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NUMERICAL SOLUTION OF THE INTERIOR DIRICHLET BOUNDARY VALUE PROBLEM FOR THE GENERALIZED LAPLACE EQUATION BY THE BOUNDARY INTEGRAL EQUATIONS METHOD

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In this paper we investigate a method that has been proposed by Rangogni and Occhi of approximating the interior Dirichlet boundary value problem for the generalized Laplace equation into the boundary value problem for simpler elliptic equations together with the boundary integral equations approach. Based on some assumptions the considered problem can be substituted by the Dirichlet problem for the Laplace, Klein-Gordon or Helmholtz equations. Small theoretical notes regarding the uniqueness of the solution to these boundary value problems are provided. Afterward, having fundamental solutions which are well known for each of these equations, we use the boundary integral equations method representing the solution as a single- or double-layer potential in conjunction with the quadrature method to obtain a fully discrete system of linear algebraic equations with unknown approximate values of the density at quadrature points. The well-posedness of the integral equations in appropriate spaces and the convergence analysis are also considered. Calculating the approximate solution of the problem for a constant-coefficient equation, the approximate solution for the generalized Laplace equation is obtained as well by making one simple additional action. Finally, several numerical examples for different domains with different discretization parameters are provided in order to show the effectiveness of this approach, especially in the case of exact reduction to a constant-coefficient equation.

Key words: Dirichlet problem, generalized Laplace equation, boundary integral equations, potentials, quadrature method.

1. Introduction

During mathematical modelling of many physical processes, the elliptic differential equations with variable coefficients often appear. In particular, a function that corresponds to variable coefficients mostly defines some specific property (for example, electrical or thermal conductivity) of a physical process in a domain.

We shall consider the generalized Laplace equation following the definition in [10, 11], although this second-order elliptic equation with variable coefficients is also called conductivity equation or EIT (electric impedance tomography) equation [8] or stationary heat transfer equation [6].

Since, in general, there is no explicit view of a fundamental solution for this differential equation or its finding can be quite complicated the parametrix (Levi function) is used instead. Using parametrix and utilizing Green's third identity (see [6]) the interior Dirichlet boundary value problem for the generalized Laplace equation in a two-dimensional domain was reduced to two systems of boundary-domain integral equations and there was shown their equivalence to the original boundary value problem. In [7] a new family of parametrices was explored for the elliptic operator to obtain an equivalent system of boundary-domain integral equations and unique solvability in appropriate spaces was investigated.

The main disadvantage of using parametrix is that we do not decrease the dimension of the problem. While there are several well-known efficient methods for obtaining an approximate solution of problems for the Laplace equation, its generalized variant is considerably more complex, and therefore requires special handling. For example, one can apply certain assumptions and approximations in order to transform this equation in a way that makes using numerical methods becomes appropriate.

In this paper we will demonstrate one of such methods, which was proposed by Rangogni and Occhi [10] and was used in combination with the boundary elements method. Here we will apply the boundary integrals approach together with the quadrature method as a means of obtaining a numerical approximation of the problem instead of the boundary element method. Note, that the quadrature method is also called the Nyström method for the integral equation of the second kind. In particular, we will be using it to the Dirichlet problem for the Laplace, Klein-Gordon and Helmholtz equations, which are involved in the aforementioned transformation process.

So, in this work we use the idea from the [10] to transform the starting equation with variable coefficients into a constant-coefficient equation for which a fundamental solution is available and then apply an effective numerical method. The first step in the procedure is to avoid the first partial derivatives of the unknown function and next step is to approximate the obtained equation.

For the outline of the work, in Section 2, we transformed the generalized Laplace equation to a constant-coefficient differential equation using the approach from the paper [10]. The fundamental solution of each equation that has been obtained and properties of potentials are provided in Section 3. In section 4, the Dirichlet boundary value problem for these equations is considered together with theoretical notes, the boundary integral equations method and full discretization. In Section 5, numerical examples for different domain configurations are considered. Some conclusions are given in Section 6.

Let D be a simply connected bounded domain in $\mathbb{R}^2$ with boundary $\Gamma \in C^2$ (see Fig. 1). We consider the following interior Dirichlet boundary value problem in the two-dimensional domain D: find such function $w \in C^2(D) \cap C(\overline{D})$ that satisfies the generalized Laplace equation

$$\nabla \cdot (\sigma \nabla w) = 0 \quad \text{in } D \tag{1}$$

and the boundary condition

$$w = g \quad \text{on} \quad \Gamma, \tag{2}$$

where function $\sigma \in C^\infty(\overline{D})$, $\sigma > 0$ and sufficiently smooth function g are known. By ν we denote the outward unit normal to the boundary and $\nabla\cdot$ and ∇ are divergence and gradient operators, respectively.

The problem can be interpreted as a stationary heat transfer problem in an isotropic medium for a planar bounded domain with prescribed temperature on the boundary and known thermal conductivity σ.

2. Reduction to a constant-coefficient equation

Let us briefly provide a scheme, described in [10, 11] how to obtain a constant-coefficient elliptic equation based on suitable assumptions. Since σ is a positive function we can rewrite the equation (1) in the form

$$\Delta w + \frac{1}{\sigma}\nabla w \cdot \nabla \sigma = 0 \quad \text{in } D. \tag{3}$$

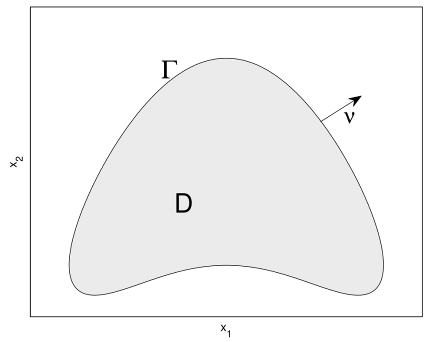

Fig. 1. An example of a domain D

Note that we get an ordinary Laplace equation if $\sigma \equiv const$. Let the solution $w(x)$ have the following form

$$w(x) = v(x)h(x), \quad x \in D. \tag{4}$$

Once we substitute (4) in (3), and introduce a condition that $h(x)$ eliminates the coefficients in front of $\frac{\partial v}{\partial x_1}$ and $\frac{\partial v}{\partial x_2}$, we get a differential equation for h. According to [10], its analytical solution is as follows:

$$h(x) = \frac{1}{\sqrt{\sigma(x)}}. \tag{5}$$

As a result, equation (3) will look like this

$$\Delta v(x) + q(x)v(x) = 0, \quad x \in D, \tag{6}$$

where

$$q(x) = \frac{|\nabla \sigma(x)|^2}{4\sigma^2(x)} - \frac{\Delta \sigma(x)}{2\sigma(x)}. \tag{7}$$

From here on out $|x|$ represents the Euclidian norm of the vector x. Taking into account that $\sigma(x) > 0$, the described transformations result in an equivalent equation. If a fundamental solution is known for (6), then this differential equation may be replaced by an equivalent boundary integral equation.

Finding the fundamental solution for the equation (6) remains a relatively complicated task due to the presence of a non-constant function q. To alleviate this issue, a method of approximating this equation was proposed in [10]. Although the following algorithm is rather primitive, it results in a constant-coefficient partial differential equation which could be solved numerically by using the boundary integral equations method.

We assume that the values of function σ within the domain D are in a narrow range, which is often encountered in real-life experiments. In that case, the derivatives of σ in q will be close to zero, and may be neglected. It also follows that for points in D, the following ratio holds

$$R = \frac{\max |q(x)|}{\min |q(x)|} \approx 1, \quad x \in D.$$

Considering this, we may replace the function q with a constant-coefficient, which will be the arithmetic mean value of q in D. We get the following expression as a result

$$\Delta u(x) + \overline{\kappa} u(x) = 0, \quad x \in D, \tag{8}$$

where

$$\overline{\kappa} = \frac{1}{meas(D)} \int_D q(x) dx,$$

$\overline{\kappa}$ is a real value and meas(D) is a measure of the domain D. Let us define $\kappa^2 = |\overline{\kappa}|$.

Therefore, we have three possible types of equations for (8) depending on the value $\overline{\kappa}$:

– $\overline{\kappa} = 0$ – Laplace equation: $\Delta u(x) = 0, \quad x \in D,$
– $\overline{\kappa} < 0$ – Klein-Gordon equation: $\Delta u(x) - \kappa^2 u(x) = 0, \quad x \in D,$
– $\overline{\kappa} > 0$ – Helmholtz equation: $\Delta u(x) + \kappa^2 u(x) = 0, \quad x \in D.$

These types of elliptic partial differential equations can be solved by using the boundary integral equations approach.

We also consider the boundary condition (2) for the generalized Laplace equation (1). For the aforementioned Dirichlet condition and taking into account the decomposition of the solution (4), we obtain an equivalent boundary condition for the transformed differential problem

$$u(x) = f(x), \quad x \in \Gamma, \tag{9}$$

where $f(x) = \sqrt{\sigma(x)}\, g(x)$.

We now have all the data required to numerically solve the problem (8)-(9) using the boundary integrals method. There are many known numerical methods that can efficiently solve problems based on this equation. In particular, the boundary elements method was successfully used in [10] and [11]. It was chosen because of its ease of use and its applicability for problems on infinite domains. The purpose of this paper is to investigate the efficiency of the boundary integrals method, which has a solid and well-researched theoretic foundation and is also applicable for different types of domains.

It is well known that the Dirichlet problem for Laplace and Klein-Gordon equations has at most one solution (see [9, 12]). But there are some problems with uniqueness in the case of the Helmholtz equation. If $\operatorname{Im} \kappa > 0$ then the interior Dirichlet problem has at most one solution (see [5]), but for real values κ this problem, in general, is not solvable uniquely. However, there are some sufficient conditions to check the uniqueness of the solution [12].

Proposition 1. *If κ^2 does not coincide with any eigenvalue of Laplace operator for the Dirichlet problem in the domain D then the Dirichlet problem (8)-(9) for the Helmholtz equation with homogeneous boundary data has the only trivial solution.*

Corollary 2. *If κ^2 does not coincide with any eigenvalue of Laplace operator for the Dirichlet problem in the domain D then the interior Dirichlet problem for the Helmholtz equation (8)-(9) has at most one solution.*

Additionally, the following statement from [5] can be used to establish the uniqueness.

Proposition 3. *Let D – bounded domain with diameter d, $u \in C^2(D) \cap C(\overline{D})$ – solution of the Helmholtz equation in D with condition $u = 0$ on Γ and let*

$$2|\kappa|^2 (e^d - 1) < 1.$$

Then $u = 0$ in D.

The fundamental solution of the Laplace equation in $\mathbb{R}^2$ has the following form

$$\Phi_1(x,y) = \frac{1}{2\pi}\ln\frac{1}{|x-y|}, \quad x \neq y.$$

For the Klein-Gordon equation, there exists the fundamental solution in the following form

$$\Phi_2(x,y) = \frac{1}{2\pi}K_0(\kappa|x-y|), \quad x,y \in \mathbb{R}^2, \quad x \neq y,$$

where K_0 is the modified Bessel function of the second kind and zeroth order [1] and has the following representation

$$K_0(z) = \ln\frac{1}{z}I_0(z) + \sum_{k=0}^{\infty}(\ln 2 + \psi(k+1))\frac{(z^2/4)^k}{(k!)^2},$$

where

$$I_0(z) = \sum_{k=0}^{\infty}\frac{(z^2/4)^k}{(k!)^2},$$

$$\psi(1) = -\gamma, \qquad \psi(n) = -\gamma + \sum_{k=1}^{n-1}k^{-1}, \ n \geq 2.$$

Here $\gamma = \lim\limits_{m\to\infty}(\sum\limits_{k=1}^{m}\frac{1}{k} - \ln m) \approx 0.5772156649...$ is the Euler's constant. While using the boundary integral equations method, we also make use of the following modified Bessel function of the first order K_1

$$K_1(z) = \frac{1}{z} - \ln\frac{1}{z}I_1(z) - \frac{z}{2}\sum_{k=0}^{\infty}\left(\ln 2 + \frac{1}{2}(\psi(k+1)+\psi(k+2))\right)\frac{(z^2/4)^k}{k!(k+1)!},$$

$$I_1(z) = \frac{z}{2}\sum_{k=0}^{\infty}\frac{(z^2/4)^k}{k!(k+1)!}.$$

Note that $I_0'(z) = I_1(z),\ K_0'(z) = K_1(z)$ and $I_0(z), I_1(z)$ are the modified Bessel function of the first kind of zeroth and first order, respectively.

In the case of the Helmholtz equation for the two-dimensional case with real-valued κ^2 the fundamental solution is a real-valued part of the Hankel function $H_0^{(1)}$ of the first kind of zeroth order (see [1, 5, 10]).

$$\Phi_3(x,y) = Re\Big(\frac{i}{4}H_0^{(1)}(\kappa|x-y|)\Big) = -\frac{1}{4}N_0(\kappa|x-y|), \quad x,y \in \mathbb{R}^2,\ x \neq y,$$

where N_0 is the Neumann function that was used as a new real-valued fundamental solution. The Bessel function of the second kind of zeroth order, denoted by Y_0, is occasionally denoted instead of N_0. The Neumann function N_0 has the following view

$$N_0(z) = \frac{2}{\pi}\left\{\ln\frac{z}{2} + \gamma\right\}J_0(z) + \frac{2}{\pi}\left\{\sum_{k=1}^{\infty}(-1)^{k+1}\eta_k\frac{\left(z^2/4\right)^k}{(k!)^2}\right\},$$

where $J_0(z)$ – the Bessel function of the first kind of zeroth order can be given by the series [1]

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k}$$

and η_k is a harmonic number, defined by the formula $\eta_k = \sum\limits_{i=1}^{k} \frac{1}{i}$.

3. Boundary integral equations method

As was mentioned previously, the main advantage of dealing with equations with constant coefficients is that their fundamental solutions are well-known. So it is possible to apply the boundary integral equations approach decreasing the dimension of the problem.

3.1. Dirichlet problem for the Laplace equation

In order to solve the boundary value problem (8)-(9), we transform it into an equivalent integral equation using the potential theory.

To construct an expression for the solution in case $\kappa^2 \equiv 0$ we use the double-layer potential

$$u(x) = \int_\Gamma \varphi_1(y) \frac{\partial \Phi_1(x,y)}{\partial \nu(y)} ds(y), \quad x \in D. \tag{10}$$

Substituting it into the boundary condition (9) an integral equation of the second kind is obtained

$$-\frac{1}{2}\varphi_1(x) + \int_\Gamma \varphi_1(y) \frac{\partial \Phi_1(x,y)}{\partial \nu(y)} ds(y) = f(x), \quad x \in \Gamma. \tag{11}$$

We assume that the curve Γ has the following parametric representation

$$\Gamma = \{x(t) = (x_1(t), x_2(t)),\ t \in [0, 2\pi]\}, \tag{12}$$

where $x_i \in C^2(\mathbb{R})$ are 2π-periodic functions, $i = 1, 2$, and $|x'(t)| > 0$.

Taking this into account the integral equation (11) leads to the following parametrized integral equation

$$-\frac{1}{2}\mu_1(t) + \frac{1}{2\pi} \int_0^{2\pi} \mu_1(\tau) L(t,\tau) d\tau = \widetilde{f}(t), \quad t \in [0, 2\pi], \tag{13}$$

where $\mu_1(t) = \varphi_1(x(t))$, $\widetilde{f}(t) = f(x(t))$ and

$$L(t,\tau) = \frac{(x(t) - x(\tau)) \cdot \nu(x(\tau))}{|x(t) - x(\tau)|^2}\ |x'(\tau)|.$$

Analyzing the kernel $L(t,\tau)$ one can observe that both input points lay on the same curve and when $\tau \to t$ it is possible to find the limit of the expression. That limit can be found analytically by using L'Hopital's rule. As a result, we get the following

$$\lim_{\tau \to t} L(t,\tau) = \frac{x''(t) \cdot \nu(x(t))}{2|x'(t)|}.$$

Hence, the modified kernel is given below

$$\widetilde{L}(t,\tau) = \begin{cases} \dfrac{(x(t)-x(\tau))\cdot\nu(x(\tau))}{|x(t)-x(\tau)|^2}\,|x'(\tau)|, & t\neq\tau, \\[2ex] \dfrac{x''(t)\cdot\nu(x(t))}{2\,|x'(t)|}, & t=\tau. \end{cases}$$

After that, we obtain the transformed parametrized integral equation that can be solved numerically

$$-\frac{1}{2}\mu_1(t) + \frac{1}{2\pi}\int_0^{2\pi}\mu_1(\tau)\widetilde{L}(t,\tau)d\tau = \widetilde{f}(t), \quad t\in[0,2\pi]. \tag{14}$$

Referring to Riesz theory for compact operators it is easy to show that considered integral equations are uniquely solvable in the space of continuous functions and the solutions depend continuously on data (see, for instance, [9]).

3.2. Dirichlet problem for the Klein-Gordon equation

Similarly to the case of the Laplace equation, we use the double-layer potential

$$u(x) = \int_\Gamma \varphi_2(y)\frac{\partial\Phi_2(x,y)}{\partial\nu(y)}ds(y), \qquad x\in D \tag{15}$$

in order to obtain an integral equation

$$-\frac{1}{2}\varphi_2(x) + \int_\Gamma \varphi_2(y)\frac{\partial\Phi_2(x,y)}{\partial\nu(y)}ds(y) = f(x), \qquad x\in\Gamma, \tag{16}$$

which considering (12) can be rewritten as

$$-\frac{1}{2}\frac{\mu_2(t)}{|x'(t)|} + \frac{1}{2\pi}\int_0^{2\pi}\mu_2(\tau)K(t,\tau)d\tau = \widetilde{f}(t), \qquad t\in[0,2\pi], \tag{17}$$

where $\mu_2(t) = \varphi_2(x(t))|x'(t)|$, $\widetilde{f}(t) = f(x(t))$, $K(t,\tau)$ is the kernel of the equation, based on the function K_1

$$K(t,\tau) = \kappa K_1(\kappa|x(t)-x(\tau)|)\frac{(x(t)-x(\tau))\cdot\nu(x(\tau))}{|x(t)-x(\tau)|}.$$

It is easy to verify that the kernel K is continuous at $\tau = t$, but to get a smoother kernel we split it making logarithmic function explicit for further quadrature application. So, for the Klein-Gordon equation we perform a decomposition using the weight functions, which become useful later for numerical computation of the integrals involved.

The decomposition itself has the following form

$$\tilde{K}(t,\tau) = K^{(1)}(t,\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right) + K^{(2)}(t,\tau),$$

where

$$K^{(1)}(t,\tau) = \begin{cases} \dfrac{\kappa}{2}I_1(\kappa|x(t)-x(\tau)|)\dfrac{(x(t)-x(\tau))\cdot\nu(x(\tau))}{|x(t)-x(\tau)|}, & t\neq\tau, \\[2ex] 0, \quad t=\tau, \end{cases}$$

$$K^{(2)}(t,\tau)=\begin{cases}K(t,\tau)-K^{(1)}(t,\tau)\ln\Big(\frac{4}{e}\sin^2\frac{t-\tau}{2}\Big), & t\neq\tau\\ \frac{x''(t)\cdot\nu(x(t))}{2|x'(t)|^2}, & t=\tau.\end{cases}$$

As a result, we get the following parametrized integral equation on $t\in[0,2\pi]$

$$-\frac{1}{2}\frac{\mu_2(t)}{|x'(t)|}+\frac{1}{2\pi}\int_0^{2\pi}\mu_2(\tau)\left(K^{(1)}(t,\tau)\ln\Big(\frac{4}{e}\sin^2\frac{t-\tau}{2}\Big)+K^{(2)}(t,\tau)\right)d\tau=\widetilde{f}(t). \quad (18)$$

Similarly, as for the Laplace equation, the well-posedness of the integral equations of the second kind in the space of continuous functions can be proved using Riesz theory.

3.3. Dirichlet problem for the Helmholtz equation

Lastly, for the Helmholtz equation we use the single-layer potential

$$u(x)=\int_\Gamma\varphi_3(y)\Phi_3(x,y)ds(y), \qquad x\in D \quad (19)$$

with continuous density φ that is a solution of the integral equation

$$\int_\Gamma\varphi_3(y)\Phi_3(x,y)ds(y)=f(x), \qquad x\in\Gamma. \quad (20)$$

Denote the series from the Neumann function N_0 as follows

$$S(z)\overset{def}{=}\frac{z^2/4}{(1!)^2}-\left(1+\frac{1}{2}\right)\frac{\left(z^2/4\right)^2}{(2!)^2}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{\left(z^2/4\right)^3}{(3!)^2}-\ldots.$$

Using parameterization of Γ the integral equation (20) can be written

$$\frac{1}{2\pi}\int_0^{2\pi}\mu_3(\tau)H(t,\tau)d\tau=\widetilde{f}(t), \quad t\in[0,2\pi], \quad (21)$$

where $\mu_3(\tau)=\varphi_3(x(\tau))|x'(\tau)|$, $\widetilde{f}(t)=f(x(t))$ and

$$H(t,\tau)=-J_0(\kappa|x(t)-x(\tau)|)\left\{\ln\left(\frac{\kappa|x(t)-x(\tau)|}{2}\right)+\gamma\right\}-S(\kappa|x(t)-x(\tau)|).$$

This kernel H has a logarithmic singularity and following the approach from [9] to make such singularity explicit, the kernel can be expressed as

$$H(t,\tau)=H^{(1)}(t,\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)+H^{(2)}(t,\tau),$$

where

$$H^{(1)}(t,\tau)=-\frac{1}{2}J_0(\kappa|(x(t))-x(\tau)|),$$

$$H^{(2)}(t,\tau)=\begin{cases}H(t,\tau)-H^{(1)}(t,\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right), & t\neq\tau,\\ \frac{1}{2}\ln\frac{4}{e\kappa^2|x'(t)|^2}-\gamma, & t=\tau.\end{cases}$$

Therefore the equation (21) can be rewritten as:

$$\frac{1}{2\pi}\int_0^{2\pi}\mu_3(\tau)\left\{H^{(1)}(t,\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)+H^{(2)}(t,\tau)\right\}d\tau=\widetilde{f}(t),\quad t\in[0,2\pi]. \quad (22)$$

For any inhomogeneity f that belongs to the linear subspace of all continuous functions $\psi \in C(\Gamma)$ for which double-layer potential with density ψ has continuous normal derivatives on both sides of Γ, the integral equation (20) of the first kind for the Dirichlet problem has a unique solution provided k is not an interior Dirichlet eigenvalue (more details, see [5]). However, even if this condition is satisfied, the problem is improperly posed in $C(\Gamma)$, but considering other spaces (for instance, Hölder spaces) and applying regularization theory [9] the well-posed problem might be obtained.

4. Full discretization

We use the quadrature method to solve the obtained integral equations (14), (18), (22). This method involves interpolation of the integrands using trigonometric polynomials and applying the appropriate quadrature formulas with further collocation of the approximating equations at the quadrature points.

To use the quadrature formula, we split the interval $[0,2\pi]$ with an even number of nodes

$$t_j=\frac{j\pi}{n},\quad j=0,...,2n-1.$$

We use the trapezoid quadrature to approximate the integral

$$\frac{1}{2\pi}\int_0^{2\pi}f(\tau)d\tau\approx\frac{1}{2n}\sum_{j=0}^{2n-1}f(t_j). \quad (23)$$

For the integral with logarithmic function that appears in (18), (22) we use trigonometric interpolation of f with the exact integration that yields the approximation

$$\frac{1}{2\pi}\int_0^{2\pi}f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)d\tau\approx\sum_{j=0}^{2n-1}f(t_j)R_j(t), \quad (24)$$

where $R_j(t)$ is the following weight function

$$R_j(t)=-\frac{1}{n}\left(\frac{1}{2}+\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_j)+\frac{1}{2n}\cos n(t-t_j)\right).$$

Applying these quadratures and collocating at quadrature points allows us to construct a fully discrete system of linear algebraic equations for approximated unknown values of the density $\tilde{\mu}_{k,j}\approx\mu_k(t_j)$ at each node within the interval, where $k=\overline{1,3}$ defines the equation ($k=1$ – Laplace equation; $k=2$ – Klein-Gordon equation; $k=3$ – Helmholtz equation).

1. Laplace equation

For the Laplace equation we have

$$-\frac{1}{2}\tilde{\mu}_{1,i}+\frac{1}{2n}\sum_{j=0}^{2n-1}\tilde{\mu}_{1,j}\tilde{L}(t_i,t_j)=\widetilde{f}(t_i),\quad i=0,...,2n-1. \quad (25)$$

Having the approximate values of the density, the approximate solution $u_n(x)$ at any point of the domain can be calculated using the formula below

$$u(x) \approx u_n(x) = \frac{1}{2n} \sum_{j=0}^{2n-1} \tilde{\mu}_{1,j} \overline{L}(x, t_j), \quad x \in D, \tag{26}$$

where

$$\overline{L}(x, t_j) = \frac{(x(t_j) - x) \cdot \nu(x(t_j))}{|x - x(t_j)|^2} \, |x'(t_j)|.$$

2. Klein-Gordon equation

   Full discretization of the equation (18) has the view

$$-\frac{\tilde{\mu}_{2,i}}{2|x'(t_i)|} + \sum_{j=0}^{2n-1} \tilde{\mu}_{2,j} \left(R_j(t_i) K^{(1)}(t_i, t_j) + \frac{1}{2n} K^{(2)}(t_i, t_j) \right) = \widetilde{f}(t_i), \; i = \overline{0, 2n-1}. \tag{27}$$

   Solving this system, obtained values $\tilde{\mu}_{2,j}$ are used in the approximation of the solution u in the domain D

$$u(x) \approx u_n(x) = \frac{1}{2n} \sum_{j=0}^{2n-1} \tilde{\mu}_{2,j} \overline{K}(x, t_j), \qquad x \in D, \tag{28}$$

   where

$$\overline{K}(x, t_j) = \kappa K_1(\kappa |x - x(t_j)|) \frac{(x - x(t_j)) \cdot \nu(t_j)}{|x - x(t_j)|}.$$

3. Helmholtz equation

   The corresponding system of linear algebraic equations for the equation (22) will be the following

$$\sum_{j=0}^{2n-1} \tilde{\mu}_{3,j} \left(R_j(t_i) H^{(1)}(t_i, t_j) + \frac{1}{2n} H^{(2)}(t_i, t_j) \right) = \widetilde{f}(t_i), \; i = 0, \ldots, 2n-1. \tag{29}$$

   Then the approximate solution of the Dirichlet problem for the Helmholtz equation can be found as

$$u(x) \approx u_n(x) = \frac{\pi}{n} \sum_{j=0}^{2n-1} \tilde{\mu}_{3,j} \overline{H}(x, t_j), \quad x \in D, \tag{30}$$

   where

$$\overline{H}(x, t_j) = -\frac{1}{4} N_0(\kappa |x - x(t_j)|).$$

Taking into account (4) and (5) and having $u_n(x)$ the approximate solution $w_n(x)$ of the considered problem (1)-(2) can be calculated by the formula

$$w_n(x) = \frac{u_n(x)}{\sqrt{\sigma(x)}}, \quad x \in D. \tag{31}$$

The convergence analysis and error estimate of the applied method to the integral equations of the second kind (for Laplace and Klein-Gordon equations) and the first kind (for the Helmholtz equation) can be carried out based on the collective compact operators theory and the estimate of the trigonometric interpolation in appropriate Banach spaces ([4], [9]). Note that if the boundary and the boundary data are analytic then the absolute error decreases exponentially, however, this estimate is correct for the original problem only in the case of exact reducing to a constant-coefficient equation.

5. Numerical examples

Example 1. Let D is bounded by the circle with radius equal to one, which means that

$$\Gamma = \{x(t) = (\cos(t), \sin(t)),\ t \in [0, 2\pi]\}.$$

Also $\sigma(x) = (2 + x_1 + x_2)^2,\ x \in \overline{D}$ and $g(x) = (x_1^2 - x_2^2)(2 + x_1 + x_2)^{-1},\ x \in \Gamma$. It is easy to verify that the exact solution $w_{ex}(x)$ of the problem (1)-(2) is $w_{ex}(x) = (x_1^2 - x_2^2)(2 + x_1 + x_2)^{-1},\ x \in D$. Straightforward calculation of q from (7) yields $q(x) \equiv 0$, so the main differential equation can be reduced to the Laplace equation. The absolute errors of the solution at two points of the domain D are provided in Table 1. Since the elliptic equation with variable coefficients was reduced to the Laplace equation without any approximations, the exponential rate of the convergence can be observed in the table. Note that the same convergence can be available for the other two types of equations in case q in (7) is calculated exactly and it is a number.

Table 1

Absolute error for Ex. 1

n	$\lvert w_{ex}(x) - w_n(x)\rvert$	
	x=(0.5,0)	x=(-0.6,0.7)
16	3.96E-10	1.48E-02
32	5.55E-17	9.90E-04
64	9.71E-17	2.30E-06
128	5.55E-17	7.62E-12

The absolute error graph based on errors at points $x^{ij} \in D$ and calculated using discretization parameter $n = 128$ is shown on Fig. 2.

The points x^{ij} are defined by the following formula

$$x^{ij} = \frac{i-1}{N}\Big(x_1(t_j), x_2(t_j)\Big),\ i = \overline{1, N}, \tag{32}$$

where $t_j = j\pi/M,\ j = \overline{0, 2M-1}$, with $N = 20$, $M = 32$.

It can be observed that the error is worse at some points near the boundary of the domain which is expected.

Example 2. We consider the solution D domain with the the boundary Γ

$$\Gamma = \{x(t) = (0.5 + 0.25\cos(t), 0.5 + 0.45\sin(t) - 0.35\sin^2(t)),\ t \in [0, 2\pi]\}.$$

The function $w_{ex}(x) = \frac{1}{2\pi}\ln\frac{1+0.5|x|^2}{0.5|x|^2},\ x \in D$ is the exact solution of the problem (1)-(2) for data $\sigma(x) = 1 + 0.5|x|^2,\ x \in \overline{D}$ and $g(x) = \frac{1}{2\pi}\ln((2 + |x|^2)|x|^{-2}),\ x \in \Gamma$. It is easy to

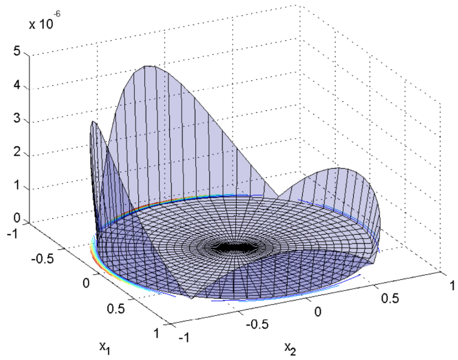

Fig. 2. Absolute error in D for Ex. 1 (n=128)

verify that

$$q(x) = 2|x|^2\Big(\frac{1}{2(2+|x|^2)^2} - \frac{1}{|x|^2(2+|x|^2)}\Big).$$

The values R and κ can be calculated approximately and we take them as follows $R \approx 1.61$, $\kappa = 0.85934$. Note, that for this example $\overline{\kappa} < 0$, so Klein-Gordon equation is being considered. The results are displayed in Table 2 as absolute errors at two points for different values of the parameter n. The convergence to the exact solution is observed.

Table 2

Absolute error for Ex. 2

n	$\lvert w_{ex}(x) - w_n(x)\rvert$	
	x=(0.4,-0.1)	x=(0.3,0.6)
16	7.38E-02	8.23E-04
32	6.93E-03	1.64E-05
64	2.75E-04	1.41E-05
128	2.15E-04	1.41E-05

The absolute errors calculated at points x^{ij} are shown on Fig. 3, where

$$x^{ij} = \Big(\frac{i-1}{N}(x_1(t_j) - 0.5) + 0.5, \frac{i-1}{N}(x_2(t_j) - 0.2) + 0.2\Big),\ i = \overline{1, N}, \qquad (33)$$

$j = \overline{0, 2M-1}$ and N, M are the same as in the previous example.

Example 3. Let the boundary Γ of the domain D has the following parametric representation

$$\Gamma = \{x(t) = (0.2\cos(t), 0.5 + 0.6\sin(t) - 0.6\sin^2(t)),\ t \in [0, 2\pi]\}.$$

Input data are $\sigma(x) = e^{x_1}\cos(x_2)+3,\ x \in \overline{D}$ and the function on the boundary $g(x) = e^{x_1}\sin(x_2),\ x \in \Gamma$. The exact solution is $w_{ex}(x) = e^{x_1}\sin(x_2),\ \ x \in D$. The function $q(x) = 0.25e^{2x_1}(e^{x_1}cos(x_2) + 3)^{-2}$ and based on $meas(D) \approx 0.378$ we can estimate $\overline{\kappa}$,

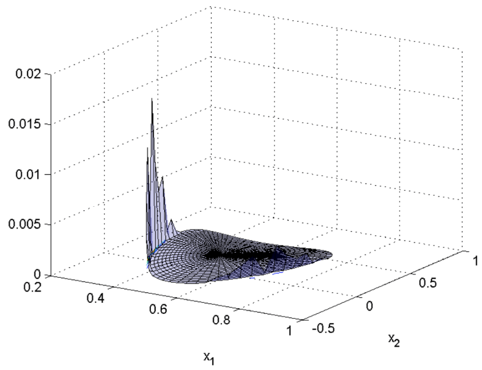

Fig. 3. Absolute error in D for Ex. 2 (n=128)

Table 3

Absolute error for Ex. 3.

n	$\lvert w_{ex}(x) - w_n(x)\rvert$	
	x=(0.1,0.5)	x=(-0.1,-0.3)
16	1.92E-04	2.20E-03
32	1.19E-06	2.31E-04
64	1.27E-06	1.82E-05
128	1.27E-06	3.71E-06

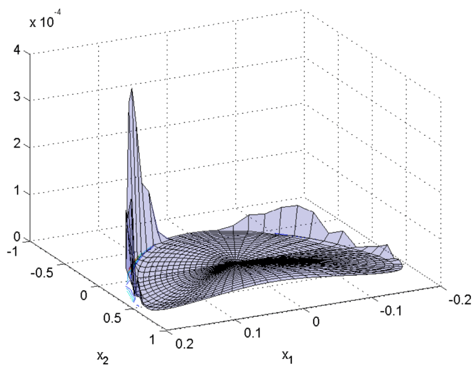

Fig. 4. Absolute error in D for Ex. 3 (n=128)

which is positive number, and set $\kappa = 0.14$. Also, R is calculated approximately with the following value $R \approx 1.9$. Recall that the greater value R, the less effective the considered approach will be. Absolute errors at two points and the graph of absolute errors calculated at points (32) are displayed in Table 3 and on Fig. 4, respectively.

6. Conclusions

In this work, the use of the method of boundary integral equations has been investigated in combination with the method of approximation of the interior Dirichlet problem described in [10] for the generalized Laplace equation. It has been shown that the considered approach gives a high rate of convergence in case of the exact reduction of the main differential equation to the three well-known types of constant-coefficient elliptic equations together with reducing the two-dimensional problem to one-dimensional. The ways to avoid restrictions that allow us to reduce the elliptic equations with variable coefficients to constant-coefficient equations may be a theme for further investigations.

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ЧИСЕЛЬНЕ РОЗВ'ЯЗУВАННЯ ВНУТРІШНЬОЇ ЗАДАЧІ ДІРІХЛЕ ДЛЯ УЗАГАЛЬНЕНОГО РІВНЯННЯ ЛАПЛАСА МЕТОДОМ ГРАНИЧНИХ ІНТЕГРАЛЬНИХ РІВНЯНЬ

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Досліджено один з методів апроксимації внутрішньої задачі Діріхле для узагальненого рівняння Лапласа до крайової задачі з простішими еліптичними рівняннями разом з методом граничних інтегральних рівнянь. Виходячи з певних припущень, розглянуту задачу можна звести до задачі Діріхле для рівняння Лапласа, Клейна-Гордона чи Гельмгольца. Після цього, маючи фундаментальні розв'язки для кожного з рівнянь, застосовуємо метод граничних інтегральних рівнянь, подаючи розв'язок задачі у вигляді потенціалу простого або подвійного шару та використовуючи метод квадратур для отримання повністю дискретизованої системи лінійних рівнянь з наближеними значеннями невідомої густини. Обчисливши наближений розв'язок задачі для рівняння з постійними коефіцієнтами, отримано також наближений розв'язок для узагальненого рівняння Лапласа. Наведено кілька чисельних прикладів з різними параметрами дискретизації, які демонструють ефективність застосованого підходу, особливо у випадку точного зведення до рівняння з постійними коефіцієнтами.

Ключові слова: крайова задача Діріхле, узагальнене рівняння Лапласа, граничні інтегральні рівняння, потенціали, метод квадратур.