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ON THE BOUNDARY-DOMAIN INTEGRALS APPROACH FOR A PARTIAL INTEGRO-DIFFERENTIAL EQUATION

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We consider in a simply connected two- or three-dimensional bounded domain the Dirichlet boundary value problem for a partial integro-differential equation, which contains the Laplace differential operator and the Fredholm integral operator over the domain. To investigate the weak solution, the formulated problem is rewritten in a variational formulation in corresponding Sobolev spaces. The uniqueness and existence of the weak solution are shown by the Lax-Milgram theorem. To establish the classical solution, the potential theory is involved. We represent the solution as a combination of the volume potential and the double-layer potential. Then the given problem is reduced to the system of boundary-domain integral equations of the second kind. We analyze the kernels of corresponding integral operators and show their compactness in the spaces of continuous functions. The well-posedness follows from the Riesz-Schauder theory.

Key words: partial integro-differential equation, weak and classical solutions, boundary-domain integral equations.

1. Introduction

Partial Integro-Differential Equation (PIDE) is an important branch of modern mathematics. PIDEs arise naturally in the study of stochastic processes with jumps. This type of processes are of particular interest in finance, population dynamics, and in some physical and biological models [5, 8, 11]. We consider the linear PIDE which contains an elliptical partial differential operator and a Fredholm integral operator, which in the general case has the following form

$$\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i=1}^{d} b_i\frac{\partial u}{\partial x_i} + cu + \int_D k(x,y)u(y)dy = f(x)$$

in some domain $D \subset \mathbb{R}^d$, $d \in \mathrm{N}$. Here a_{ij}, b_i, c, k and f are given functions in D.

This equation can be interpreted as some kind of more general Fokker-Plank equation. As a simple case we investigate the following Dirichlet boundary value problem for the Laplace equation with a compact perturbation.

Let $D \subset \mathbb{R}^d$, $d = 2, 3$ be a simply connected bounded domain with the boundary Γ. We search the function $u : D \to \mathbb{R}$, which satisfies the partial integro-differential equation

$$-\Delta u(x) + \int_D k(x,y)u(y)dy = f(x), \quad x \in D \tag{1}$$

and the Dirichlet boundary condition

$$u(x) = 0, \quad x \in \Gamma, \tag{2}$$

where $k : D \times D \to \mathbb{R}$ and $f : D \to \mathbb{R}$ are given functions.

Our goal is to establish the solvability of the boundary value problem (1)-(2) in weak and classical senses and to reduce this problem to boundary-domain integral equations (BDIE).

The outline of the paper is as follows. In the section 2 we use the Lax-Milgram theorem to prove the existence and uniqueness of the weak solution for the problem (1)-(2). In the main section 3 the problem (1)-(2) is reduced to the system of BDIE with the help of combination of volume and double layer potentials for the Laplace equation. By the Riesz-Schauder theory we show the well possedness of this system in corresponding spaces of continuous functions.

Note that the use of the integral equation approach doesn't decrease the dimension of the problem. But it gives us the possibility to apply some mesh-less numerical method to received BDIEs in contrast with the problem (1)-(2). This point will be investigated in our future work.

2. Weak solution

We denote $\langle \cdot, \cdot \rangle$ the inner product in the Lebesgue space $L^2(D)$ and $k_{min} = \min\limits_{x,y \in \bar{D}} K(x,y)$. We make the following assumption during this section: $\Gamma \in C^1$, $f \in L^2(D)$, $k \in C(D \times D)$ and if $k_{min} < 0$

$$|k_{min}| \leq \frac{1}{\lambda_1 |D|}, \tag{3}$$

where λ_1 is the smallest eigenvalue of the Laplace operator with the Dirichlet boundary condition on Γ.

Let $H^1(D)$ denotes the Sobolev space of all functions belong to $L^2(D)$ for which their first order weak derivatives also belong to $L^2(D)$ and $H_0^1(D)$ is the closure in $H^1(D)$ of the set of all functions with compact support in D. It is known [6], that the function v from $H^1(D)$ belong to $H_0^1(D)$ if and only if $v = 0$ on Γ. We use also the notation $V = H_0^1(D)$ and introduce the integral operator

$$(Ku)(x) = \int_D k(x,y)u(y)dy, \quad x \in D.$$

Note that the operator $K : L^2(D) \to L^2(D)$ is compact. According to the smooth properties of the kernel k, we can estimate for all $u, v \in H^1(D)$

$$\langle Ku, v \rangle \leq \|k\|_\infty |D| \|u\|_{H^1} \|v\|_{H^1}. \tag{4}$$

Assume that $u \in H^2(D)$ solves the problem (1)-(2). We multiply the equation (1) by $v \in V$ and integrate over D. The use of Green-Ostrogradski formula with the boundary value condition (2) leads to the variational equation

$$\sum_{i=1}^{2} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle + \langle Ku, v \rangle = \langle f, v \rangle, \quad v \in V. \tag{5}$$

Let us rewrite the equation (5) in an abstract form. We have a bilinear form $a : V \times V \to \mathbb{R}$

$$a(u,v) = \sum_{\ell=1}^{2} \left\langle \frac{\partial u}{\partial x_\ell}, \frac{\partial v}{\partial x_\ell} \right\rangle + \langle Ku, v \rangle$$

and a linear form $\ell : V \to \mathbb{R}$

$$\ell(v) = \langle f, v \rangle .$$

Thus the variational formulation then have the form

$$a(u,v) = \ell(v) \quad \text{for all} \;\; v \in V. \tag{6}$$

Theorem 1. *There exists a unique solution* $u \in H_0^1(D)$ *of the variational problem* (6).

Proof. We can rewrite the bilinear form a as

$$a(u,v) = a_L(u,v) + \langle Ku, v \rangle$$

with a bilinear form $a_L : V \times V \to \mathbb{R}$

$$a_L(u,v) = \sum_{i=1}^{2} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle .$$

It is well known (see for example [4]) that a_L is continuous and V-elliptic, i.e.

$$|a_L(u,v)| \le C_1 \|u\|_V \|v\|_V \quad \text{and} \quad a_L(u,u) \ge \frac{1}{\lambda_1}\|u\|_V^2 .$$

Taking into account the inequalities (4), we have immediately

$$|a(u,v)| \le (C_1 + \|k\|_\infty |D|)\, \|u\|_V \|v\|_V .$$

To prove that a is V-elliptic we different two cases.
1. Let $k_{min} \ge 0$. Then $\langle Ku, u \rangle \ge 0$ and we have that $a(u,u) \ge a_L(u,u)$ for all $u \in V$ and therefore a is V-elliptic.
2. Let $k_{min} < 0$. Then $\langle Ku, u \rangle \ge k_{min}|D| \|u\|_V^2$ and as result $a(u,u) \ge (\lambda_1^{-1} + k_{min}|D|)\|u\|_V^2$. According to the condition (3), the bilinear form a is again V-elliptic.

Thus our bilinear form a is continuous and V-elliptic and the linear form ℓ is continuous. Then the statement of the theorem follows from the Lax-Milgram theorem. □

The condition (3) can be simplified for the two-dimensional case as follows. Let our simply connected domain D contains the disc with maximal radius R. Then the minimal Dirichlet eigenvalue can be estimated as [1]

$$\lambda_1 \ge \frac{A}{R^2}$$

with $A = 0.6197$. Thus the requested property (3) for the function k looks as $|k_{min}| \le 0.52$.

The solution of (6) is called a weak solution of (1)-(2). Following [6] if $\Gamma \in C^2$ and $f \in L^2(D)$ the weak solution of the variational problem $a_L(u,v) = \langle f, v \rangle$, $v \in V$ belongs to $H^2(D)$. If we get then back to the problem (1)–(2) it is clear then that their has a unique solution $u \in H^2(D) \cap H_0^1(D)$. More regularity assumptions on the right hand side f implies the existence of the classical solution of the considered problem. But we would like to investigate the solvability of (1)–(2) in the classical sense involving integral equations. This give us some way to find the numerical solution of (1)–(2) in the future.

Note, that it is possible to replace the integral in (1) with a more general (nonlinear) operator, for results in this direction see [3,13] with a general setting covered in [2,12].

3. Classical solution

Now we assume during this section that $\Gamma \in C^2$, $f \in C(D)$ and $k \in C(D \times D)$. We look for the classical solution $u \in C^2(D) \cap C(\bar{D})$ of the problem (1)-(2). We would like to apply for it the integral equation technique. Let

$$\Phi(x,y) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x-y|}, x,y \in \mathbb{R}^2, x \neq y, \\ \\ \frac{1}{4\pi} \frac{1}{|x-y|}, x,y \in \mathbb{R}^3, x \neq y \end{cases}$$

be a fundamental solution of the Laplace operator and by ν we denote the outward unit normal to the boundary Γ.
According to properties of volume and double layer potentials for the Laplace equation, we have the following result.

Theorem 2. *The solution of the problem* (1)–(2) *can be presented in the form*

$$u(x) = \int_D \varphi(y)\Phi(x,y)dy + \int_\Gamma \psi(y) \frac{\partial \Phi}{\partial \nu(y)}(x,y)ds(y), \quad x \in D,$$

where the unknown densities $\varphi \in C(D)$ and $\psi \in C(\Gamma)$ satisfy the system of BDIE

$$\begin{cases} \varphi(x) - \int_D \varphi(y) G(x,y)dy - \int_\Gamma \psi(y) \frac{\partial G}{\partial \nu(y)}(x,y)ds(y) = -f(x), \quad x \in D, \\ \\ \psi(x) - 2\int_D \varphi(y)\Phi(x,y)dy - 2\int_\Gamma \psi(y) \frac{\partial \Phi}{\partial \nu(y)}(x,y)ds(y) = 0, \quad x \in \Gamma \end{cases} \tag{7}$$

with

$$G(x,y) = \int_D k(x,z)\Phi(z,y)dz.$$

To analyze the solvability of the system (7) we need to know the smooth properties of the kernel G.

Lemma 3. *The kernel G is continuous, i.e. $G \in C(D \times D)$.*

Proof. Clearly it is sufficient to consider the simple case of a ball (a disc in $\mathbb{R}^2$) D_R with the radius R. We have

$$G(x,y) = k(x,z^*)I(y), \quad \text{with } I(y) = \int_{D_R} \Phi(z,y)dz, \;\; z^* \in D_R.$$

The value of the integral only depends on the distance $\varrho = |x|$. Firstly we analyze the two dimensional case. Let $x = (0, \varrho)$. Now introduce the polar coordinate $z_1 = r\cos\phi$, $z_2 = r\sin\phi$ and reduce the integral I to

$$I(y) = \tilde{I}(\varrho) = -\frac{1}{\pi}\int_0^R r \int_0^{2\pi} \ln\left(r^2 + \varrho^2 - 2r\varrho\sin\phi\right) d\phi dr.$$

The following table integral can be found in [10]

$$\int_0^{2\pi} \ln(a + c\sin t)dt = 2\pi \ln \frac{a + \sqrt{a^2 - c^2}}{2}$$

for $a^2 > c^2$. Then we have immediately $\tilde{I}(\varrho) = 2(R^2 - \varrho^2 - R^2 \ln R)$.
For the three-dimensional case we assume $x = (0, 0, \varrho)$ and introduce the spherical coordinates $z_1 = r \sin\theta \cos\phi$, $z_2 = r \sin\theta \sin\phi$ and $z_3 = r \cos\theta$. Then we have

$$I(y) = \tilde{I}(\varrho) = \frac{1}{2} \int_0^R r^2 \int_0^\pi \frac{\sin\theta}{\sqrt{r^2 + \varrho^2 - 2r\varrho\cos\theta}} d\theta dr.$$

The straightforward calculation gives $\tilde{I}(\varrho) = \dfrac{R^2}{2} - \dfrac{\varrho^2}{6}$. □

Now we investigate the well-possedness of the BDIE (7).

Theorem 4. *The system of BDIE* (7) *is uniquely solvable.*

Proof. Firstly we consider the homogeneous system of BDIEs and construct with their solutions $\tilde{\varphi} \in C(D)$ and $\tilde{\psi} \in C(\Gamma)$ the combination of potentials

$$v(x) = \int_D \tilde{\varphi}(y)\Phi(x,y)dy + \int_\Gamma \tilde{\psi}(y) \frac{\partial \Phi}{\partial \nu(y)}(x,y)ds(y), \quad x \in D. \tag{8}$$

Clearly the function v satisfies the homogeneous problem

$$-\Delta v(x) + \int_D k(x,y)v(y)dy = 0 \quad x \in D, \qquad v(x) = 0, \quad x \in \Gamma.$$

From the maximum principle for the PIDE [7] we conclude that $v = 0$ in D.
Next we act by the Laplace operator to the representation (8) and receive that $-\Delta v(x) = \tilde{\varphi}(x)$, $x \in D$. Thus $\tilde{\varphi} = 0$ in D. Then we have for $\tilde{\psi}$ the following boundary integral equation of the second kind

$$\tilde{\psi}(x) - 2\int_0^{2\pi} \tilde{\psi}(y) \frac{\partial \Phi}{\partial \nu(y)}(x,y)ds(y) = 0, \quad x \in \Gamma.$$

It is well known [9] that $\tilde{\psi} = 0$.
Now we introduce the integral operators

$$(A_{11}\varphi)(x) = \int_D \varphi(y)G(x,y)dy, \quad (A_{12}\varphi)(x) = \int_\Gamma \varphi(y)\frac{\partial G}{\partial \nu(y)}(x,y)dy, \ \ x \in D,$$

$$(A_{21}\varphi)(x) = 2\int_D \varphi(y)\Phi(x,y)dy, \quad (A_{22}\varphi)(x) = 2\int_\Gamma \varphi(y)\frac{\partial \Phi}{\partial \nu(y)}(x,y)dy, \ \ x \in \Gamma.$$

Then we can rewrite the system (7) in the following operator form

$$\begin{cases} (I - A_{11})\,\varphi - A_{12}\psi = -f & \text{in } D, \\ -A_{21}\varphi + (I - A_{22})\,\psi = 0 & \text{on } \Gamma. \end{cases} \tag{9}$$

According to lemma 3, the kernels $G(x,y)$ for all $x, y \in D$ and $\frac{\partial G}{\partial \nu(y)}(x,y)$ for all $x \in D$ and $y \in \Gamma$ are continuous. Also the kernel $\Phi(x,y)$ for all $x \in \Gamma$ and $y \in D$ is continuous too, and $\frac{\partial \Phi}{\partial \nu(y)}(x,y)$ for all $x, y \in \Gamma$ is continous for $\mathbb{R}^2$ case and has a week singularity in $\mathbb{R}^3$. Therefore integral operators $A_{11} : C(D) \to C(D)$, $A_{12} : C(\Gamma) \to C(D)$, $A_{21} : C(D) \to C(\Gamma)$ and $A_{22} : C(\Gamma) \to C(\Gamma)$ in the system (9) are compact. As a result, we can apply the Riesz-Schauder solvability theory to the operator equation (9) (see for example [9]). Thus from the proved uniqueness of the solution of (7) follows their existence and stability. □

Now we get the solvability of the problem (1)–(2) in the classical sense.

Theorem 5. *The problem* (1)–(2) *has a unique solution.*

4. Conclusions

We investigated the Dirichlet boundary value problem for a simplest case of a PIDE named the Fokker-Plank equation. This equation contains a combination of a partial elliptic operator and a Fredholm integral operator. Uniqueness and existence of the weak solution in Sobolev space are shown by the Lax-Milgram theorem. Also we proved the well-posedness of this problem in the classical sense. To do it we reduced this problem to the system of BDIEs and recall the Riesz-Schauder solvability theory. Our future plans are connected with the numerical solution of the received BDIE.

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ПРО МЕТОД ГРАНИЧНО-ПРОСТОРОВИХ ІНТЕГРАЛІВ ДЛЯ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ В ЧАСТИННИХ ПОХІДНИХ

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У однозв'язній обмеженій області розглянуто крайову задачу Діріхле для інтегро-диференціального рівняння в часткових похідних, яке містить диференціальний оператор Лапласа та інтегральний оператор по заданій області. Досліджено існування та єдиність слабкого так класичного розв'язків. У випадку слабкого розв'язку використано теорему Лакса-Мілграма, а у випадку класичного – теорію Рісса-Шаудера. У підсумку розглядувана крайова задача редукована до системи коректних гранично-просторових інтегральних рівнянь.

Ключові слова: інтегро-диференціальне рівняння в частинних похідних, слабкий і класичний розв'язки, гранично-просторове інтегральне рівняння.