UDC 519.833+519.837.3+519.6

TWO-PERSON GAMES ON A PRODUCT OF STAIRCASE-FUNCTION CONTINUOUS AND FINITE SPACES

V. Romanuke

O.S. Popov Odessa National Academy of Telecommunications,
Kuzneczna str., 1, Odessa, Ukraine, 65029,
e-mail: *romanukevadimv@gmail.com*

A tractable method of solving two-person games defined on a product of staircase-function spaces is presented. The spaces can be finite and continuous as well. The method is based on stacking equilibria of “short” two-person games, each defined on an interval where the pure strategy value is constant. First, a two-person game is formalized, in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time. The time can be thought of as it is discrete. The four theorems allowing to fulfill the stacking are proved for the case of pure-strategy equilibria. Second, the set of possible values of the player's pure strategy is discretized so that the game becomes defined on a product of staircase-function finite spaces. To formalize a method of solving two-person games defined on a product of staircase-function finite spaces, it is then proved that the game is solved as a stack of respective equilibria in the “short” bimatrix games. The equilibria in this case are considered in general terms, so they can be in mixed strategies as well. The stack is any combination (succession) of the respective equilibria of the “short” bimatrix games. Apart from the stack, there are no other equilibria in this “long” bimatrix game. The stack is always possible, even when only time is discrete (and the set of pure strategy possible values is continuous). An example is presented to show how the stacking is fulfilled for a case of when every “short” bimatrix game has a single pure-strategy equilibrium. The presented method, further “breaking” the initial (“long”) game defined on a product of staircase-function finite spaces, makes it completely tractable.

Key words: game theory, payoff functional, staircase-function strategy, bimatrix game.

1. Introduction

Two-person games are models of processes where two sides (personified and referred to as persons or players) struggle for optimizing the distribution of the limited resources [1, 2]. Bimatrix games are the simplest two-person games wherein equilibrium, efficiency, profitability, and eventual optimality of their solutions are well-studied [1, 3, 4]. Infinite or continuous two-person games (where the players' payoff functions are surfaces, which may have also discontinuities, defined on finite-dimensional Euclidean subspaces) are more complicated as, opposed to bimatrix games, an equilibrium is not always determinable and feasible [4, 5]. Therefore, the best choice is to approximate such games to finite ones, which are easily rendered to bimatrix games [4, 6]. Nevertheless, even a bimatrix game solution, if it is in mixed strategies, is not always practicable due to finite horizon of the game iterations (actions, plays, etc.) [1, 2, 7, 8]. Moreover, if the game has more than one solution, a problem of the solution selection comes open [5, 9, 10]. Furthermore, if at least two solutions are symmetric, they may be quite unstable due to cooperation between the players is excluded [1, 5, 10, 11].

If the player's pure strategy is a function (commonly, it is a function of time), this is a far more complicated case of the two-person game. In such games, the player's payoff is a functional [8, 12, 13]. Each player's functional maps every pair of functions (pure strategies of the players defined on a time interval) into a real value. When each of the players possesses a finite set of such function-strategies, the game might be rendered down to a bimatrix game [3, 4, 6, 8]. Obviously, such rendering is impossible if the set of the player's function-strategies is either infinite or continuous.

If to break a time interval, on which the pure strategy is defined, into a set of subintervals, on which the strategy could be approximately considered constant, the game is not simplified much because of the continuity of possible values of the strategy on a subinterval. However, the continuity might be removed also by sampling [3, 4, 6, 14]. The set of function-strategies becomes thus finite.

2. Motivation

In practical reality, the number of factual actions of the players in any game has a natural limit regardless of the form of pure strategies used in the game [1, 2, 7, 8, 15]. Nevertheless, if the rules of a system which is game-modeled are defined and administered beforehand, the administrator is likely to define (or constrain) the form of the strategies players will use [12, 16, 17].

In the simplest case, the player's pure strategy is a short action whose duration is negligible and thus is represented as just a time point. This case is exhaustively studied as bimatrix games [1, 5, 10, 15, 18]. In a more complicated case, the player's pure strategy is a function of time [8, 13], so the player's action is a complex process. A way to appropriately administer the players' actions is to constrain them to staircase functions whose points of discontinuities (breakpoints) have to be the same for both the players [13, 16, 19]. Along with the discrete time, possible values of the player's pure strategy should be discrete as well. Then the set of the player's possible (complex) actions is finite, indeed. So, the game can be represented as a bimatrix game, in which the player's selection of a pure strategy means using a staircase function on a time interval whereon every pure strategy is defined. Obviously, the number of the player's pure strategies in the bimatrix staircase-function game grows immensely as the number of breakpoints ("stair" intervals) or/and the number of possible values of the player's pure strategy increases. For instance, if the number of intervals is 5, and the number of possible values of the player's pure strategy is just 4, then there are $4^5 = 1024$ possible pure strategies at this player, where every strategy is a 5-interval 4-staircased function of time. The respective bimatrix 1024×1024 game even in this trivialized case appears to be big enough. In a more real example, when every strategy, say, is a 10-interval 8-staircased function of time, the respective bimatrix $1073741824 \times 1073741824$ staircase-function game appears to be intractably gigantic. Indeed, every player possessing more than a billion pure strategies is not capable of making proper decisions. All the more so since there are 1152921504606846976 (more than a quintillion, i. e., 10^{18}) situations in the game. This means that, instead of rendering to a bimatrix game, a tractable method of solving two-person games defined on a product of staircase-function finite spaces should be suggested.

3. Goals and tasks to be fulfilled

Issuing from the impracticability of rendering finite two-person games with staircase-function strategies to bimatrix games, the goal is to develop a tractable method of solving two-person games defined on a product of staircase-function finite spaces. For achieving the goal, the following six tasks are to be fulfilled:

1. To formalize a two-person game, in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time. Herein, the time can be thought of as it is discrete.
2. To discretize the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces.
3. To formalize a method of solving two-person games defined on a product of staircase-function finite spaces.
4. To exemplify it.
5. To discuss applicability and significance of the method.
6. To make an appropriate conclusion on it.

4. A two-person game with staircase-function strategies

In a two-person game, in which the player's pure strategy is a function of time, let each of the players use time-varying strategies defined almost everywhere on interval $[t_1;\ t_2]$ by $t_2 > t_1$. Denote a strategy of the first player by $x(t)$ and a strategy of the second player by $y(t)$. These functions are presumed to be bounded, i. e.

$$a_{\min} \leqslant x(t) \leqslant a_{\max} \ \text{by} \ a_{\min} < a_{\max} \tag{1}$$

and

$$b_{\min} \leqslant y(t) \leqslant b_{\max} \ \text{by} \ b_{\min} < b_{\max}, \tag{2}$$

defined almost everywhere on $[t_1;\ t_2]$. Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space of functions of time:

$$X = \{x(t)\,,\ t \in [t_1;\ t_2]\,,\ t_1 < t_2 :\ a_{\min} \leqslant x(t) \leqslant a_{\max} \ \text{by} \ a_{\min} < a_{\max}\} \subset \\ \subset \mathbb{L}_2\,[t_1;\ t_2] \tag{3}$$

and

$$Y = \{y(t)\,,\ t \in [t_1;\ t_2]\,,\ t_1 < t_2 :\ b_{\min} \leqslant y(t) \leqslant b_{\max} \ \text{by} \ b_{\min} < b_{\max}\} \subset \\ \subset \mathbb{L}_2\,[t_1;\ t_2] \tag{4}$$

are the sets of the players' pure strategies.

The first player's payoff in situation $\{x(t)\,,\ y(t)\}$ is $K(x(t)\,,\ y(t))$ presumed to be an integral functional [19, 20]:

$$K(x(t)\,,\ y(t)) = \int\limits_{[t_1;\ t_2]} f(x(t)\,,\ y(t)\,,\ t)\, d\mu(t), \tag{5}$$

where $f(x(t)\,,\ y(t)\,,\ t)$ is a function of $x(t)$ and $y(t)$ explicitly including t. The second player's payoff in situation $\{x(t)\,,\ y(t)\}$ is $H(x(t)\,,\ y(t))$ presumed to be an integral functional also:

$$H(x(t)\,,\ y(t)) = \int\limits_{[t_1;\ t_2]} g(x(t)\,,\ y(t)\,,\ t)\, d\mu(t), \tag{6}$$

where $g(x(t),\ y(t),\ t)$ is a function of $x(t)$ and $y(t)$ explicitly including t. Therefore, the continuous two-person game

$$\langle \{X,\ Y\},\ \{K(x(t),\ y(t)),\ H(x(t),\ y(t))\} \rangle \tag{7}$$

is defined on product

$$X \times Y \subset \mathbb{L}_2[t_1;\ t_2] \times \mathbb{L}_2[t_1;\ t_2] \tag{8}$$

of rectangular functional spaces (3) and (4) of players' pure strategies.

First, it is presumed that game (7) is administered so that the players are forced to use pure strategies $x(t)$ and $y(t)$ such that they both change their values for a finite number of times. Denote by N the number of intervals at which the player's pure strategy is constant, where $N \in \mathbb{N} \backslash \{1\}$. Then the player's pure strategy is a staircase function having only N different values. If $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$ are time points at which the staircase-function strategy changes its value, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2, \tag{9}$$

then

$$x_i = x\left(\tau^{(i)}\right) \text{ by } i = \overline{0,\ N} \tag{10}$$

are the values of the first player's strategy, and

$$y_i = y\left(\tau^{(i)}\right) \text{ by } i = \overline{0,\ N} \tag{11}$$

are the values of the second player's strategy. The staircase-function strategies are right-continuous [20, 21]:

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)}+\varepsilon\right) = x\left(\tau^{(i)}\right) \tag{12}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)}+\varepsilon\right) = y\left(\tau^{(i)}\right) \tag{13}$$

for $i = \overline{1,\ N-1}$, whereas

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)}-\varepsilon\right) \neq x\left(\tau^{(i)}\right) \tag{14}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)}-\varepsilon\right) \neq y\left(\tau^{(i)}\right) \tag{15}$$

for $i = \overline{1,\ N-1}$. As an exception,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(N)}-\varepsilon\right) = x\left(\tau^{(N)}\right) \tag{16}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(N)}-\varepsilon\right) = y\left(\tau^{(N)}\right), \tag{17}$$

so $x_{N-1} = x_N$ and $y_{N-1} = y_N$. Then constant values (10) and (11) by (9) mean that game (7) can be thought of as it is a succession of N continuous games

$$\langle \{[a_{\min};\ a_{\max}],\ [b_{\min};\ b_{\max}]\},\ \{K(\alpha_i,\ \beta_i),\ H(\alpha_i,\ \beta_i)\} \rangle \tag{18}$$

defined on product

$$[a_{\min};\ a_{\max}] \times [b_{\min};\ b_{\max}] \tag{19}$$

by

$$\alpha_i = x(t) \in [a_{\min};\ a_{\max}] \ \text{ and } \ \beta_i = y(t) \in [b_{\min};\ b_{\max}]$$
$$\forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \ \text{ for } \ i = \overline{1,\ N-1} \ \text{ and } \ \forall\, t \in \left[\tau^{(N-1)};\ \tau^{(N)}\right], \tag{20}$$

where the factual first player's payoff in situation $\{\alpha_i,\ \beta_i\}$ is

$$K(\alpha_i,\ \beta_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) \ \ \forall\, i = \overline{1,\ N-1} \tag{21}$$

and

$$K(\alpha_N,\ \beta_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t), \tag{22}$$

so

$$\begin{aligned} K(x(t),\ y(t)) &= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \\ &+ \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) \end{aligned} \tag{23}$$

instead of (5), and the factual second player's payoff in situation $\{\alpha_i,\ \beta_i\}$ is

$$H(\alpha_i,\ \beta_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) \ \ \forall\, i = \overline{1,\ N-1} \tag{24}$$

and

$$H(\alpha_N,\ \beta_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t), \tag{25}$$

so

$$\begin{aligned} H(x(t),\ y(t)) &= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \\ &+ \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t) \end{aligned} \tag{26}$$

instead of (6). In other words, if every optimal (with respect to equilibrium [1, 15]) situation in pure strategies in game (7) on product (8) by conditions (1) – (6) is (or forced to be) of staircase functions satisfying conditions (9) – (17), then this game is equivalent to the succession of N games (18) by (9) – (17) and (20) – (26). In this case game (7) can be represented by the succession of games (18).

Theorem 1. *If each of N games (18) by (9) – (17) and (20) – (26) has a single equilibrium situation in pure strategies, and game (7) on product (8) by conditions (1) – (6) is equivalent to the succession of these games, then the equilibrium situation in pure strategies in game (7) is determined by independently finding pure-strategy equilibria in N games (18), whereupon these equilibria are successively stacked.*

Proof. First, the equivalency means that game (7) has only staircase pure-strategy equilibria. Next, it should be proved that game (7) has a pure-strategy equilibrium situation, which is a successive stack of the N "short" games (18). Let $\{\alpha_i^*,\ \beta_i^*\}_{i=1}^N$ be pure-strategy equilibria in games (18) by (9) – (17) and (20) – (26). Then

$$K(\alpha_i,\ \beta_i^*) \leqslant K(\alpha_i^*,\ \beta_i^*)$$
$$\forall\, \alpha_i \in [a_{\min};\ a_{\max}] \ \text{ and } \forall\, i = \overline{1,\ N} \tag{27}$$

and

$$H(\alpha_i^*,\ \beta_i) \leqslant H(\alpha_i^*,\ \beta_i^*)$$
$$\forall\, \beta_i \in [b_{\min};\ b_{\max}] \ \text{ and } \forall\, i = \overline{1,\ N}, \tag{28}$$

i. e.,

$$\begin{aligned} K(\alpha_i,\ \beta_i^*) &= \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} f(\alpha_i,\ \beta_i^*,\ t)\, d\mu(t) \leqslant \\ &\leqslant \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} f(\alpha_i^*,\ \beta_i^*,\ t)\, d\mu(t) = \\ &= K(\alpha_i^*,\ \beta_i^*) \ \ \forall\, i = \overline{1,\ N-1}, \end{aligned} \tag{29}$$

$$\begin{aligned} K(\alpha_N,\ \beta_N^*) &= \int\limits_{[\tau^{(N-1)};\ \tau^{(N)}]} f(\alpha_N,\ \beta_N^*,\ t)\, d\mu(t) \leqslant \\ &\leqslant \int\limits_{[\tau^{(N-1)};\ \tau^{(N)}]} f(\alpha_N^*,\ \beta_N^*,\ t)\, d\mu(t) = \\ &= K(\alpha_N^*,\ \beta_N^*)\,, \end{aligned} \tag{30}$$

and

$$\begin{aligned} H(\alpha_i^*,\ \beta_i) &= \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} g(\alpha_i^*,\ \beta_i,\ t)\, d\mu(t) \leqslant \\ &\leqslant \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} g(\alpha_i^*,\ \beta_i^*,\ t)\, d\mu(t) = \\ &= H(\alpha_i^*,\ \beta_i^*) \ \ \forall\, i = \overline{1,\ N-1}, \end{aligned} \tag{31}$$

$$H\left(\alpha_N^*,\ \beta_N\right) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(\alpha_N^*,\ \beta_N,\ t\right) d\mu\left(t\right) \leqslant$$
$$\leqslant \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(\alpha_N^*,\ \beta_N^*,\ t\right) d\mu\left(t\right) =$$
$$= H\left(\alpha_N^*,\ \beta_N^*\right). \tag{32}$$

So,

$$\sum_{i=1}^{N} K\left(\alpha_i,\ \beta_i^*\right) \leqslant \sum_{i=1}^{N} K\left(\alpha_i^*,\ \beta_i^*\right) \tag{33}$$

and

$$\sum_{i=1}^{N} H\left(\alpha_i^*,\ \beta_i\right) \leqslant \sum_{i=1}^{N} H\left(\alpha_i^*,\ \beta_i^*\right). \tag{34}$$

Therefore, the successive stack of pure-strategy equilibria $\{\alpha_i^*,\ \beta_i^*\}_{i=1}^{N}$ *is a pure-strategy equilibrium in game (7). Obviously, games (18) can be solved independently, whose equilibria are stacked afterwards to form the pure-strategy equilibrium in game (7).* $\square$

In fact, Theorem 1 claims that if each of N "short" games (18) has a single pure-strategy equilibrium, then the solution of game (7) can be determined in a simpler way, by solving games (18) and successively stacking their equilibria. They are solved in parallel (independently), without caring of the succession. The question of whether the stacked equilibrium in game (7) is single or not is answered by the following assertion.

Theorem 2. *If each of N games (18) by (9) – (17) and (20) – (26) has a single equilibrium situation in pure strategies, and game (7) on product (8) by conditions (1) – (6) is equivalent to the succession of these games, then the equilibrium situation in pure strategies in game (7) is single being the successive stack of the "short" games equilibria.*

Proof. *Now, the pure-strategy equilibrium in game (7) is constructed according to Theorem 1, i. e., it is the successive stack of pure-strategy equilibria* $\{\alpha_i^*,\ \beta_i^*\}_{i=1}^{N}$. *Let this equilibrium be referred to as the* $\{\alpha_i^*,\ \beta_i^*\}_{i=1}^{N}$*-stack equilibrium. Suppose that there is another pure-strategy equilibrium in game (7). First, let this equilibrium differ from the* $\{\alpha_i^*,\ \beta_i^*\}_{i=1}^{N}$*-stack equilibrium in just that the first player uses some* $\alpha_k^{(0)} \in [a_{\min};\ a_{\max}]$ *instead of* α_k^* *by some* $k \in \left\{\overline{1,\ N}\right\}$. *So, this is the*

$$\left\{\{\alpha_i^*,\ \beta_i^*\}_{i \in \left\{\overline{1,\ N}\right\} \backslash \{k\}} \bigcup \left\{\alpha_k^{(0)},\ \beta_k^*\right\}\right\} \text{-stack equilibrium,}$$

which means that

$$\sum_{i \in \left\{\overline{1,\ N}\right\} \backslash \{k\}} K\left(\alpha_i,\ \beta_i^*\right) + K\left(\alpha_k,\ \beta_k^*\right) \leqslant$$
$$\leqslant \sum_{i \in \left\{\overline{1,\ N}\right\} \backslash \{k\}} K\left(\alpha_i^*,\ \beta_i^*\right) + K\left(\alpha_k^{(0)},\ \beta_k^*\right) \tag{35}$$

and

$$\sum_{i\in\left\{\overline{1,\ N}\right\}\backslash\{k\}} H\left(\alpha_i^*,\ \beta_i\right)+H\left(\alpha_k^{(0)},\ \beta_k\right)\leqslant$$
$$\leqslant\sum_{i\in\left\{\overline{1,\ N}\right\}\backslash\{k\}} H\left(\alpha_i^*,\ \beta_i^*\right)+H\left(\alpha_k^{(0)},\ \beta_k^*\right), \tag{36}$$

i. e.,

$$K\left(\alpha_i,\ \beta_i^*\right)\leqslant K\left(\alpha_i^*,\ \beta_i^*\right)$$
$$\forall\ \alpha_i\in\left[a_{\min};\ a_{\max}\right]\ \text{ and }\ \forall\, i\in\left\{\overline{1,\ N}\right\}\setminus\{k\} \tag{37}$$

and

$$K\left(\alpha_k,\ \beta_k^*\right)\leqslant K\left(\alpha_k^{(0)},\ \beta_k^*\right)\ \ \forall\ \alpha_k\in\left[a_{\min};\ a_{\max}\right] \tag{38}$$

along with (27) and

$$H\left(\alpha_i^*,\ \beta_i\right)\leqslant H\left(\alpha_i^*,\ \beta_i^*\right)$$
$$\forall\ \beta_i\in\left[b_{\min};\ b_{\max}\right]\ \text{ and }\ \forall\, i\in\left\{\overline{1,\ N}\right\}\setminus\{k\} \tag{39}$$

and

$$H\left(\alpha_k^{(0)},\ \beta_k\right)\leqslant H\left(\alpha_k^{(0)},\ \beta_k^*\right)\ \ \forall\ \beta_k\in\left[b_{\min};\ b_{\max}\right] \tag{40}$$

along with (28). Inequalities (38) and (40) imply that $\left\{\alpha_k^{(0)},\ \beta_k^*\right\}$ *is a pure-strategy equilibrium at the k-th interval (in the k-th game), which is impossible due to every interval has a single pure-strategy equilibrium. The impossibility of the other pure-strategy equilibrium for the second player in such a case is proved symmetrically.*

Second, suppose that the other pure-strategy equilibrium differs from the $\left\{\alpha_i^*,\ \beta_i^*\right\}_{i=1}^{N}$ *– stack equilibrium in that the first player uses some* $\alpha_k^{(0)}\in\left[a_{\min};\ a_{\max}\right]$ *instead of* α_k^* *by some* $k\in\left\{\overline{1,\ N}\right\}$ *and the second player uses some* $\beta_h^{(0)}\in\left[b_{\min};\ b_{\max}\right]$ *instead of* β_h^* *by some* $h\in\left\{\overline{1,\ N}\right\}$*. So, this is the*

$$\left\{\left\{\alpha_i^*,\ \beta_i^*\right\}_{i\in\left\{\overline{1,\ N}\right\}\backslash\{k\}}\bigcup\left\{\alpha_k^{(0)},\ \beta_k^{(0)}\right\}\right\}\text{-stack equilibrium} \tag{41}$$

if $h=k$*, and is the*

$$\left\{\left\{\alpha_i^*,\ \beta_i^*\right\}_{i\in\left\{\overline{1,\ N}\right\}\backslash\{k,\ h\}}\bigcup\left\{\alpha_k^{(0)},\ \beta_k^*\right\}\bigcup\left\{\alpha_h^*,\ \beta_h^{(0)}\right\}\right\}$$
$$\text{-stack equilibrium} \tag{42}$$

if $h\neq k$*. Thus, (41) means that*

$$\sum_{i\in\left\{\overline{1,\ N}\right\}\backslash\{k\}} K\left(\alpha_i,\ \beta_i^*\right)+K\left(\alpha_k,\ \beta_k^{(0)}\right)\leqslant$$
$$\leqslant\sum_{i\in\left\{\overline{1,\ N}\right\}\backslash\{k\}} K\left(\alpha_i^*,\ \beta_i^*\right)+K\left(\alpha_k^{(0)},\ \beta_k^{(0)}\right) \tag{43}$$

and

$$\sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k\}} H\left(\alpha_i^*,\ \beta_i\right)+H\left(\alpha_k^{(0)},\ \beta_k\right)\leqslant$$
$$\leqslant\sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k\}} H\left(\alpha_i^*,\ \beta_i^*\right)+H\left(\alpha_k^{(0)},\ \beta_k^{(0)}\right), \tag{44}$$

i. e., inequalities (37) and inequality

$$K\left(\alpha_k,\ \beta_k^{(0)}\right)\leqslant K\left(\alpha_k^{(0)},\ \beta_k^{(0)}\right)\ \ \forall\ \alpha_k\in\left[a_{\min};\ a_{\max}\right] \tag{45}$$

hold along with (27) and inequalities (39) and inequality

$$H\left(\alpha_k^{(0)},\ \beta_k\right)\leqslant H\left(\alpha_k^{(0)},\ \beta_k^{(0)}\right)\ \ \forall\ \beta_k\in\left[b_{\min};\ b_{\max}\right] \tag{46}$$

hold along with (28). Inequalities (45) and (46) imply that $\left\{\alpha_k^{(0)},\ \beta_k^{(0)}\right\}$ *is a pure-strategy equilibrium at the* k*-th interval (in the* k*-th game), which is impossible. If (42) is true, then*

$$\sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k,\ h\}} K\left(\alpha_i,\ \beta_i^*\right)+K\left(\alpha_k,\ \beta_k^*\right)+K\left(\alpha_h,\ \beta_h^{(0)}\right)\leqslant$$
$$\leqslant\sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k,\ h\}} K\left(\alpha_i^*,\ \beta_i^*\right)+K\left(\alpha_k^{(0)},\ \beta_k^*\right)+K\left(\alpha_h^*,\ \beta_h^{(0)}\right) \tag{47}$$

and

$$\sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k,\ h\}} H\left(\alpha_i^*,\ \beta_i\right)+H\left(\alpha_k^{(0)},\ \beta_k\right)+H\left(\alpha_h^*,\ \beta_h\right)\leqslant$$
$$\leqslant\sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k,\ h\}} H\left(\alpha_i^*,\ \beta_i^*\right)+H\left(\alpha_k^{(0)},\ \beta_k^*\right)+H\left(\alpha_h^*,\ \beta_h^{(0)}\right), \tag{48}$$

i. e., inequalities

$$K\left(\alpha_i,\ \beta_i^*\right)\leqslant K\left(\alpha_i^*,\ \beta_i^*\right)$$
$$\forall\ \alpha_i\in\left[a_{\min};\ a_{\max}\right]\ \text{ and }\forall\, i\in\left\{\overline{1,\ N}\right\}\setminus\{k,\ h\} \tag{49}$$

and inequality

$$K\left(\alpha_k,\ \beta_k^*\right)+K\left(\alpha_h,\ \beta_h^{(0)}\right)\leqslant K\left(\alpha_k^{(0)},\ \beta_k^*\right)+K\left(\alpha_h^*,\ \beta_h^{(0)}\right)$$
$$\forall\ \alpha_k\in\left[a_{\min};\ a_{\max}\right]\ \text{ and }\forall\ \alpha_h\in\left[a_{\min};\ a_{\max}\right] \tag{50}$$

hold along with (27) and inequalities

$$H\left(\alpha_i^*,\ \beta_i\right)\leqslant H\left(\alpha_i^*,\ \beta_i^*\right)$$
$$\forall\ \beta_i\in\left[b_{\min};\ b_{\max}\right]\ \text{ and }\forall\, i\in\left\{\overline{1,\ N}\right\}\setminus\{k,\ h\} \tag{51}$$

and inequality

$$H\left(\alpha_k^{(0)},\ \beta_k\right) + H\left(\alpha_h^*,\ \beta_h\right) \leqslant H\left(\alpha_k^{(0)},\ \beta_k^*\right) + H\left(\alpha_h^*,\ \beta_h^{(0)}\right)$$
$$\forall\, \beta_k \in [b_{\min};\ b_{\max}] \ \text{ and } \forall\, \beta_h \in [b_{\min};\ b_{\max}] \tag{52}$$

hold along with (28). Plugging $\alpha_h = \alpha_h^*$ in the left side of inequality (50) and plugging $\beta_h = \beta_h^{(0)}$ in the left side of inequality (52) gives inequalities (38) and (40), which are impossible due to $\left\{\alpha_k^{(0)},\ \beta_k^*\right\}$ is not a pure-strategy equilibrium. Therefore, the supposition about (41) and (42) is contradictory.

Next, suppose that the other pure-strategy equilibrium differs from the $\{\alpha_i^*,\ \beta_i^*\}_{i=1}^N$-stack equilibrium in that the first player uses some $\alpha_{k_1}^{(0)} \in [a_{\min};\ a_{\max}]$ instead of $\alpha_{k_1}^*$ by some $k_1 \in \left\{\overline{1,\ N}\right\}$ and some $\alpha_{k_2}^{(0)} \in [a_{\min};\ a_{\max}]$ instead of $\alpha_{k_2}^*$ by some $k_2 \in \left\{\overline{1,\ N}\right\}$. The respective

$$\left\{\{\alpha_i^*,\ \beta_i^*\}_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k_1,\ k_2\}} \bigcup \left\{\alpha_{k_1}^{(0)},\ \beta_{k_1}^*\right\} \bigcup \left\{\alpha_{k_2}^{(0)},\ \beta_{k_2}^*\right\}\right\}$$
$$\text{-stack equilibrium} \tag{53}$$

means that

$$\sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k_1,\ k_2\}} K\left(\alpha_i,\ \beta_i^*\right) + K\left(\alpha_{k_1},\ \beta_{k_1}^*\right) + K\left(\alpha_{k_2},\ \beta_{k_2}^*\right) \leqslant$$
$$\leqslant \sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k_1,\ k_2\}} K\left(\alpha_i^*,\ \beta_i^*\right) + K\left(\alpha_{k_1}^{(0)},\ \beta_{k_1}^*\right) + K\left(\alpha_{k_2}^{(0)},\ \beta_{k_2}^*\right) \tag{54}$$

and

$$\sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k_1,\ k_2\}} H\left(\alpha_i^*,\ \beta_i\right) + H\left(\alpha_{k_1}^{(0)},\ \beta_{k_1}\right) + H\left(\alpha_{k_2}^{(0)},\ \beta_{k_2}\right) \leqslant$$
$$\leqslant \sum_{i\in\left\{\overline{1,\ N}\right\}\setminus\{k_1,\ k_2\}} H\left(\alpha_i^*,\ \beta_i^*\right) + H\left(\alpha_{k_1}^{(0)},\ \beta_{k_1}^*\right) + H\left(\alpha_{k_2}^{(0)},\ \beta_{k_2}^*\right), \tag{55}$$

i. e., inequalities

$$K\left(\alpha_i,\ \beta_i^*\right) \leqslant K\left(\alpha_i^*,\ \beta_i^*\right)$$
$$\forall\, \alpha_i \in [a_{\min};\ a_{\max}] \ \text{ and } \forall\, i \in \left\{\overline{1,\ N}\right\} \setminus \{k_1,\ k_2\} \tag{56}$$

and inequality

$$K\left(\alpha_{k_1},\ \beta_{k_1}^*\right) + K\left(\alpha_{k_2},\ \beta_{k_2}^*\right) \leqslant K\left(\alpha_{k_1}^{(0)},\ \beta_{k_1}^*\right) + K\left(\alpha_{k_2}^{(0)},\ \beta_{k_2}^*\right)$$
$$\forall\, \alpha_{k_1} \in [a_{\min};\ a_{\max}] \ \text{ and } \forall\, \alpha_{k_2} \in [a_{\min};\ a_{\max}] \tag{57}$$

hold along with (27) and inequalities

$$H\left(\alpha_i^*,\ \beta_i\right) \leqslant H\left(\alpha_i^*,\ \beta_i^*\right)$$
$$\forall\, \beta_i \in [b_{\min};\ b_{\max}] \ \text{ and } \forall\, i \in \left\{\overline{1,\ N}\right\} \setminus \{k_1,\ k_2\} \tag{58}$$

and inequality

$$H\left(\alpha_{k_1}^{(0)},\ \beta_{k_1}\right)+H\left(\alpha_{k_2}^{(0)},\ \beta_{k_2}\right)\leqslant H\left(\alpha_{k_1}^{(0)},\ \beta_{k_1}^{*}\right)+H\left(\alpha_{k_2}^{(0)},\ \beta_{k_2}^{*}\right)$$
$$\forall\,\beta_{k_1}\in[b_{\min};\ b_{\max}]\ \text{ and }\forall\,\beta_{k_2}\in[b_{\min};\ b_{\max}] \tag{59}$$

hold along with (28). Plugging $\alpha_{k_2}=\alpha_{k_2}^{(0)}$ *in the left side of inequality (57) and plugging* $\beta_{k_2}=\beta_{k_2}^{*}$ *in the left side of inequality (59) gives inequalities*

$$K\left(\alpha_{k_1},\ \beta_{k_1}^{*}\right)\leqslant K\left(\alpha_{k_1}^{(0)},\ \beta_{k_1}^{*}\right)\ \ \forall\,\alpha_{k_1}\in[a_{\min};\ a_{\max}] \tag{60}$$

and

$$H\left(\alpha_{k_1}^{(0)},\ \beta_{k_1}\right)\leqslant H\left(\alpha_{k_1}^{(0)},\ \beta_{k_1}^{*}\right)\ \ \forall\,\beta_{k_1}\in[b_{\min};\ b_{\max}], \tag{61}$$

which are impossible due to $\left\{\alpha_{k_1}^{(0)},\ \beta_{k_1}^{*}\right\}$ *is not a pure-strategy equilibrium. Therefore, the supposition about (53) is contradictory. The impossibility of the other pure-strategy equilibrium for the second player in such a case (of two intervals) is proved symmetrically. The impossibility of other pure-strategy equilibria differing from the* $\{\alpha_i^{*},\ \beta_i^{*}\}_{i=1}^{N}$*-stack equilibrium in that the players use some other values at intervals is proved symmetrically as well.* □

So, Theorem 2 along with Theorem 1 allows obtaining the single pure-strategy solution of game (7) directly from equilibria in games (18). Does the equilibrium singularity in games (18) change when the single pure-strategy equilibrium of game (7) is already known? This question is answered by the following assertion.

Theorem 3. *If game (7) on product (8) by conditions (1) – (6) and (9) – (17) has a single equilibrium situation in pure strategies, then each of* N *games (18) by (9) – (17) and (20) – (26) has a single pure-strategy equilibrium, which is the respective interval part of the game (7) equilibrium.*

Proof. *Let game (7) have a single* $\{\alpha_i^{*},\ \beta_i^{*}\}_{i=1}^{N}$*-stack equilibrium. This implies that inequalities (33) and (34) hold. Plugging* $\alpha_i=\alpha_i^{*}\ \forall\,i\in\left\{\overline{1,\ N}\right\}\setminus\{k_*\}$ *in the left side of inequality (33) and plugging* $\beta_i=\beta_i^{*}\ \forall\,i\in\left\{\overline{1,\ N}\right\}\setminus\{k_*\}$ *in the left side of inequality (34) gives inequalities*

$$K\left(\alpha_{k_*},\ \beta_{k_*}^{*}\right)\leqslant K\left(\alpha_{k_*}^{*},\ \beta_{k_*}^{*}\right)\ \ \forall\,\alpha_{k_*}\in[a_{\min};\ a_{\max}] \tag{62}$$

and

$$H\left(\alpha_{k_*}^{*},\ \beta_{k_*}\right)\leqslant H\left(\alpha_{k_*}^{*},\ \beta_{k_*}^{*}\right)\ \ \forall\,\beta_{k_*}\in[b_{\min};\ b_{\max}], \tag{63}$$

whence $\left\{\alpha_{k_*}^{*},\ \beta_{k_*}^{*}\right\}$ *is a pure-strategy equilibrium at the* k_**-th interval (in the* k_**-th game) for every* $k_*\in\left\{\overline{1,\ N}\right\}$.

Suppose that $\exists\ k_0\in\left\{\overline{1,\ N}\right\}$ *such that* $\left\{\alpha_{k_0}^{(0)},\ \beta_{k_0}^{*}\right\}$ *is an equilibrium by* $\alpha_{k_0}^{(0)}\neq\alpha_{k_0}^{*}$. *Then inequalities*

$$K\left(\alpha_{k_0},\ \beta_{k_0}^{*}\right)\leqslant K\left(\alpha_{k_0}^{(0)},\ \beta_{k_0}^{*}\right)\ \ \forall\,\alpha_{k_0}\in[a_{\min};\ a_{\max}] \tag{64}$$

and

$$H\left(\alpha_{k_0}^{(0)},\ \beta_{k_0}\right)\leqslant H\left(\alpha_{k_0}^{(0)},\ \beta_{k_0}^{*}\right)\ \ \forall\,\beta_{k_0}\in[b_{\min};\ b_{\max}] \tag{65}$$

hold, whence inequalities

$$\sum_{k_*\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}} K\left(\alpha_{k_*},\ \beta^*_{k_*}\right)+K\left(\alpha_{k_0},\ \beta^*_{k_0}\right)\leqslant$$
$$\leqslant \sum_{k_*\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}} K\left(\alpha^*_{k_*},\ \beta^*_{k_*}\right)+K\left(\alpha^{(0)}_{k_0},\ \beta^*_{k_0}\right) \tag{66}$$

and

$$\sum_{k_*\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}} H\left(\alpha^*_{k_*},\ \beta_{k_*}\right)+H\left(\alpha^{(0)}_{k_0},\ \beta_{k_0}\right)\leqslant$$
$$\leqslant \sum_{k_*\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}} H\left(\alpha^*_{k_*},\ \beta^*_{k_*}\right)+H\left(\alpha^{(0)}_{k_0},\ \beta^*_{k_0}\right) \tag{67}$$

must hold as well. However, inequalities (66) and (67) imply that there is the

$$\left\{\{\alpha^*_i,\ \beta^*_i\}_{i\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}}\bigcup\left\{\alpha^{(0)}_{k_0},\ \beta^*_{k_0}\right\}\right\}\text{-stack equilibrium,}$$

which is impossible. Supposing that $\left\{\alpha^{(0)}_{k_0},\ \beta^{(0)}_{k_0}\right\}$ *is an equilibrium by* $\alpha^{(0)}_{k_0}\neq\alpha^*_{k_0}$ *and* $\beta^{(0)}_{k_0}\neq\beta^*_{k_0}$ *leads to inequalities*

$$K\left(\alpha_{k_0},\ \beta^{(0)}_{k_0}\right)\leqslant K\left(\alpha^{(0)}_{k_0},\ \beta^{(0)}_{k_0}\right)\ \ \forall\,\alpha_{k_0}\in[a_{\min};\ a_{\max}] \tag{68}$$

and

$$H\left(\alpha^{(0)}_{k_0},\ \beta_{k_0}\right)\leqslant H\left(\alpha^{(0)}_{k_0},\ \beta^{(0)}_{k_0}\right)\ \ \forall\,\beta_{k_0}\in[b_{\min};\ b_{\max}]\,, \tag{69}$$

whence impossible inequalities

$$\sum_{k_*\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}} K\left(\alpha_{k_*},\ \beta^*_{k_*}\right)+K\left(\alpha_{k_0},\ \beta^{(0)}_{k_0}\right)\leqslant$$
$$\leqslant \sum_{k_*\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}} K\left(\alpha^*_{k_*},\ \beta^*_{k_*}\right)+K\left(\alpha^{(0)}_{k_0},\ \beta^{(0)}_{k_0}\right) \tag{70}$$

and

$$\sum_{k_*\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}} H\left(\alpha^*_{k_*},\ \beta_{k_*}\right)+H\left(\alpha^{(0)}_{k_0},\ \beta_{k_0}\right)\leqslant$$
$$\leqslant \sum_{k_*\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}} H\left(\alpha^*_{k_*},\ \beta^*_{k_*}\right)+H\left(\alpha^{(0)}_{k_0},\ \beta^{(0)}_{k_0}\right) \tag{71}$$

imply the impossibility of the

$$\left\{\{\alpha^*_i,\ \beta^*_i\}_{i\in\left\{\overline{1,\,N}\right\}\backslash\{k_0\}}\bigcup\left\{\alpha^{(0)}_{k_0},\ \beta^{(0)}_{k_0}\right\}\right\}\text{-stack equilibrium.}$$

The impossibility of other pure-strategy equilibrium cases in "short" games (18) is proved symmetrically. □

The case when every "short" game has just a single pure-strategy equilibrium seems to be rarer than, say, the case with multiple equilibria. This, however, does not diminish the importance of Theorem 1 along with Theorem 2 and Theorem 3. These assertions allow to build a simpler proof of a more generalized assertion.

Theorem 4. *If each of N games (18) by (9) – (17) and (20) – (26) has a nonempty set of equilibrium situations in pure strategies, and game (7) on product (8) by conditions (1) – (6) is equivalent to the succession of these games, then every pure-strategy equilibrium in game (7) is a stack of any respective N equilibria in games (18). Apart from the stack, there are no other pure-strategy equilibria in game (7).*

Proof. Let the i-th game have J_i equilibria $\left\{\alpha^*_{ij_i},\ \beta^*_{ij_i}\right\}_{j_i=1}^{J_i}$ by $J_i \in \mathbb{N}$, where $\alpha^*_{ij_i} \in [a_{\min};\ a_{\max}]$, $\beta^*_{ij_i} \in [b_{\min};\ b_{\max}]$. Then

$$K\left(\alpha_i,\ \beta^*_{ij_i}\right) \leqslant K\left(\alpha^*_{ij_i},\ \beta^*_{ij_i}\right) \ \ \forall\, \alpha_i \in [a_{\min};\ a_{\max}] \tag{72}$$

and

$$H\left(\alpha^*_{ij_i},\ \beta_i\right) \leqslant H\left(\alpha^*_{ij_i},\ \beta^*_{ij_i}\right) \ \ \forall\, \beta_i \in [b_{\min};\ b_{\max}]\,, \tag{73}$$

whence

$$\sum_{i=1}^{N} K\left(\alpha_i,\ \beta^*_{ij_i}\right) \leqslant \sum_{i=1}^{N} K\left(\alpha^*_{ij_i},\ \beta^*_{ij_i}\right) \tag{74}$$

and

$$\sum_{i=1}^{N} H\left(\alpha^*_{ij_i},\ \beta_i\right) \leqslant \sum_{i=1}^{N} H\left(\alpha^*_{ij_i},\ \beta^*_{ij_i}\right). \tag{75}$$

Inequalities (74) and (75) directly imply the

$$\left\{\alpha^*_{ij_i},\ \beta^*_{ij_i}\right\}_{i=1}^{N}\text{-stack equilibrium} \tag{76}$$

for every $j_i \in \left\{\overline{1,\ J_i}\right\}$ by $i = \overline{1,\ N}$. Apart from stacks (76), there are no other pure-strategy equilibria in game (7) owing to Theorem 3 along with Theorem 2. □

It is quite obvious that Theorems 1 – 4 are valid for any two-person games whose players are constrained (forced) to use staircase-function strategies, i. e., they are valid for bimatrix games (with staircase-function strategies) as well. It remains only to study a possibility of equilibria in mixed strategies in such bimatrix games.

5. Representation by a succession of bimatrix games

Along with discrete time intervals, players may be forced to act within a finite subset of possible values of their pure strategies. That is, these values are

$$a_{\min} = a^{(0)} < a^{(1)} < a^{(2)} < \ldots < a^{(M-1)} < a^{(M)} = a_{\max} \tag{77}$$

and

$$b_{\min} = b^{(0)} < b^{(1)} < b^{(2)} < \ldots < b^{(Q-1)} < b^{(Q)} = b_{\max} \tag{78}$$

for the first and second players, respectively ($M \in \mathbb{N}$ and $Q \in \mathbb{N}$). Then the succession of N continuous games (18) by (9) – (17) and (20) – (26) becomes a succession of N bimatrix games

$$\left\langle \left\{ \left\{a^{(m-1)}\right\}_{m=1}^{M+1},\ \left\{b^{(q-1)}\right\}_{q=1}^{Q+1} \right\},\ \{\mathbf{K}_i,\ \mathbf{H}_i\} \right\rangle \tag{79}$$

with first player's payoff matrices $\mathbf{K}_i = [k_{imq}]_{(M+1)\times(Q+1)}$ whose elements are

$$k_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\,(t) \text{ for } i = \overline{1,\ N-1} \tag{80}$$

and

$$k_{Nmq} = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\,(t), \tag{81}$$

and with second player's payoff matrices $\mathbf{H}_i = [h_{imq}]_{(M+1)\times(Q+1)}$ whose elements are

$$h_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\,(t) \text{ for } i = \overline{1,\ N-1} \tag{82}$$

and

$$h_{Nmq} = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\,(t). \tag{83}$$

It is well-known that a finite two-person game always has an equilibrium either in pure or mixed strategies. So, if game (7) is made equivalent to a series of bimatrix games (or, in other words, is represented by a succession of bimatrix games), then it is easy to see that, unlike the representation with continuous games (18) by (9) – (17) and (20) – (26), the game always has a solution (at least, in mixed strategies).

Theorem 5. *If game (7) on product (8) by conditions (1) – (6) is equivalent to the succession of N bimatrix games (79) by (80) – (83), then the game is always solved as a stack of respective equilibria in these bimatrix games. Apart from the stack, there are no other equilibria in game (7).*

Proof. An equilibrium situation in the bimatrix game always exists, either in pure or mixed strategies. Denote by

$$\mathbf{U}_i = \left[u_i^{(m)}\right]_{1\times(M+1)}$$

and

$$\mathbf{Z}_i = \left[z_i^{(q)}\right]_{1\times(Q+1)}$$

the mixed strategies of the first and second players, respectively, in bimatrix game (79). The respective sets of mixed strategies of the first and second players are

$$\boldsymbol{\mathcal{U}} = \left\{\mathbf{U}_i \in \mathbb{R}^{M+1} : u_i^{(m)} \geqslant 0,\ \sum_{m=1}^{M+1} u_i^{(m)} = 1\right\} \tag{84}$$

and

$$\boldsymbol{\mathcal{Z}} = \left\{\mathbf{Z}_i \in \mathbb{R}^{Q+1} : z_i^{(q)} \geqslant 0,\ \sum_{q=1}^{Q+1} z_i^{(q)} = 1\right\}, \tag{85}$$

so $\mathbf{U}_i \in \boldsymbol{\mathcal{U}}$, $\mathbf{Z}_i \in \boldsymbol{\mathcal{Z}}$, and $\{\mathbf{U}_i,\ \mathbf{Z}_i\}$ is a situation in game (79), where J_i equilibria exist, $J_i \in \mathbb{N}$. Let $\left\{\mathbf{U}^*_{ij_i},\ \mathbf{Z}^*_{ij_i}\right\}_{i=1}^{N}$ be equilibria in N games (79) by (80) – (83), where

$$\mathbf{U}^*_{ij_i} = \left[u_{ij_i}^{(m)*}\right]_{1\times(M+1)} \in \boldsymbol{\mathcal{U}} \tag{86}$$

and

$$\mathbf{Z}_{ij_i}^{*} = \left[z_{ij_i}^{(q)*} \right]_{1\times(Q+1)} \in \boldsymbol{\mathcal{Z}}. \tag{87}$$

Henceforward, the proof is similar to that in Theorem 4. For equilibria $\left\{\mathbf{U}_{ij_i}^{*},\ \mathbf{Z}_{ij_i}^{*}\right\}_{i=1}^{N}$ *by (86) and (87), inequalities*

$$\begin{gathered}
\mathbf{U}_i \cdot \mathbf{K}_i \cdot \left(\mathbf{Z}_{ij_i}^{*}\right)^T = \\
= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} u_i^{(m)} z_{ij_i}^{(q)*} = \\
= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} u_i^{(m)} z_{ij_i}^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant \\
\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} u_{ij_i}^{(m)*} z_{ij_i}^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) = \\
= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} u_{ij_i}^{(m)*} z_{ij_i}^{(q)*} = \\
= \mathbf{U}_{ij_i}^{*} \cdot \mathbf{K}_i \cdot \left(\mathbf{Z}_{ij_i}^{*}\right)^T \quad \forall\, \mathbf{U}_i = \left[u_i^{(m)}\right]_{1\times(M+1)} \in \boldsymbol{\mathcal{U}} \ \text{ for } \ i = \overline{1,\ N-1},
\end{gathered} \tag{88}$$

$$\begin{gathered}
\mathbf{U}_N \cdot \mathbf{K}_N \cdot \left(\mathbf{Z}_{Nj_N}^{*}\right)^T = \\
= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} u_N^{(m)} z_{Nj_N}^{(q)*} = \\
= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} u_N^{(m)} z_{Nj_N}^{(q)*} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant \\
\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} u_{Nj_N}^{(m)*} z_{Nj_N}^{(q)*} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) = \\
= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} u_{Nj_N}^{(m)*} z_{Nj_N}^{(q)*} = \\
= \mathbf{U}_{Nj_N}^{*} \cdot \mathbf{K}_N \cdot \left(\mathbf{Z}_{Nj_N}^{*}\right)^T \quad \forall\, \mathbf{U}_N = \left[u_N^{(m)}\right]_{1\times(M+1)} \in \boldsymbol{\mathcal{U}}
\end{gathered} \tag{89}$$

and inequalities

$$\begin{gathered}
\mathbf{U}_{ij_i}^{*} \cdot \mathbf{H}_i \cdot \mathbf{Z}_i^T = \\
= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} u_{ij_i}^{(m)*} z_i^{(q)} =
\end{gathered}$$

$$= \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{ij_i}^{(m)*} z_i^{(q)} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{ij_i}^{(m)*} z_{ij_i}^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} h_{imq} u_{ij_i}^{(m)*} z_{ij_i}^{(q)*} =$$

$$= \mathbf{U}_{ij_i}^{*} \cdot \mathbf{H}_i \cdot \left(\mathbf{Z}_{ij_i}^{*}\right)^T \quad \forall\, \mathbf{Z}_i = \left[z_i^{(q)}\right]_{1\times(Q+1)} \in \boldsymbol{\mathcal{Z}} \ \text{ for } \ i = \overline{1,\ N-1}, \tag{90}$$

$$\mathbf{U}_{Nj_N}^{*} \cdot \mathbf{H}_N \cdot \mathbf{Z}_N^T =$$

$$= \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} h_{Nmq} u_{Nj_N}^{(m)*} z_N^{(q)} =$$

$$= \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{Nj_N}^{(m)*} z_N^{(q)} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{Nj_N}^{(m)*} z_{Nj_N}^{(q)*} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} h_{Nmq} u_{Nj_N}^{(m)*} z_{Nj_N}^{(q)*} =$$

$$= \mathbf{U}_{Nj_N}^{*} \cdot \mathbf{H}_N \cdot \left(\mathbf{Z}_{Nj_N}^{*}\right)^T \quad \forall\, \mathbf{Z}_N = \left[z_N^{(q)}\right]_{1\times(Q+1)} \in \boldsymbol{\mathcal{Z}} \tag{91}$$

hold. So, inequalities

$$\sum_{i=1}^{N-1} \mathbf{U}_i \cdot \mathbf{K}_i \cdot \left(\mathbf{Z}_{ij_i}^{*}\right)^T + \mathbf{U}_N \cdot \mathbf{K}_N \cdot \left(\mathbf{Z}_{Nj_N}^{*}\right)^T =$$

$$= \sum_{i=1}^{N-1}\sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} k_{imq} u_i^{(m)} z_{ij_i}^{(q)*} + \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} k_{Nmq} u_N^{(m)} z_{Nj_N}^{(q)*} =$$

$$= \sum_{i=1}^{N-1}\left(\sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_i^{(m)} z_{ij_i}^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right)\right) +$$

$$+ \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_N^{(m)} z_{Nj_N}^{(q)*} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1}\left(\sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{ij_i}^{(m)*} z_{ij_i}^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right)\right) +$$

$$+ \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{Nj_N}^{(m)*} z_{Nj_N}^{(q)*} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$\begin{aligned} &= \sum_{i=1}^{N-1}\sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} k_{imq} u_{ij_i}^{(m)*} z_{ij_i}^{(q)*} + \\ &+ \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} k_{Nmq} u_{Nj_N}^{(m)*} z_{Nj_N}^{(q)*} = \\ &= \sum_{i=1}^{N-1} \mathbf{U}_{ij_i}^{*} \cdot \mathbf{K}_i \cdot \left(\mathbf{Z}_{ij_i}^{*}\right)^T + \\ &+ \mathbf{U}_{Nj_N}^{*} \cdot \mathbf{K}_N \cdot \left(\mathbf{Z}_{Nj_N}^{*}\right)^T \end{aligned} \tag{92}$$

and

$$\begin{aligned} &\sum_{i=1}^{N-1} \mathbf{U}_{ij_i}^{*} \cdot \mathbf{H}_i \cdot \mathbf{Z}_i^T + \\ &+ \mathbf{U}_{Nj_N}^{*} \cdot \mathbf{H}_N \cdot \mathbf{Z}_N^T = \\ &= \sum_{i=1}^{N-1}\sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} h_{imq} u_{ij_i}^{(m)*} z_i^{(q)} + \\ &+ \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} h_{Nmq} u_{Nj_N}^{(m)*} z_N^{(q)} = \end{aligned}$$

$$= \sum_{i=1}^{N-1}\left(\sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{ij_i}^{(m)*} z_i^{(q)} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right)\right) +$$

$$+ \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{Nj_N}^{(m)*} z_N^{(q)} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1}\left(\sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{ij_i}^{(m)*} z_{ij_i}^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right)\right) +$$

$$+ \sum_{m=1}^{M+1}\sum_{q=1}^{Q+1} u_{Nj_N}^{(m)*} z_{Nj_N}^{(q)*} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$
\begin{aligned}
&= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} u_{ij_i}^{(m)*} z_{ij_i}^{(q)*} + \\
&+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Nmq} u_{Nj_N}^{(m)*} z_{Nj_N}^{(q)*} = \\
&= \sum_{i=1}^{N-1} \mathbf{U}_{ij_i}^{*} \cdot \mathbf{H}_i \cdot \left(\mathbf{Z}_{ij_i}^{*}\right)^T + \\
&+ \mathbf{U}_{Nj_N}^{*} \cdot \mathbf{H}_N \cdot \left(\mathbf{Z}_{Nj_N}^{*}\right)^T
\end{aligned}
\tag{93}
$$

hold as well. Therefore, the stack of successive equilibria $\left\{\mathbf{U}_{ij_i}^{*},\ \mathbf{Z}_{ij_i}^{*}\right\}_{i=1}^{N}$ *is an equilibrium in game (7). The sub-assertion of that, apart from such stacks, there are no other equilibria in game (7) is proved similarly to Theorem 3 along with Theorem 2.* □

Clearly, inequalities (72) and (73) by $i = \overline{1,\ N}$ are a partial case of inequalities (88)–(91). Inequalities (74) and (75) are a partial case of inequalities (92) and (93). In a way, Theorem 5 is a generalization of Theorem 4 for the case of finite game (7), which is correspondingly defined a product of staircase-function finite spaces. Nevertheless, stacking up pure-strategy equilibria and mixed-strategy equilibria of $(M+1)\times(Q+1)$ bimatrix games (79) can be cumbersome. The best case is when every “short” game has a single pure-strategy equilibrium.

6. Exemplification

To exemplify how the suggested method solves bimatrix games defined on a product of staircase-function spaces (which are obviously finite), consider a case in which $t \in [0.9\pi;\ 2\pi]$, the set of pure strategies of the first player is

$$X = \{x(t),\ t \in [0.9\pi;\ 2\pi]:\ 5 \leqslant x(t) \leqslant 8\} \subset \mathbb{L}_2\,[0.9\pi;\ 2\pi] \tag{94}$$

and the set of pure strategies of the second player is

$$Y = \{y(t),\ t \in [0.9\pi;\ 2\pi]:\ 3 \leqslant y(t) \leqslant 5\} \subset \mathbb{L}_2\,[0.9\pi;\ 2\pi]\,. \tag{95}$$

The first player's payoff functional is

$$K(x(t),\ y(t)) = \int\limits_{[0.9\pi;\ 2\pi]} \sin(xt)\cos(0.009xyt)\,d\mu(t) \tag{96}$$

and the second player's payoff functional is

$$H(x(t),\ y(t)) = \int\limits_{[0.9\pi;\ 2\pi]} \sin(0.2xyt)\,d\mu(t). \tag{97}$$

The players are forced to use pure strategies $x(t)$ and $y(t)$ such that

$$x(t) \in \{5 + 0.2\cdot(m-1)\}_{m=1}^{16} \subset [5;\ 8] \tag{98}$$

and

$$y(t) \in \{3 + 0.2\cdot(q-1)\}_{q=1}^{11} \subset [3;\ 5]\,, \tag{99}$$

and they can change their values only at time points

$$\left\{\tau^{(i)}\right\}_{i=1}^{10} = \{0.9\pi + 0.1i\pi\}_{i=1}^{10}. \tag{100}$$

Consequently, this game can be thought of as it is defined on rectangular lattice

$$\{5 + 0.2 \cdot (m-1)\}_{m=1}^{16} \times \{3 + 0.2 \cdot (q-1)\}_{q=1}^{11} \subset [5;\ 8] \times [3;\ 5], \tag{101}$$

that is this game is a succession of 11 finite 16×11 (bimatrix) games

$$\left\langle \left\{ \left\{a^{(m-1)}\right\}_{m=1}^{16},\ \left\{b^{(q-1)}\right\}_{q=1}^{11} \right\},\ \{\mathbf{K}_i,\ \mathbf{H}_i\} \right\rangle =$$
$$= \left\langle \left\{ \{5 + 0.2 \cdot (m-1)\}_{m=1}^{16},\ \{3 + 0.2 \cdot (q-1)\}_{q=1}^{11} \right\},\ \{\mathbf{K}_i,\ \mathbf{H}_i\} \right\rangle \tag{102}$$

with first player's payoff matrices $\left\{\mathbf{K}_i = [k_{imq}]_{16\times 11}\right\}_{i=1}^{11}$ whose elements are

$$k_{imq} = \int\limits_{[0.9\pi+0.1\cdot(i-1)\pi;\ 0.9\pi+0.1i\pi)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu(t) =$$
$$= \int\limits_{[0.9\pi+0.1\cdot(i-1)\pi;\ 0.9\pi+0.1i\pi)} f\left(5 + 0.2 \cdot (m-1),\ 3 + 0.2 \cdot (q-1),\ t\right) d\mu(t) =$$
$$= \int\limits_{[0.9\pi+0.1\cdot(i-1)\pi;\ 0.9\pi+0.1i\pi)} \sin\big(5t + 0.2 \cdot (m-1)\,t\big) \times$$
$$\times \cos\Big(0.009t\big(5 + 0.2 \cdot (m-1)\big)\big(3 + 0.2 \cdot (q-1)\big)\Big) d\mu(t) =$$
$$= \int\limits_{[0.9\pi+0.1\cdot(i-1)\pi;\ 0.9\pi+0.1i\pi)} \sin\big((4.8 + 0.2m)\,t\big) \times$$
$$\times \cos\big(0.009t\,(4.8 + 0.2m)\,(2.8 + 0.2q)\big) d\mu(t) \quad \text{for} \quad i = \overline{1,\ 10} \tag{103}$$

and

$$k_{11mq} = \int\limits_{[1.9\pi;\ 2\pi]} \sin\big((4.8 + 0.2m)\,t\big) \times$$
$$\times \cos\big(0.009t\,(4.8 + 0.2m)\,(2.8 + 0.2q)\big) d\mu(t), \tag{104}$$

and with second player's payoff matrices $\left\{\mathbf{H}_i = [h_{imq}]_{16\times 11}\right\}_{i=1}^{11}$ whose elements are

$$h_{imq} = \int\limits_{[0.9\pi+0.1\cdot(i-1)\pi;\ 0.9\pi+0.1i\pi)} g\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu(t) =$$

$$= \int\limits_{[0.9\pi+0.1\cdot(i-1)\pi;\ 0.9\pi+0.1i\pi)} g\left(5 + 0.2 \cdot (m-1),\ 3 + 0.2 \cdot (q-1),\ t\right) d\mu(t) =$$

$$= \int\limits_{[0.9\pi+0.1\cdot(i-1)\pi;\ 0.9\pi+0.1i\pi)} \sin\Big(0.2t\big(5+0.2\cdot(m-1)\big)\big(3+0.2\cdot(q-1)\big)\Big)d\mu(t) =$$

$$= \int\limits_{[0.9\pi+0.1\cdot(i-1)\pi;\ 0.9\pi+0.1i\pi)} \sin\big(0.2t\,(4.8+0.2m)\,(2.8+0.2q)\big)d\mu(t)$$

$$\text{for } i=\overline{1,\ 10} \tag{105}$$

and

$$h_{11mq} = \int\limits_{[1.9\pi;\ 2\pi]} \sin\big(0.2t\,(4.8+0.2m)\,(2.8+0.2q)\big)d\mu(t). \tag{106}$$

The 16×11 bimatrix games (102) with (103) – (106) are solved in pure strategies, whereas every game has a single pure-strategy equilibrium. The stack of the 11 first player's equilibrium strategies in each of those 16×11 bimatrix games is shown in Fig. 1. The stack of the 11 second player's equilibrium strategies is similarly shown in Fig. 2. These stacks are the factual single pure-strategy equilibrium (of staircase time functions) in the initial game. The players' payoffs are shown in Fig. 3. The eventual payoff of the first player is approximately 1.83203383, whereas the second player receives approximately 3.03719908558.

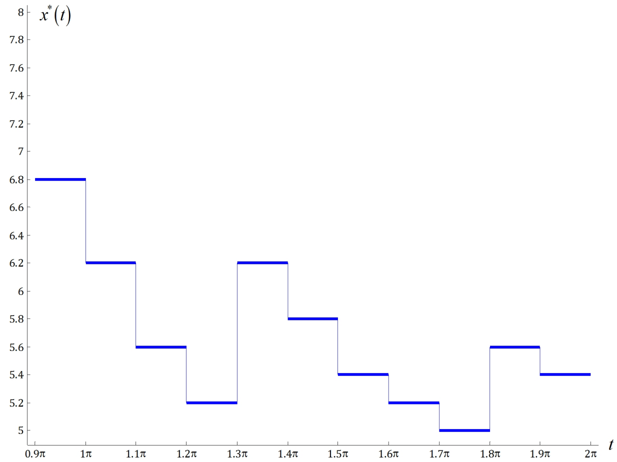

Fig. 1. The stack of the 11 first player's equilibrium pure strategies as the equilibrium staircase-function pure strategy $x^*(t)$

The example clearly shows (especially when seeing Fig. 1 and Fig. 2) that solving a succession of bimatrix games is far easier than tackling games whose players' pure strategies look like those staircase functions in Fig. 1 and Fig. 2. Even if not every bimatrix game has a single equilibrium, a solution of the initial game is built in the same way as (94) – (106). The only difference is that then there will be multiple stacked equilibria, which commonly induce instability of the players' behavior [5, 9, 10]. The behavior instability is a serious problem in noncooperative games having multiple equilibria differing in the player's payoffs [1, 15, 22]. It is particularly solved by equilibria refinement with using domination efficiency along with maximin and the superoptimality rule [10].

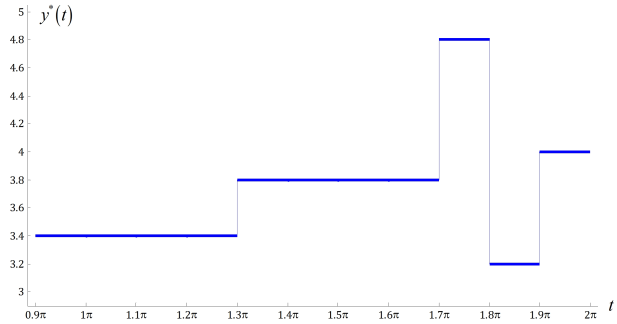

Fig. 2. The stack of the 11 second player's equilibrium pure strategies as the equilibrium staircase-function pure strategy $y^*(t)$

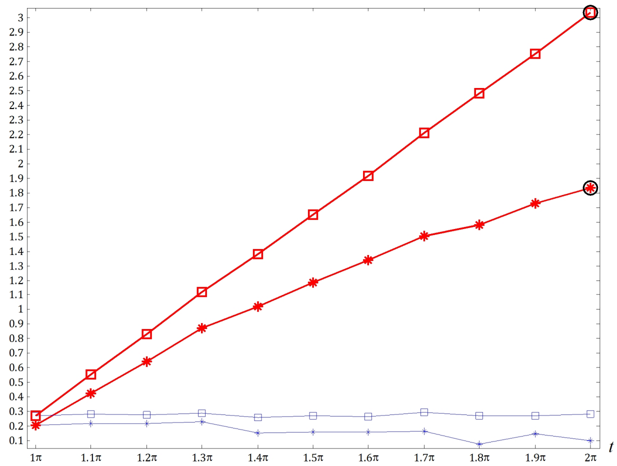

Fig. 3. The first player's payoffs (stars) and the second player's payoffs (squares) at the end of every interval and their cumulative sum (thicker polyline)

7. Discussion

Just like in the considered example, stacking the "short" games' pure-strategy equilibria (by Theorem 4) is fulfilled trivially. When there is at least an equilibrium in mixed strategies for an interval (that actually falls within conditions of Theorem 5), the stacking is fulfilled as well implying that the resulting pure-mixed-strategy solution (equilibrium) of game (7) is realized successively, interval by interval, spending the same amount of time to implement both pure strategy and mixed strategy solutions (equilibria).

Continuous games are ever struggled to be approximated or rendered to finite games not just for the sake of simplicity itself [3, 4, 6]. The matter is the finite approximation or rendering makes solutions tractable so that they can be easily implemented and practiced [7, 8, 19, 22]. However, even a finite (that is, bimatrix) game may be not tractable due to gigantic number of situations in game (as it is exemplified in Motivation). So, the presented method, further "breaking" the initial game defined on a product of staircase-function finite spaces, makes it completely tractable. The tractability does not depend on the number of (time) intervals. Unless the sets of possible values of players' pure strategies are of order of hundreds or thousands (when searching for equilibria in a "short" bimatrix game may take a few seconds and more), the method is entirely applicable. Moreover, the presented method is a significant contribution to the mathematical game theory and practice for avoiding too complicated solutions resulting from game continuities and functional spaces of pure strategies. This is similar to preventing Einstellung effect in modeling [13, 23]. The "breaking" of the initial game defined on a product of staircase-function finite spaces into a succession of "short" bimatrix games herein "deeinstellungizes" such noncooperative two-person games.

8. Conclusion

A two-person game defined on a product of staircase-function finite spaces is equivalent to a bimatrix game. However, players' payoff matrices in this game are built very slowly, so it is impracticable to find any solutions in such games. On the other hand, the two-person game is equivalent to the succession of "short" bimatrix games, each defined on an interval where the pure strategy value is constant. Thus, owing to Theorem 5 (along with Theorem 3 and Theorem 4), the solution (equilibrium) of the initial game can be obtained by stacking the solutions (equilibria) of the "short" bimatrix games. The stack is always possible, even when only time is discrete (and the set of pure strategy possible values is continuous). Moreover, any combination of the respective equilibria of the "short" bimatrix games is an equilibrium of the initial two-person game.

A similar question of solving games on a product of staircase-function finite spaces should be studied for the case of three players. Then the presented assertions and conclusions might be just adapted to trimatrix games, which model processes of practically optimizing the distribution of the limited resources among three sides as well as bimatrix games do for two sides. Theorems 1 – 3, however, are expected to have less practical impact for trimatrix games due to the equilibrium singleness is less likely in this case.

References

1. Vorob'yov N.N. Game theory fundamentals. Noncooperative games / N.N. Vorob'yov – Moscow: Nauka, 1984. – 496 p. (in Russian).
2. Nisan N. Algorithmic Game Theory / N. Nisan, T. Roughgarden, É. Tardos, V.V. Vazirani. – Cambridge, UK: Cambridge University Press, 2007. – 778 p.
3. Romanuke V.V. Sampling the continuous two-sided noncooperative game on unit hypercube and multidimensional matrix reshaping for solving the corresponding bimatrix game / V.V. Romanuke // Visnyk of Kremenchuk National University of Mykhaylo Ostrogradskyy. – 2014. – Iss. 6 (89) – P. 1, 32–37.
4. Romanuke V.V. Approximation of isomorphic infinite two-person noncooperative games via variously sampling the players' payoff functions and reshaping payoff matrices into bimatrix game / V.V. Romanuke, V.G. Kamburg // Applied Computer Systems. – 2016. – Vol. 20. – P. 5–14.
5. Romanuke V.V. A couple of collective utility and minimum payoff parity loss rules for refining Nash equilibria in bimatrix games with asymmetric payoffs / V.V. Romanuke // Visnyk of Kremenchuk National University of Mykhaylo Ostrogradskyy. – 2018. – Iss. 1 (108). – P. 38–43.
6. Romanuke V.V. Approximation of unit-hypercubic infinite two-sided noncooperative game via dimension-dependent irregular samplings and reshaping the multidimensional payoff matrices into flat matrices for solving the corresponding bimatrix game / V.V. Romanuke // Computer Modelling and New Technologies. – 2015. – Vol. 19, № 3A. – P. 7–16.

7. Romanuke V.V. Convergence and estimation of the process of computer implementation of the optimality principle in matrix games with apparent play horizon / V.V. Romanuke // Journal of Automation and Information Sciences. – 2013. – Vol. 45, Iss. 10. – P. 49–56.
8. Romanuke V.V. Theoretic-game methods of identification of models for multistage technical control and run-in under multivariate uncertainties (a Dissertation for the Doctoral Degree of Technical Sciences in Speciality 01.05.02 Mathematical Modeling and Computational Methods) / V.V. Romanuke. – Vinnytsia: Vinnytsia National Technical University, 2014. – 449 p. (in Ukrainian).
9. Harsanyi J.C. A General Theory of Equilibrium Selection in Games / J.C. Harsanyi, R. Selten. – Cambridge Mass.: The MIT Press, 1988. – 396 p.
10. Romanuke V.V. Pure strategy Nash equilibria refinement in bimatrix games by using domination efficiency along with maximin and the superoptimality rule / V.V. Romanuke // KPI Science News. – 2018. – № 3. – P. 42–52.
11. Romanuke V.V. Maximization of collective utility and minimization of payoff parity losses for ordering and scaling efficient Nash equilibria in trimatrix games with asymmetric payoffs / V.V. Romanuke // Herald of Khmelnytskyi national university. Technical sciences. – 2018. – № 3. – P. 279–281.
12. Yang J. Group formation in the spatial public goods game with continuous strategies / J. Yang, Y.-S. Chen, Y. Sun, H.-X. Yang, Y. Liu // Physica A: Statistical Mechanics and its Applications. – 2018. – Iss. 505. – P. 737–743.
13. Romanuke V.V. Finite approximation of continuous noncooperative two-person games on a product of linear strategy functional spaces / V.V. Romanuke // Journal of Mathematics and Applications. – 2020. – Vol. 43. – P. 123–138.
14. Romanuke V.V. Approximate equilibrium situations with possible concessions in finite noncooperative game by sampling irregularly fundamental simplexes as sets of players' mixed strategies / V.V. Romanuke // Journal of Uncertain Systems. – 2016. – Vol. 10, № 4. – P. 269–281.
15. Vorob'yov N.N. Game theory for economists-cyberneticists / N.N. Vorob'yov. – Moscow: Nauka, 1985. – 272 p. (in Russian).
16. Romanuke V.V. Adaptive finite approximation of continuous noncooperative games / V.V. Romanuke // Journal of Automation and Information Sciences. – 2020. – Vol. 52, Iss. 10. – P. 31–41.
17. Hirshleifer D. Mood beta and seasonalities in stock returns / D. Hirshleifer, D. Jiang, Y.M. DiGiovanni // Journal of Financial Economics. – 2020. – Vol. 137, Iss. 1. – P. 272–295.
18. Kontogiannis S.C. Polynomial algorithms for approximating Nash equilibria of bimatrix games / S.C. Kontogiannis, P.N. Panagopoulou, P.G. Spirakis // Theoretical Computer Science. – 2009. – Vol. 410, Iss. 17. – P. 1599–1606.
19. Romanuke V. V. Theory of Antagonistic Games / V.V. Romanuke. – Lviv: New World. – 2000, 2010. – 294 p. (in Ukrainian).
20. Edwards R.E. Functional Analysis: Theory and Applications / R.E. Edwards. – New-York City, New-York, USA: Holt, Rinehart and Winston, 1965. – 781 p.
21. Romanuke V.V. Equidistantly discrete on the argument axis functions and their representation in the Walsh series and some other orthonormal bases series / V.V. Romanuke // Optoelectronic Information-Power Technologies. – 2009. – № 2 (18). – P. 35–48.
22. Romanuke V.V. Ecological-economic balance in fining environmental pollution subjects by a dyadic 3-person game model / V.V. Romanuke // Applied Ecology and Environmental Research. – 2019. – Vol. 17, № 2. – P. 1451–1474.
23. Loesche F. Mindset and Einstellung Effect / F. Loesche, T. Ionescu // Encyclopedia of Creativity. – Cambridge, Massachusetts, USA: Academic Press. – 2020. – P. 174–178.

Article: received 08.09.2021
revised 20.10.2021
printing adoption 10.11.2021

ІГРИ ДВОХ ОСІБ НА ДОБУТКУ НЕПЕРЕРВНИХ І СКІНЧЕННИХ ПРОСТОРІВ СХОДИНКОВИХ ФУНКЦІЙ

В. Романюк

Одеська національна академія зв'язку ім. О. С. Попова,
вул. Кузнечна, 1, Одеса, Україна, 65029,
e-mail: romanukevadimv@gmail.com

Запропоновано доцільний метод розв'язування ігор двох осіб на добутку просторів сходинкових функцій. Ці простори можуть бути скінченними та нескінченними. Метод заснований на склеюванні рівноваг "коротких" ігор двох осіб, кожна з яких визначена на інтервалі, де значення чистої стратегії постійні. Спочатку формалізовано гру двох осіб, у якій стратегії гравців є сходинковими функціями. У такій грі множиною чистих стратегій гравця є континуум сходинкових функцій часу. Час приймається дискретним. Доведено чотири теореми, які дають змогу виконувати склейку для випадку рівноваг у чистих стратегіях. Далі множина можливих значень чистої стратегії гравця дискретизується так, що гра стає визначеною на добутку скінченних просторів сходинкових функцій. Для формалізації методу розв'язування ігор двох осіб на добутку скінченних просторів сходинкових функцій доводиться твердження про те, що відповідна гра розв'язується як склейка відповідних рівноваг "коротких" біматричних ігор. У такому випадку рівноваги розглядаються загалом, позаяк вони можуть бути подані й у змішаних стратегіях. Склейка є довільною комбінацією (послідовністю) відповідних рівноваг "коротких" біматричних ігор. Крім такої склейки, у "довгій" біматричній грі інших рівноваг немає. Ця склейка завжди можлива, навіть якщо дискретним є лише час (і множина можливих значень чистої стратегії є неперервною). Наведено приклад, який демонструє, як виконується склеювання для випадку, коли кожна "коротка" біматрична гра має єдину рівновагу у чистих стратегіях. Такий метод, далі "розбиваючи" вихідну ("довгу") біматричну гру, визначену на добутку скінченних просторів сходинкових функцій, надає їй повної змістовності.

Ключові слова: теорія ігор, функціонал виграшів, стратегія у формі сходинкової функції, біматрична гра.