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SCHEME FOR APPROXIMATE SOLVING SYSTEMS OF SECOND-ORDER MATRIX EQUATIONS

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In view of the growing role of unmanned aerial vehicles in the military sphere and the increasing automation of enterprises, the problem of improving control quality is urgent. Matrix equations and systems of matrix equations are widely used in some applied disciplines, in particular, in optimization problems of control systems. However, there is no universal approach to solving all problems of this class: only methods of solving the most popular matrix equations, such as the Sylvester, Riccati, and Lyapunov equations, have been developed. There is an opportunity to explore this topic in more detail in the papers of Beavers A.N., Denman E.D., Boichuk A.A., Krivosheya S.A. and Kramer K. In one of our previous articles, a method for solving systems of algebraic equations over the field of real numbers was proposed. In this paper, the previously considered approach is generalized, and an approximate solution scheme for systems of polynomial matrix equations of the second degree with many unknowns is presented. A recurrent formula for calculating the approximate solution of systems in the form of a continued matrix fraction is also given. The convergence of the proposed method is investigated based on Vorpitsky's sufficient condition. The results of numerical experiments that confirm the validity of theoretical calculations and the effectiveness of the proposed scheme for the approximate solution of matrix equations are presented.

Key words: iterative method, systems of matrix equations, matrix equations of the second degree.

1. Introduction

A lot of applied problems [1], [3], [4] come down to solving systems of algebraic equations. Therefore, it is difficult to overestimate the role of methods for finding solutions of such systems in applied mathematics. In [6], a new iterative scheme for solving systems of polynomial equations of the second degree by developing a solution into a chain matrix fraction was presented. A similar approach can be used to solve systems of matrix equations that arise in many theoretical and applied disciplines, for example, in the theory of linear Hamiltonian systems, calculus of variations, in the optimal control problems, filtering problems, in control theory stabilization of linear systems problems and others [5].

Let us consider a system of n polynomial-matrix equations of the second degree:

$$\begin{aligned}
&A_{1,X_1X_1}X_1^2 + A_{1,X_1X_2}X_1X_2 + \ldots + A_{1,X_1X_n}X_1X_n + A_{1,X_2X_1}X_2X_1 + \\
&\qquad + A_{1,X_2X_n}X_2X_n + \ldots + A_{1,X_nX_n}X_n^2 + A_{1,X_1}X_1 + A_{1,X_2}X_2 + \ldots + \\
&\qquad + A_{1,X_n}X_n + A_{1,1} = 0; \\
&A_{2,X_1X_1}X_1^2 + A_{2,X_1X_2}X_1X_2 + \ldots + A_{2,X_1X_n}X_1X_n + A_{2,X_2X_1}X_2X_1 + \\
&\qquad + A_{2,X_2X_n}X_2X_n + \ldots + A_{2,X_nX_n}X_n^2 + A_{2,X_1}X_1 + A_{2,X_2}X_2 + \ldots + \\
&\qquad + A_{2,X_n}X_n + A_{2,1} = 0;
\end{aligned} \qquad (1)$$

$$\begin{aligned}
&\vdots\\
&A_{n,X_1X_1}X_1^2 + A_{n,X_1X_2}X_1X_2 + \ldots + A_{n,X_1X_n}X_1X_n + A_{n,X_2X_1}X_2X_1 +\\
&\quad + A_{n,X_2X_n}X_2X_n + \ldots + A_{n,X_nX_n}X_n^2 + A_{n,X_1}X_1 + A_{n,X_2}X_2 + \ldots +\\
&\quad + A_{n,X_n}X_n + A_{n,1} = 0;
\end{aligned}$$

Unknown $X_i (i = 1, 2, \ldots, n)$ and coefficients A_{l,X_iX_j}, A_{l,X_i} $(l = 1, 2;\ i, j = 1, 2, \ldots, n)$ are real square noncommutative matrices, which have dimension $m \times m$.

2. Computational scheme of the method

We apply some elementary transformations to the system of matrix polynomial equations (1) and present it in the form

$$\begin{pmatrix} A_{1,X_1X_1}X_1 + A_{1,X_2X_1}X_2 + \ldots + A_{1,X_nX_1}X_n + A_{1,X_1} \\ A_{2,X_1X_1}X_1 + A_{2,X_2X_1}X_2 + \ldots + A_{2,X_nX_1}X_n + A_{2,X_1} \\ \vdots \\ A_{n,X_1X_1}X_1 + A_{n,X_2X_1}X_2 + \ldots + A_{n,X_nX_1}X_n + A_{n,X_1} \end{pmatrix} X_1 +$$

$$+ \begin{pmatrix} A_{1,X_1X_2}X_1 + A_{1,X_2X_2}X_2 + \ldots + A_{1,X_nX_2}X_n + A_{1,X_2} \\ A_{2,X_1X_2}X_1 + A_{2,X_2X_2}X_2 + \ldots + A_{2,X_nX_2}X_n + A_{2,X_2} \\ \vdots \\ A_{n,X_1X_2}X_1 + A_{n,X_2X_2}X_2 + \ldots + A_{n,X_nX_2}X_n + A_{n,X_2} \end{pmatrix} X_2 + \ldots +$$

$$+ \begin{pmatrix} A_{1,X_1X_n}X_1 + A_{1,X_2X_n}X_2 + \ldots + A_{1,X_nX_n}X_n + A_{1,X_n} \\ A_{2,X_1X_n}X_1 + A_{2,X_2X_n}X_2 + \ldots + A_{2,X_nX_n}X_n + A_{2,X_n} \\ \vdots \\ A_{n,X_1X_n}X_1 + A_{n,X_2X_n}X_2 + \ldots + A_{n,X_nX_n}X_n + A_{n,X_n} \end{pmatrix} X_n = - \begin{pmatrix} A_{1,1} \\ A_{2,1} \\ \vdots \\ A_{n,1} \end{pmatrix},$$

or

$$\left(\left(\begin{pmatrix} A_{1,X_1X_1} & A_{1,X_1X_2} & \cdots & A_{1,X_1X_n} \\ A_{2,X_1X_1} & A_{2,X_1X_2} & \cdots & A_{2,X_1X_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,X_1X_1} & A_{n,X_1X_2} & \cdots & A_{n,X_1X_n} \end{pmatrix} X_1 + \begin{pmatrix} A_{1,X_2X_1} & A_{1,X_2X_2} & \cdots & A_{1,X_2X_n} \\ A_{2,X_2X_1} & A_{2,X_2X_2} & \cdots & A_{2,X_2X_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,X_2X_1} & A_{n,X_2X_2} & \cdots & A_{n,X_2X_n} \end{pmatrix} X_2 + \right.\right.$$

$$+ \ldots + \begin{pmatrix} A_{1,X_nX_1} & A_{1,X_nX_2} & \cdots & A_{1,X_nX_n} \\ A_{2,X_nX_1} & A_{2,X_nX_2} & \cdots & A_{2,X_nX_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,X_nX_1} & A_{n,X_nX_2} & \cdots & A_{n,X_nX_n} \end{pmatrix} X_n +$$

$$\left.\left.\begin{pmatrix} A_{1,X_1} & A_{1,X_2} & \cdots & A_{1,X_n} \\ A_{2,X_1} & A_{2,X_2} & \cdots & A_{2,X_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,X_1} & A_{n,X_2} & \cdots & A_{n,X_n} \end{pmatrix}\right)\right) \circ \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = - \begin{pmatrix} A_{1,1} \\ A_{2,1} \\ \cdots \\ A_{n,1} \end{pmatrix}. \qquad (2)$$

Let us introduce the following notation

$$B_1=\begin{pmatrix} A_{1,X_1X_1} & A_{1,X_1X_2} & \dots & A_{1,X_1X_n} \\ A_{2,X_1X_1} & A_{2,X_1X_2} & \dots & A_{2,X_1X_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,X_1X_1} & A_{n,X_1X_2} & \dots & A_{n,X_1X_n} \end{pmatrix}, B_2=\begin{pmatrix} A_{1,X_2X_1} & A_{1,X_2X_2} & \dots & A_{1,X_2X_n} \\ A_{2,X_2X_1} & A_{2,X_2X_2} & \dots & A_{2,X_2X_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,X_2X_1} & A_{n,X_2X_2} & \dots & A_{n,X_2X_n} \end{pmatrix},$$

$$\dots\dots\dots$$

$$B_n=\begin{pmatrix} A_{1,X_nX_1} & A_{1,X_nX_2} & \dots & A_{1,X_nX_n} \\ A_{2,X_nX_1} & A_{2,X_nX_2} & \dots & A_{2,X_nX_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,X_nX_1} & A_{n,X_nX_2} & \dots & A_{n,X_nX_n} \end{pmatrix}, B_{n+1}=\begin{pmatrix} A_{1,X_1} & A_{1,X_2} & \dots & A_{1,X_n} \\ A_{2,X_1} & A_{2,X_2} & \dots & A_{2,X_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,X_1} & A_{n,X_2} & \dots & A_{n,X_n} \end{pmatrix},$$

$$Y=-\begin{pmatrix} A_{1,1} & A_{2,1} & \dots & A_{n,1} \end{pmatrix}^T.$$

Then (2) will look like

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \left(\begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix} \circ \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} + B_{n+1} \right)^{-1} Y. \tag{3}$$

Product $\begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix} \circ \begin{pmatrix} X_1 & X_2 & \dots & X_n \end{pmatrix}^T$ denotes the sum $\sum\limits_{i=1}^{n} B_i X_i$.

Thus to find the solution of the equation (1) based on (3) we can develop the following algorithm:

1. Set the admissible error $\varepsilon > 0$;
2. Set the initial approximation, valid square matrices $X_i^{(0)}$ $(i = 1, 2, \dots, n)$;
3. Establish counter $k = 1$;
4. Calculate $X_i^{(1)}$ $(i = 1, 2, \dots, n)$ by the fomula

$$\begin{pmatrix} X_1^{(1)} \\ X_2^{(1)} \\ \vdots \\ X_n^{(1)} \end{pmatrix} = \left(\begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix} \circ \begin{pmatrix} X_1^{(0)} \\ X_2^{(0)} \\ \vdots \\ X_n^{(0)} \end{pmatrix} + B_{n+1} \right)^{-1} Y. \tag{4}$$

5. Check the condition $\left\| X_i^{(1)} - X_i^{(0)} \right\| \le \varepsilon$ $(i = 1, 2, \dots, n)$. If the condition is not satisfied, set the counter $k = k + 1$, assign to matrices $X_i^{(0)}$ values of matrices $X_i^{(1)} (i = 1, 2, \dots, n)$ and go to the step 4, otherwise return $X_i^{(1)} (i = 1, 2, \dots, n)$.

3. Convergence of scheme for systems of matrix equations of the second degree

Let A be certain real nondegenerate square matrix with dimension $mn \times mn$, vector F be a vector with dimension $mn \times m$ over a field of real numbers. The operation $A^{-1}F$ will sometimes be denoted as $\frac{F}{A}$ for convenience. Then (3) can also be written as

$$\begin{pmatrix} X_1 & X_2 & \dots & X_n \end{pmatrix}^T = \frac{Y}{B_{n+1} + \begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix} \circ \begin{pmatrix} X_1 & X_2 & \dots & X_n \end{pmatrix}^T}. \quad (5)$$

By using (5)the solution can be written as the following continued fraction

$$\begin{pmatrix} X_1 & X_2 & \dots & X_n \end{pmatrix}^T = \frac{Y|}{|B_{n+1}} + \begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix} \circ \frac{Y|}{|B_{n+1}} + $$
$$+ \begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix} \circ \frac{Y|}{|B_{n+1}} + \dots \quad (6)$$

in a compact form of Pringsheim's notation.

The convergence of the iterative process (4) requires the convergence of the operator continued fraction (6).

Vorpitsky's sufficient condition for convergence was generalized in paper [2]. This feature can be used in the analysis of the convergence of the matrix continued fraction (6):

Theorem 1. *The matrix continued fraction*

$$\sum_{k_1=1}^{n} \frac{A_{k_1}|}{|E} + \sum_{k_2=1}^{n} \frac{A_{k_1k_2}|}{|E} + \dots + \sum_{k_l=1}^{n} \frac{A_{k_1k_2\dots k_l}|}{|E} + \dots$$

is absolutely convergent if the condition

$$\|A_{k_1k_2\dots k_l}\| \le \frac{1}{4n} \quad (i = 1, 2, 3, \dots;\ k_l = 1, 2, \dots, n)$$

is satisfied.

Let us apply the theorem (1) to the continued fraction (6). Obviously, this continued fraction will coincide absolutely if the condition

$$\left\| \begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix} \circ (B_{n+1})^{-1} Y \right\| \le \frac{1}{4}. \quad (7)$$

is satisfied.

Substituting the values of $B_1, B_2, \dots, B_{n+1}, Y$ into the inequality (7), we obtain a sufficient condition for the convergence of the matrix continued fraction (5).

4. Computational experiments

The above algorithm was implemented in the FreeMat environment. Let us consider the following prepared test examples to demonstrate its applicability and effectiveness:

Example 1. Let us consider a system of matrix equations that has the form

$$\begin{cases} A_{1,X_1X_1}X_1^2 + A_{1,X_1X_2}X_1X_2 + A_{1,X_1X_3}X_1X_3 + A_{1,X_2X_1}X_2X_1 + \\ \quad +A_{1,X_2X_2}X_2^2 + A_{1,X_2X_3}X_2X_3 + A_{1,X_3X_1}X_3X_1 + A_{1,X_3X_2}X_3X_2 + \\ \quad +A_{1,X_3X_3}X_3^2 + A_{1,X_1}X_1 + A_{1,X_2}X_2 + A_{1,X_3}X_3 + A_{1,1} = 0; \\ A_{2,X_1X_1}X_1^2 + A_{2,X_1X_2}X_1X_2 + A_{2,X_1X_3}X_1X_3 + A_{2,X_2X_1}X_2X_1 + \\ \quad +A_{2,X_2X_2}X_2^2 + A_{21,X_2X_3}X_2X_3 + A_{2,X_3X_1}X_3X_1 + A_{2,X_3X_2}X_3X_2 + \\ \quad +A_{2,X_3X_3}X_3^2 + A_{2,X_1}X_1 + A_{2,X_2}X_2 + A_{2,X_3}X_3 + A_{2,1} = 0; \\ A_{3,X_1X_1}X_1^2 + A_{3,X_1X_2}X_1X_2 + A_{3,X_1X_3}X_1X_3 + A_{3,X_2X_1}X_2X_1 + \\ \quad +A_{3,X_2X_2}X_2^2 + A_{3,X_2X_3}X_2X_3 + A_{3,X_3X_1}X_3X_1 + A_{3,X_3X_2}X_3X_2 + \\ \quad +A_{3,X_3X_3}X_3^2 + A_{3,X_1}X_1 + A_{3,X_2}X_2 + A_{3,X_3}X_3 + A_{3,1} = 0, \end{cases} \tag{8}$$

where

$$A_{1,X_1X_1} = \begin{pmatrix} -10 & 3 \\ 3 & -2.11 \end{pmatrix},\ A_{1,X_1X_2} = \begin{pmatrix} 2 & 0 \\ -28 & -3 \end{pmatrix},\ A_{1,X_1X_3} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix},$$

$$A_{1,X_2X_1} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},\ A_{1,X_2X_2} = \begin{pmatrix} 2 & 1 \\ -1.1 & -3 \end{pmatrix},\ A_{1,X_2X_3} = \begin{pmatrix} -3 & -1 \\ 1 & 2 \end{pmatrix},$$

$$A_{1,X_3X_1} = \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix},\ A_{1,X_3X_2} = \begin{pmatrix} -4 & 1 \\ -2 & 2 \end{pmatrix},\ A_{1,X_3X_3} = \begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix},$$

$$A_{1,X_1} = \begin{pmatrix} 1.5 & -5 \\ 0 & -2.2 \end{pmatrix},\ A_{1,X_2} = \begin{pmatrix} 0 & 0.1 \\ 3.2 & -2.5 \end{pmatrix},\ A_{1,X_3} = \begin{pmatrix} 1.4 & -3 \\ -5.3 & 1.1 \end{pmatrix},$$

$$A_{11} = \begin{pmatrix} 30.3 & -5.8 \\ -110.8 & 60.24 \end{pmatrix}$$

$$A_{2,X_1X_1} = \begin{pmatrix} -2 & 1 \\ 22 & -1 \end{pmatrix},\ A_{2,X_1X_2} = \begin{pmatrix} -3 & -2 \\ -2 & 1 \end{pmatrix},\ A_{2,X_1X_3} = \begin{pmatrix} -3 & -3 \\ 3 & -1 \end{pmatrix},$$

$$A_{2,X_2X_1} = \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix},\ A_{2,X_2X_2} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ A_{2,X_2X_3} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix},$$

$$A_{2,X_3X_1} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix},\ A_{2,X_3X_2} = \begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix},\ A_{2,X_3X_3} = \begin{pmatrix} 3 & -2 \\ 41 & \end{pmatrix},$$

$$A_{2,X_1} = \begin{pmatrix} 0 & -3 \\ 3 & 2 \end{pmatrix},\ A_{2,X_2} = \begin{pmatrix} -1 & 4 \\ 10 & 1 \end{pmatrix}, A_{2,X_3} = \begin{pmatrix} 5 & 1 \\ 1 & 0.5 \end{pmatrix},$$

$$A_{2,1} = \begin{pmatrix} -12 & 0 \\ -110.5 & -53.5 \end{pmatrix},$$

$$A_{3,X_1X_1} = \begin{pmatrix} -2 & -13 \\ -1 & 1 \end{pmatrix},\ A_{3,X_1X_2} = \begin{pmatrix} 3 & -4 \\ 0 & 5 \end{pmatrix},\ A_{3,X_1X_3} = \begin{pmatrix} 2 & -7 \\ 4 & 5 \end{pmatrix},$$

$$A_{3,X_2X_1} = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix},\ A_{3,X_2X_2} = \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix},\ A_{3,X_2X_3} = \begin{pmatrix} -3 & -2 \\ -3 & 1 \end{pmatrix},$$

$$A_{3,X_3X_1} = \begin{pmatrix} 2 & 2 \\ -5 & -8 \end{pmatrix},\ A_{3,X_3X_2} = \begin{pmatrix} -4 & -5 \\ 0 & 1 \end{pmatrix},\ A_{3,X_3X_3} = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix},$$

Table 1

Results of approximate solving the matrix equation (8)

ε	Number of iterations, n	Approximate solution, X_n	Error
0.1	68	$X_1 = \begin{pmatrix} 8.7144 & 16.8973 \\ -4.1322 & -8.3556 \end{pmatrix}$ $X_2 = \begin{pmatrix} -2.9606 & -4.1526 \\ 1.2979 & 2.3091 \end{pmatrix}$ $X_3 = \begin{pmatrix} 0.0285 & -1.9902 \\ -0.1192 & 0.8400 \end{pmatrix}$	0.07881265
0.01	84	$X_1 = \begin{pmatrix} 8.6991 & 16.8649 \\ -4.1245 & -8.3401 \end{pmatrix}$ $X_2 = \begin{pmatrix} -2.9580 & -4.1451 \\ 1.2962 & 2.3058 \end{pmatrix}$ $X_3 = \begin{pmatrix} 0.0330 & -1.9804 \\ -0.1216 & 0.8363 \end{pmatrix}$	0.00877181
0.001	99	$X_1 = \begin{pmatrix} 8.6977 & 16.8618 \\ -4.1237 & -8.3386 \end{pmatrix}$ $X_2 = \begin{pmatrix} -2.9578 & -4.1444 \\ 1.2961 & 2.3055 \end{pmatrix}$ $X_3 = \begin{pmatrix} 0.0334 & -1.9795 \\ -0.1218 & 0.8359 \end{pmatrix}$	0.00098755
0.0001	115	$X_1 = \begin{pmatrix} 8.6975 & 16.8614 \\ -4.1237 & -8.3385 \end{pmatrix}$ $X_2 = \begin{pmatrix} -2.9577 & -4.1443 \\ 1.2960 & 2.3054 \end{pmatrix}$ $X_3 = \begin{pmatrix} 0.0335 & -1.9794 \\ -0.1218 & 0.8358 \end{pmatrix}$	0.00009654
0.00001	131	$X_1 = \begin{pmatrix} 8.6975 & 16.8614 \\ -4.1236 & -8.3384 \end{pmatrix}$ $X_2 = \begin{pmatrix} -2.9577 & -4.1443 \\ 1.2960 & 2.3054 \end{pmatrix}$ $X_3 = \begin{pmatrix} 0.0335 & -1.9794 \\ -0.1218 & 0.8358 \end{pmatrix}$	0.00000943

$$A_{3,X_1} = \begin{pmatrix} 1 & 4 \\ 0 & -1.1 \end{pmatrix}, \ A_{3,X_2} = \begin{pmatrix} -1.2 & 3.9 \\ 0 & 21 \end{pmatrix}, \ A_{3,X_3} = \begin{pmatrix} 1.1 & 0 \\ 1 & 4.5 \end{pmatrix},$$

$$A_{3,1} = \begin{pmatrix} 32.8 & 53.5 \\ -41.5 & -41.3 \end{pmatrix}.$$

Let us put the initial approximation

$$X_1 = \begin{pmatrix} 1.8 & 0 \\ 0 & 1.8 \end{pmatrix}, \ X_2 = \begin{pmatrix} -1.8 & 1 \\ 1 & 1.8 \end{pmatrix}, \ X_3 = \begin{pmatrix} -0.8 & -0.8 \\ -0.8 & 0.8 \end{pmatrix}.$$

Then by using the iterative process (4), we obtain the results in the table 1.

Example 2. Let us suppose that we have a system of matrix equations of the second order

$$\begin{cases} A_{1,X_1X_1}X_1^2 + A_{1,X_1X_2}X_1X_2 + A_{1,X_1X_3}X_1X_3 + A_{1,X_2X_1}X_2X_1+ \\ \quad +A_{1,X_2X_2}X_2^2 + A_{1,X_2X_3}X_2X_3 + A_{1,X_3X_1}X_3X_1 + A_{1,X_3X_2}X_3X_2+ \\ \quad +A_{1,X_3X_3}X_3^2 + A_{1,X_1}X_1 + A_{1,X_2}X_2 + A_{1,X_3}X_3 + A_{1,1} = 0; \\ A_{2,X_1X_1}X_1^2 + A_{2,X_1X_2}X_1X_2 + A_{2,X_1X_3}X_1X_3 + A_{2,X_2X_1}X_2X_1+ \\ \quad +A_{2,X_2X_2}X_2^2 + A_{21,X_2X_3}X_2X_3 + A_{2,X_3X_1}X_3X_1 + A_{2,X_3X_2}X_3X_2+ \\ \quad +A_{2,X_3X_3}X_3^2 + A_{2,X_1}X_1 + A_{2,X_2}X_2 + A_{2,X_3}X_3 + A_{2,1} = 0; \\ A_{3,X_1X_1}X_1^2 + A_{3,X_1X_2}X_1X_2 + A_{3,X_1X_3}X_1X_3 + A_{3,X_2X_1}X_2X_1+ \\ \quad +A_{3,X_2X_2}X_2^2 + A_{3,X_2X_3}X_2X_3 + A_{3,X_3X_1}X_3X_1 + A_{3,X_3X_2}X_3X_2+ \\ \quad +A_{3,X_3X_3}X_3^2 + A_{3,X_1}X_1 + A_{3,X_2}X_2 + A_{3,X_3}X_3 + A_{3,1} = 0, \end{cases} \tag{9}$$

where

$$A_{1,X_1X_1}=\begin{pmatrix} -11 & 3 & 0 \\ 3 & -2.11 & 0 \\ 0 & 0 & 2 \end{pmatrix}, A_{1,X_1X_2}=\begin{pmatrix} 2 & 0 & 0 \\ -31 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{1,X_1X_3}=\begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{1,X_2X_1}=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{1,X_2X_2}=\begin{pmatrix} 2 & 1 & 0 \\ -1.1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{1,X_2X_3}=\begin{pmatrix} -3 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{1,X_3X_1}=\begin{pmatrix} 0 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, A_{1,X_3X_2}=\begin{pmatrix} -4 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{1,X_3X_3}=\begin{pmatrix} 3 & -2 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$A_{1,X_1}=\begin{pmatrix} 1.4 & -5 & 0 \\ -0.03 & -2.2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{1,X_2}=\begin{pmatrix} -0.02 & 0.1 & 0 \\ 3.2 & -2.5 & 0 \\ 0 & 0 & 10 \end{pmatrix}, A_{1,X_3}=\begin{pmatrix} 1.4 & -3 & 0 \\ -5.3 & 1.1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{11} = \begin{pmatrix} 34.46 & -5.780 & 0 \\ -124.74 & 62.24 & 0 \\ 0 & 0 & -29 \end{pmatrix},$$

$$A_{2,X_1X_1}=\begin{pmatrix} -2 & 1 & 0 \\ 22 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{2,X_1X_2}=\begin{pmatrix} -3 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, A_{2,X_1X_3}=\begin{pmatrix} -4 & -3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{2,X_2X_1}=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 2 \end{pmatrix}, A_{2,X_2X_2}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{2,X_2X_3}=\begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{2,X_3X_1}=\begin{pmatrix} 1 & 1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{2,X_3X_2}=\begin{pmatrix} -2 & -1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{2,X_3X_3}=\begin{pmatrix} 3 & -2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Table 2

Results of approximate solving the matrix equation (9)

ε	Number of iterations, n	Approximate solution, X_n	Error
0.1	224	$X_1=\begin{pmatrix} 27.5273 & 61.1999 & 0 \\ -12.2029 & -27.2371 & 0 \\ 0 & 0 & 2.7373 \end{pmatrix}$ $X_2=\begin{pmatrix} -9.2275 & -19.1793 & 0 \\ 4.1112 & 8.7282 & 0 \\ 0 & 0 & 0.1747 \end{pmatrix}$ $X_3=\begin{pmatrix} -5.5102 & -15.0802 & 0 \\ 2.4213 & 6.6774 & 0 \\ 0 & 0 & 0.7347 \end{pmatrix}$	0.02382976
0.01	234	$X_1=\begin{pmatrix} 27.5274 & 61.2002 & 0 \\ -12.2030 & -27.2373 & 0 \\ 0 & 0 & 2.7373 \end{pmatrix}$ $X_2=\begin{pmatrix} -9.2275 & -19.1794 & 0 \\ 4.1113 & 8.7282 & 0 \\ 0 & 0 & 0.1747 \end{pmatrix}$ $X_3=\begin{pmatrix} -5.5102 & -15.0803 & 0 \\ 2.4213 & 6.6774 & 0 \\ 0 & 0 & 0.7347 \end{pmatrix}$	0.00834828
0.001	240	$X_1=\begin{pmatrix} 27.5275 & 61.2003 & 0 \\ -12.2030 & -27.2373 & 0 \\ 0 & 0 & 2.7373 \end{pmatrix}$ $X_2=\begin{pmatrix} -9.2275 & -19.1794 & 0 \\ 4.1113 & 8.7282 & 0 \\ 0 & 0 & 0.1747 \end{pmatrix}$ $X_3=\begin{pmatrix} -5.5103 & -15.0804 & 0 \\ 2.4213 & 6.6774 & 0 \\ 0 & 0 & 0.7347 \end{pmatrix}$	0.00018689
0.0001	248	$X_1=\begin{pmatrix} 27.5275 & 61.2003 & 0 \\ -12.2030 & -27.2373 & 0 \\ 0 & 0 & 2.7373 \end{pmatrix}$ $X_2=\begin{pmatrix} -9.2275 & -19.1794 & 0 \\ 4.1113 & 8.7282 & 0 \\ 0 & 0 & 0.1747 \end{pmatrix}$ $X_3=\begin{pmatrix} -5.5103 & -15.0804 & 0 \\ 2.4213 & 6.6774 & 0 \\ 0 & 0 & 0.7347 \end{pmatrix}$	0.00006919

$$A_{2,X_1}=\begin{pmatrix} -0.006 & -3.5 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, A_{2,X_2}=\begin{pmatrix} -1 & 4 & 0 \\ 9 & -0.0091 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{2,X_3}=\begin{pmatrix} 5.1 & 1.2 & 0 \\ 1.1 & 1.5 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

Table 2

Results of approximate solving the matrix equation (9). Continue

ε	Number of iterations, n	Approximate solution, X_n	Error
0.00001	253	$X_1 = \begin{pmatrix} 27.5275 & 61.2003 & 0 \\ -12.2030 & -27.2373 & 0 \\ 0 & 0 & 2.7373 \end{pmatrix}$ $X_2 = \begin{pmatrix} -9.2275 & -19.1794 & 0 \\ 4.1113 & 8.7282 & 0 \\ 0 & 0 & 0.1747 \end{pmatrix}$ $X_3 = \begin{pmatrix} -5.5103 & -15.0804 & 0 \\ 2.4213 & 6.6774 & 0 \\ 0 & 0 & 0.7347 \end{pmatrix}$	0.00000813

$$A_{2,1} = \begin{pmatrix} -13.688 & 8.9 & 0 \\ -108.3909 & -47.3818 & 0 \\ 0 & 0 & -17 \end{pmatrix},$$

$$A_{3,X_1X_1} = \begin{pmatrix} -2 & -13 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{3,X_1X_2} = \begin{pmatrix} 3 & -4 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}, A_{3,X_1X_3} = \begin{pmatrix} 2 & -7 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{3,X_2X_1} = \begin{pmatrix} 4 & 2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{3,X_2X_2} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -5 & 0 \\ 0 & 0 & 15 \end{pmatrix}, A_{3,X_2X_3} = \begin{pmatrix} -3 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{3,X_3X_1} = \begin{pmatrix} 2 & 2 & 0 \\ -6 & -7 & 0 \\ 0 & 0 & 2 \end{pmatrix}, A_{3,X_3X_2} = \begin{pmatrix} -4 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{3,X_3X_3} = \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{3,X_1} = \begin{pmatrix} 1 & 4.1 & 0 \\ 0.074 & -1.1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{3,X_2} = \begin{pmatrix} -1.2 & 3.9 & 0 \\ 0.11 & 210 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{3,X_3} = \begin{pmatrix} 1.1 & 0.078 & 0 \\ 1 & 4.5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{3,1} = \begin{pmatrix} 32.878 & 53.222 & 0 \\ -43.428 & -49.41 & 0 \\ 0 & 0 & -18 \end{pmatrix}.$$

Let us put the initial approximation

$$X_1 = \begin{pmatrix} 1.8 & 0 & 0 \\ 0 & 1.8 & 0 \\ 0 & 0 & 1.8 \end{pmatrix}, X_2 = \begin{pmatrix} -1.8 & 1 & 0 \\ 1 & 1.8 & 0 \\ 0 & 0 & 0.8 \end{pmatrix}, X_3 = \begin{pmatrix} -0.9 & -0.9 & 0 \\ -1 & 0.9 & 0 \\ 0 & 0 & 0.9 \end{pmatrix}.$$

Then by using the iterative process (4), we obtain the results in the table 2.

5. Conclusions

In this paper systems of polynomial matrix equations of the second degree with many unknowns are considered. The scheme for solving nonlinear systems is proposed and recurrent relations for finding their approximate solutions over the field of noncommutative matrices are obtained. The convergence of the continued fractions used in the computational scheme is investigated. Numerical experiments that confirm the effectiveness of the proposed approach are conducted.

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СХЕМА НАБЛИЖЕНОГО РОЗВ'ЯЗУВАННЯ СИСТЕМ МАТРИЧНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ

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Матричні рівняння та системи матричних рівнянь широко використовують в задачах оптимізації систем керування. Проте універсального підходу до розв'язування всіх задач цього класу не існує: розроблено лише методи розв'язування найбільш популярних матричних рівнянь, наприклад, рівнянь Ріккаті та Ляпунова. Узагальнено розглянутий раніше метод розв'язування систем алгебричних рівнянь над полем дійсних чисел і запропоновано схему для систем поліноміальних матричних рівнянь другого ступеня з багатьма невідомими. Також наведена рекурентна формула для обчислення наближеного розв'язку систем у вигляді ланцюгового матричного дробу. Досліджено збіжність запропонованого методу. Наведено результати чисельних експериментів, що підтверджують обґрунтованість теоретичних розрахунків та ефективність запропонованої схеми.

Ключові слова: ітераційний метод, системи матричних рівнянь, матричні рівняння другого порядку.