k)}\right\|_\infty\frac{t^k}{k!} \le \sum_{k=0}^{n/2} a^k\left(\frac{(diam\Omega)^{n+1-2k}}{(n+1-2k)!}\right)\frac{T^k}{k!}\left\|\varphi^{(n+1)}\right\|_\infty.$$

Let us denote $M = \max\{a, diam\Omega, T\}$ then we derive the estimation for $\|u_{n/2}-u_h\|_{V_h}$:

$$\|u_{n/2}-u_h\|_{V_h} \le \sum_{k=0}^{n/2} M^k\left(\frac{M^{n+1-2k}}{(n+1-2k)!}\right)\frac{M^k}{k!}\left\|\varphi^{(n+1)}\right\|_\infty =$$
$$= \left\|\varphi^{(n+1)}\right\|_\infty\cdot\sum_{k=0}^{n/2}\frac{M^{n+1}}{(n+1-2k)!} \le \frac{M^{n+1}2^{n-1}}{\left(\frac{n}{2}-1\right)!}\left\|\varphi^{(n+1)}\right\|_\infty = \frac{(2M)^{n+1}}{4\left(\frac{n}{2}-1\right)!}\left\|\varphi^{(n+1)}\right\|_\infty.$$

As a conclusion of the above findings we can verify that $\lim_{n\to\infty}\|u_{n/2}-u_h\|_{V_h} = 0$, in fact:

$$\lim_{n\to\infty}\|u_{n/2}-u_h\|_{V_h} \le \left\|\varphi^{(n+1)}\right\|_\infty\lim_{n\to\infty}\left(\frac{(2M)^{n+1}}{4\left(\frac{n}{2}-1\right)!}\right) = 0.$$

Finally we have the estimation (9) which implies the convergence of the proposed in (7) numerical scheme, namely $\lim\limits_{n\to\infty} \|u - u_h\|_{V_h} = 0$, since

$$\lim_{n\to\infty} \|u - u_h\|_{V_h} \le \left(\lim_{n\to\infty} \|u - u_{n/2}\|_{V_h} + \lim_{n\to\infty} \|u_{n/2} - u_h\|_{V_h}\right) = 0.$$

□

## 5. Numerical example

Let us proceed to the analysis of numerical results. For that purpose, we consider a cylindric domain $Q_T := (0,1) \times (0,1)$, i.e $x \in (0,1)$, $t \in (0,1)$. and a model problem:

$$\begin{cases} \text{find function } u = u(x,t) \text{ such, that:} \\ \dfrac{\partial u}{\partial t} = -\dfrac{\partial^2 u}{\partial x^2},\ (x,t) \in Q_T, \\ u|_{t=0} = \sin x, \end{cases} \tag{10}$$

having the exact solution $u(x,t) = e^t \sin(x)$.

The norm of the error of approximating the exact solution $u - u_h = u(x,t) - u_h(x,t)$ in the functional space $L^2(Q_T)$ is calculated by the formula

$$\|u - u_h\|^2_{L^2(Q_T)} = \int\limits_{Q_T} (u - u_h)^2\, dxdt,$$

in the functional space $L^\infty(Q_{T,h})$ is calculated at the discretization nodes:

$$\|u - u_h\|_{L^\infty(Q_{T,h})} = \sup_{(x,t)\in Q_{T,h}} |u(x,t) - u_h(x,t)|,$$

and the norm in the Sobolev's space $W^{1,2}(Q_T)$ [6] is calculated according to

$$\|u - u_h\|^2_{W^{1,2}(Q_T)} = \int\limits_{Q_T} \left[(u - u_h)^2 + \left(\frac{\partial u}{\partial x} - \frac{\partial u_h}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t} - \frac{\partial u_h}{\partial t}\right)^2\right] dxdt.$$

The exact solution is known for the problem (10), thus we use the following rule for evaluating the rate of convergence: $p_h = \log_2\left(\dfrac{\|u - u_h\|}{\|u - u_{h/2}\|}\right)$. If we get value $\|u - u_h\| = 0$ and $\|u - u_{h/2}\| = 0$, thus the value 0/0 is denoted by NaN (*not a number*).

The model problem is investigated by explicit scheme of finite differences method (FDM), the method of Lie-algebraic discrete approximations (MLADA), Generalized method of Lie-algebraic discrete approximations (GMLADA) and Direct method of Lie-algebraic discrete approximations (DMLADA). The solution of Cauchy problem with the system of differential equations was performed using Mathematica. Let us denote the step of discretization by space variable by $\Delta x = \frac{1}{(n_x - 1)}$, and $\Delta t = \frac{1}{(n_t - 1)}$ as the step of discretization by time variable. If discretization steps by both variables are equal then we use $h = \Delta x = \Delta t$ for FDM and GMLADA. Nevertheless $h$ denotes the step of discretization by space variable for MLADA, because time step is chosen automatically while solving the Cauchy problem with the system of differential equation by means of Wolfram Mathematica software.

Table. 1. Error estimations in $L^2(Q_T)$ space.

| Step $h$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| $h = 1/2$ | 0.129611 | 0.256518 | 0.0844799 | 0.0873493 |
| $h = 1/4$ | 0.0713553 | 0.0808159 | 0.020411 | 0.0204055 |
| $h = 1/8$ | 0.0380837 | 0.00404986 | 0.000689873 | 0.000689873 |
| $h = 1/16$ | $3.50191 \cdot 10^{10}$ | 1827.91 | $1.43891 \cdot 10^{-7}$ | $1.43891 \cdot 10^{-7}$ |

Table. 2. Error estimations in $L^\infty(Q_{T,h})$ space.

| Step $h$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| $h = 1/2$ | 0.420034 | 0.976364 | 0.36965 | 0.369653 |
| $h = 1/4$ | 0.241512 | 0.356581 | 0.099300 | 0.0992996 |
| $h = 1/8$ | 0.130843 | 0.0212378 | 0.0038209 | 0.00382094 |
| $h = 1/16$ | $7.08679 \cdot 10^{11}$ | 12995.3 | $9.1009 \cdot 10^{-6}$ | $9.10089 \cdot 10^{-6}$ |

Table. 3. Error estimations in $W^{1,2}(Q_T)$ space.

| Step $h$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| $h = 1/2$ | 0.413269 | 0.984519 | 0.374173 | 0.378134 |
| $h = 1/4$ | 0.2323471 | 0.360836 | 0.104918 | 0.104912 |
| $h = 1/8$ | 0.124427 | 0.0239794 | 0.00464834 | 0.00464834 |
| $h = 1/16$ | $2.09896 \cdot 10^{12}$ | 22343.5 | $1.49632 \cdot 10^{-6}$ | $1.49632 \cdot 10^{-6}$ |

Table. 4. Rates of convergence in $L^2(Q_T)$ space.

| Step $h$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| $h = 1/2$ | 0.861099 | 1.66635 | 2.04926 | 2.09784 |
| $h = 1/4$ | 0.905849 | 4.3187 | 4.88687 | 4.88648 |
| $h = 1/8$ | −39.7421 | −18.7839 | 12.2271 | 12.2271 |

Table. 5. Rates of convergence in $L^\infty(Q_T)$ space.

| Step $h$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| $h = 1/2$ | 0.798413 | 1.45319 | 1.89631 | 1.89631 |
| $h = 1/4$ | 0.884252 | 4.06952 | 4.69979 | 4.69979 |
| $h = 1/8$ | −42.3004 | −19.2229 | 12.0356 | 12.0356 |

Table. 6. Rates of convergence in $W^{1,2}(Q_T)$ space.

| Step $h$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| $h = 1/2$ | 0.830802 | 1.44808 | 1.83445 | 1.84972 |
| $h = 1/4$ | 0.900985 | 3.91147 | 4.4964 | 4.49632 |
| $h = 1/8$ | −43.9394 | −19.8296 | 11.6011 | 11.6011 |

From the above tables we can see the increase of errors in MLADA. This is caused by the stiff system of ordinary differential equations to which the partial differential equation was reduced to. Such systems need either increasing the count of nodes or usage of some special numerical techniques.

Table. 7. Count of arithmetic operations for $n_x = n_t = 3$

| Step $h = 1/16$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| Error in $L^2(\Omega_T)$ space | 0.0419804 | 0.129574 | 0.0479767 | 0.0507986 |
| Additions, substractions | 42 | 141 | 1081 | 42 |
| Multiplications | 50 | 157 | 1173 | 51 |
| Divisions | 18 | 3 | 42 | 2 |
| Time (ms) | 5 | 1 | 41 | 7 |

Table. 8. Count of arithmetic operations for $n_x = n_t = 5$

| Step $h = 1/16$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| Error in $L^2(\Omega_T)$ space | 0.0199765 | 0.051718 | 0.0146769 | 0.0146827 |
| Additions, substractions | 146 | 607 | 92921 | 150 |
| Multiplications | 152 | 663 | 93785 | 175 |
| Divisions | 66 | 5 | 420 | 4 |
| Time (ms) | 5 | 1 | 84 | 7 |

Table. 9. Count of arithmetic operations for $n_x = n_t = 9$

| Step $h = 1/16$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| Error in $L^2(\Omega_T)$ space | 0.00965197 | 0.00343672 | 0.000637523 | 0.000637523 |
| Additions, substractions | 546 | 3291 | 14007601 | 774 |
| Multiplications | 524 | 3499 | 14018193 | 855 |
| Divisions | 258 | 9 | 5256 | 8 |
| Time (ms) | 6 | 2 | 998 | 15 |

Table. 10. Count of arithmetic operations for $n_x = n_t = 17$

| Step $h = 1/16$ | FDM | MLADA | GMLADA | DMLADA |
|---|---|---|---|---|
| Error in $L^2(\Omega_T)$ space | $4.19664 \cdot 10^{11}$ | 18151.3 | $1.85966 \cdot 10^{-6}$ | $1.85966 \cdot 10^{-6}$ |
| Additions, substractions | 2114 | 21043 | 2767151201 | 4998 |
| Multiplications | 1940 | 21843 | 2767300001 | 5287 |
| Divisions | 1026 | 17 | 74256 | 16 |
| Time (ms) | 7 | 4 | 203068 | 22 |

The main benefit of using the proposed numerical scheme is reduced count of arithmetic operations maintaining the same computational properties as a generalized method of Lie-algebraic discrete approximations. Provided tables demonstrate such behavior.

Moreover, proposed approach is reusable for backward heat equation and original heat equation in that sense that the same numerical scheme can be used and only single change of coefficient is needed, i.e. $a := 1$ for original heat equation and $a := -1$ for backward heat equation. The same behavior was observed for Generalized method of Lie algebraic discrete approximations [23].

## 6. Conclusions

We have applied the direct method of Lie-algebraic discrete approximations for solving the Cauchy problem for backward heat equation in this paper. Different numerical schemes (finite difference method, classical method of Lie-algebraic discrete approximations, generalized method of Lie-algebraic discrete approximations and direct method of Lie-algebraic discrete approximations) are compared for solving the Cauchy problem for backward heat equation. One can obtain numerical result with the same high precision and with significantly less computational costs in compare to the generalized method of Lie-algebraic discrete approximations because that method approximates the solution instead of the differential operator of the equation.

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## ПРЯМИЙ МЕТОД ЛІ-АЛГЕБРИЧНИХ ДИСКРЕТНИХ АПРОКСИМАЦІЙ ДЛЯ РОЗВ'ЯЗУВАННЯ ОБЕРНЕНОГО РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ

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Запропоновано та обґрунтовано прямий метод Лі-алгебричних дискретних апроксимацій для чисельного розв'язування задачі Коші для оберненого рівняння теплопровідності. Ідея прямого методу Лі-алгебричних дискретних апроксимацій полягає у тому, що з використанням аналітичних підходів, зокрема метода малого параметра, або розкладу у ряд Тейлора, побудовано наближений аналітичний розв'язок задачі у вигляді степеневого ряду за часовою змінною. Після цього побудовано його дискретний відповідник з використанням квазізображень елементів алгебри Лі. Доведено, що обчислювальна схема має факторіальний порядок збіжності. З'ясовано можливість повторного використання обчислювальної схеми для рівняння теплопровідності.

*Ключові слова*: прямий метод Лі-алгебричних дискретних апроксимацій, обернене рівняння теплопровідності, скінченновимірне квазізображення, поліном Лагранжа, метод малого параметра, факторіальна збіжність.