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ON THE INTEGRAL EQUATION APPROACH FOR THE NUMERICAL SOLUTION OF THE DIRICHLET ELASTOSTATIC PROBLEM IN THE DOMAIN WITH A CRACK

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The Dirichlet boundary value problem for the elastostatic equation in the planar bounded domain with a crack inside is considered. Using indirect integral equation approach this problem can be reduced to the system of boundary integral equations of the first kind with unknown densities. These densities are vector values and defined on the domain boundary and the crack. Method incorporates the usage of a matrix of the fundamental solution to the elastostatic equation as a kernel of the single layer elastic potential. Further, having a parametric representation of the boundary curve and the crack, we parametrize system of integral equations. Parametric representation of the crack can be extended to be defined on the same segment as the boundary curve. Some kernels contain logarithmic singularities. It is straightforward to verify that they can be allocated explicitly in an additive way employing certain weight functions. The density over the crack has square root singularity that also can be handled within so-called *cosine* substitution. The well-posedness of the system in corresponding Hölder spaces is shown. The partial discretization is performed by the quadrature method based on trigonometrical quadrature rules. In order to obtain completely discrete system, we collocate the partially discrete system at equidistant nodal points that are used in the quadrature formulas. One of the main advantage of this approach is that it has spectral properties. The convergence analysis and error estimate are completed. We presented numerical examples for two different configurations of domain and input data with known and unknown exact solution, respectively. Numerical experiments show the applicability of the method, its effectiveness and confirm the obtained theoretical results regarding the error estimation.

Key words: elastostatics, Dirichlet problem, integral equation approach, trigonometrical quadratures, domain with crack.

1. Introduction

Elastostatic boundary value problems can be considered as mathematical model for various physical process, which are important in different applications [9, 12]. For the theoretical investigation of such kind of problems usually the integral equation method is involved. This approach can be used also for the numerical solution of elastostatic problems. In the case of solution domains with a crack the integral equation techniques have some additional specific difficulties. Various researches of the scattering problems are presented for time-harmonic elastic waves with a Neumann boundary condition on the two-dimensional crack [5] and mixed boundary value problems for Laplace equation in a planar domain contained a cut [2]. In all cases the theoretical investigation and the numerical solution are based on boundary integral equations of the first kind. These equations are well posed in corresponding spaces (see f.e. [8]).

In this paper we extent these methods to the case of elastostatic equation in a bounded domain contained a crack with Dirichlet boundary value condition. We describe the numerical solution of integral equations arising from an indirect potential approach by a trigonometrical quadrature method and show its convergence and error analysis. The main advantage of proposed approach is that it belongs to spectral methods and in the case of analytical input data we have the exponential convergence. The similar approach was used for a number of elastostatic problems in [1, 3, 6].

Let $D \subset \mathbb{R}^2$ be a simply connected bounded domain with an orientable simply closed boundary $\Gamma_2 \in C^1$, that has the regular parametric representation and $\Gamma_1 \in C^1$ be a regular non-intersecting C^1-smooth open arc with fixed endpoints x^*_{-1} and x^*_1, that is located in D (see Fig. 1).

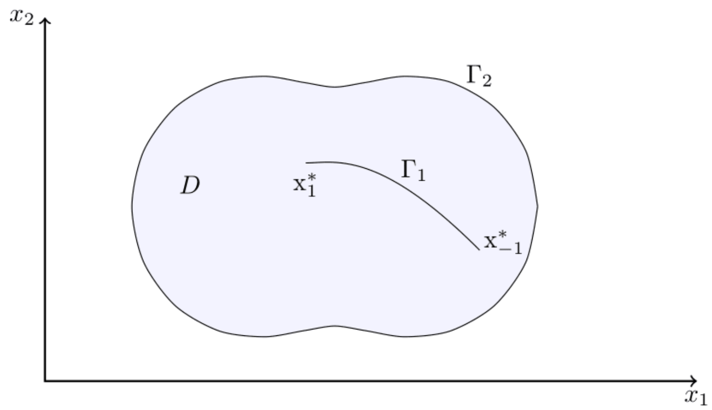

Fig. 1. Example of a planar domain D with a crack Γ_1

We consider a boundary value problem for the elastostatic equation

$$\mu\Delta u + (\lambda + \mu)\text{graddiv}u = 0 \quad \text{in } D \setminus \Gamma_1 \tag{1}$$

with Dirichlet boundary value conditions

$$u = f_1 \quad \text{on } \Gamma_1, \qquad u = f_2 \quad \text{on } \Gamma_2. \tag{2}$$

Here μ and λ are Lamé constants with $\lambda > -\mu$ and $\mu > 0$ characterizing physical properties of the body and f_ℓ, $\ell = 1, 2$ are given smooth functions. For regularity, we require $u \in C^2(D \setminus \Gamma_1)$, where $u : D \setminus \Gamma_1 \to \mathbb{R}^2$ and $f_\ell : \Gamma_\ell \to \mathbb{R}^2$ and the corresponding function spaces have to be understood as vector valued. According to [10] the following uniqueness result holds.

Proposition 1. *The boundary value problem (1), (2) has at most one solution.*

2. Indirect integral equation method

We present the solution of (1), (2) as a sum of elastostatic single layer potentials with vector densities ϕ_ℓ on Γ_ℓ for $\ell = 1, 2$

$$u(x) = \sum_{k=1}^{2} \int_{\Gamma_k} \Phi(x, y)\phi_k(y)\, ds(y), \quad x \in D \setminus \Gamma_1. \tag{3}$$

Here Φ is the fundamental solution of the equation (1) [12]

$$\Phi(x,y) = \frac{c_1}{2\pi}\,\Psi(x,y)\,I + \frac{c_2}{2\pi}\,J(x-y),$$

where

$$c_1 = \frac{\lambda+3\mu}{\mu(\lambda+2\mu)}, \qquad c_2 = \frac{\lambda+\mu}{\mu(\lambda+2\mu)}$$

and

$$\Psi(x,y) = \ln\frac{1}{|x-y|}\,, \quad x \neq y,$$

matrix J denotes dyadic product

$$J(w) = \frac{w\,w^\top}{|w|^2},$$

I is the identity matrix. The density ϕ_2 is continuous and the density ϕ_1 is assumed to be of the form

$$\phi_1(x) = \frac{\tilde{\phi}_1(x)}{\sqrt{|x-x^*_{-1}||x-x^*_1|}},\ x \in \Gamma_1 \setminus \{x^*_{-1}, x^*_1\}$$

with $\tilde{\phi}_1 \in C(\Gamma_1)$.

Then according to the properties of the elastostatic single layer potential we receive the following result.

Theorem 2. *The combination of elastostatic single layer potentials (3) solves the boundary value problem (1), (2), if the densities ϕ_ℓ, $\ell = 1, 2$ satisfy the integral equations system*

$$\sum_{k=1}^{2}\int_{\Gamma_k}\Phi(x,y)\phi_k(y)\,ds(y) = f_\ell(x), \quad x \in \Gamma_\ell, \quad \ell = 1,2. \tag{4}$$

The proof of this theorem is gained by matching the representation against the given boundary data involving classical jump relations for elastic single-layer potentials (for formulas, see [9,12]). To investigate the solvability of the system (4) and for the further numerical solution we parametrize the system (4) with the separation of the singularities in the kernels as some weight functions. The density singularity is handled by employing the cosine-substitution.

Assume that the boundaries Γ_ℓ, $\ell = 1, 2$ have parametric representations

$$\Gamma_\ell = \{x_\ell(t) = (x_{\ell,1}(t), x_{\ell,2}(t)) : t \in I_\ell\}, \quad \ell = 1,2$$

with $I_1 = [-1,1]$ and $I_2 = [0,2\pi]$. Then the system (4) leads to the system of parametrized integral equations

$$\frac{1}{2\pi}\int_{-1}^{1} H_{\ell,1}(t,\tau)\mu_1(\tau)d\tau + \frac{1}{2\pi}\int_{0}^{2\pi} H_{\ell,2}(t,\tau)\tilde{\mu}_2(\tau)d\tau = \hat{f}_\ell(t), \tag{5}$$

where

$$\mu_1(t) = \phi_1(x_1(t))|x_1'(t)|,\ \tilde{\mu}_2(t) = \phi_2(x_2(t))|x_2'(t)|,\ \hat{f}_\ell(t) = f_\ell(x_\ell(t)),\ t \in I_\ell,\ \ell = 1,2$$

and the kernels have form

$$H_{\ell,k}(t,\tau) = 2\pi\Phi(x_\ell(t), x_k(\tau)), \quad \ell, k = 1, 2.$$

We manage the square root singularity in the density μ_1 by using the cosine-substitution in the corresponding integrals. To handle the logarithmic singularities in the kernels $H_{\ell,\ell}$, we make suitable transformations and apply special quadrature rules. For that reason, after substitution $t = \cos s$ in the first integral equation and $\tau = \cos\sigma$ in the integrals containing function μ_1 (for more details see [2, 4, 13]) the system (5) can be reduced to the equivalent system

$$\begin{aligned} &\frac{1}{2\pi}\int\limits_0^{2\pi}\Big[-c_1\ln\frac{4}{e}\sin^2\frac{s-\sigma}{2}I + \ell\tilde{H}_{\ell,\ell}(s,\sigma)\Big]\tilde{\mu}_\ell(\sigma)d\sigma \\ &+\frac{1}{2\pi}\int\limits_0^{2\pi}\tilde{H}_{\ell,3-\ell}(s,\sigma)\tilde{\mu}_{3-\ell}(\sigma)d\sigma = \tilde{f}_\ell(s),\ s\in I_2,\ \ell=1,2, \end{aligned} \tag{6}$$

where

$$\tilde{\mu}_1(s) = \tilde{\phi}_1(\tilde{x}_1(s))|\tilde{x}_1'(s)|,\ \tilde{x}_1(s) = x_1(\cos s),\ \tilde{f}_1(s) = 2\hat{f}_1(\cos s),\ \tilde{f}_2(t) = 2\hat{f}_2(s)$$

and obtained kernels are given as

$$\tilde{H}_{1,2}(s,\sigma) = 2H_{1,2}(\cos s,\sigma),$$

$$\tilde{H}_{2,1}(s,\sigma) = H_{2,1}(s,\cos\sigma),$$

$$\tilde{H}_{1,1}(s,\sigma) = \frac{c_1}{2}\ln\frac{4}{e^2}(\cos s - \cos\sigma)^2 I + H_{1,1}(\cos s,\cos\sigma),$$

and

$$\tilde{H}_{2,2}(s,\sigma) = \frac{c_1}{2}\ln\frac{4}{e}\sin^2\frac{s-\sigma}{2}I + H_{2,2}(s,\sigma),$$

with diagonal terms

$$\tilde{H}_{1,1}(s,s) = -c_1\ln\frac{e|\tilde{x}_1'(s)|}{2}I + c_2\tilde{J}_1(s,s)$$

and

$$\tilde{H}_{2,2}(s,s) = -\frac{c_1}{2}\ln e|x_2'(s)|^2 I + c_2\tilde{J}_2(s,s).$$

Here we used the notations

$$\tilde{J}_1(s,s) = \frac{\tilde{x}_1(s)\tilde{x}_1(s)^\top}{|\tilde{x}_1'(s)|^2}, \quad \tilde{J}_2(s,s) = \frac{x_2(s)x_2(s)^\top}{|x_2'(s)|^2}.$$

After these transformations we get that $\tilde{\mu}_1(-s) = \tilde{\mu}_1(s)$, $\tilde{H}_{1,1}(-s,-\sigma) = \tilde{H}_{1,1}(s,\sigma)$, $\tilde{H}_{1,2}(-s,\sigma) = \tilde{H}_{1,2}(s,\sigma)$ and $\tilde{H}_{2,1}(s,-\sigma) = \tilde{H}_{2,1}(s,\sigma)$.

To establish the solvability of the system (6) we consider the standard Hölder spaces $C^{m,\alpha}[0,2\pi]$ with $m \in \mathbb{N}\cup\{0\}$ and $0<\alpha<1$ and $C_e^{m,\alpha}[0,2\pi]$ the subspaces of even functions from $C^{m,\alpha}[0,2\pi]$.

From the uniqueness Proposition 1 and the properties of integral operators in the system (6) holds the following result.

Theorem 3. *Let $\Gamma_\ell \in C^p$, $\ell = 1,2$, $p \in \mathbb{N}$. For $m < p$, $\tilde{f}_1 \in C_e^{m,\alpha}[0,2\pi]$, $\tilde{f}_2 \in C^{m,\alpha}[0,2\pi]$, the system of integral equations (6) is uniquely solvable with $\tilde{\mu}_1 \in C_e^{m-1,\alpha}[0,2\pi]$, $\tilde{\mu}_2 \in C^{m-1,\alpha}[0,2\pi]$.* The well-posedness of system (6) can be carried out also in a Sobolev space setting. Note that as compared with previous investigations (see [8]) the cosine transformation makes the existence analysis more simple and clear.

3. Full discretization

In order to obtain completely discrete system we apply the quadrature method based on trigonometrical quadrature rules [4, 11] with $2n$ equidistant nodal points

$$s_j = \frac{j\pi}{n}, \quad j = \overline{0, 2n-1}. \tag{7}$$

They are constructed via interpolation in the $2n$-dimensional space T_n of trigonometric polynomials of the form

$$v(s) = \sum_{m=0}^{n} a_m \cos ms + \sum_{m=1}^{n-1} b_m \sin ms, \; a_m, b_m \in \mathbb{R}$$

For the integrals in (6) we use the following interpolatory quadratures

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\, d\sigma \approx \frac{1}{2n}\sum_{k=0}^{2n-1} f(s_k),$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma) \ln\frac{4}{e}\sin^2\frac{s-\sigma}{2} d\sigma \approx \sum_{k=0}^{2n-1} R_k(s)\, f(s_k),$$

with weight functions

$$R_k(s) = -\frac{1}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(s-s_k) - \frac{1}{2n^2}\cos n(s-s_k).$$

Thus, applying these quadratures to approximate the integrals in the system (6) and collocating the received approximate equations at the same points we get the $(6n+2) \times (6n+2)$ linear system

$$\begin{cases} \displaystyle\sum_{j=0}^{n} A_{ij}^{11}\tilde{\mu}_{1j} + \sum_{j=0}^{2n-1} A_{ij}^{12}\tilde{\mu}_{2j} = \tilde{f}_{1i}, \quad i = \overline{0, n}, \\ \displaystyle\sum_{j=0}^{n} A_{ij}^{21}\tilde{\mu}_{1j} + \sum_{j=0}^{2n-1} A_{ij}^{22}\tilde{\mu}_{2j} = \tilde{f}_{2i}, \quad i = \overline{0, 2n-1}, \end{cases} \tag{8}$$

where

$$\tilde{\mu}_{1k} \approx \tilde{\mu}_1(s_k), \quad \tilde{f}_{1k} = \tilde{f}_1(s_k), \quad k = \overline{0, n},$$
$$\tilde{\mu}_{2k} \approx \tilde{\mu}_2(s_k), \quad \tilde{f}_{2k} = \tilde{f}_2(s_k), \quad k = \overline{0, 2n-1},$$

and the matrix coefficients are denoted as

$$A^{11}_{i\ell} = -\frac{c_1}{2} R_i I + \frac{1}{2n} \tilde{H}_{11}(s_i, s_\ell), \quad \ell = 0, n$$

for $i = \overline{0,n}$,

$$A^{11}_{ij} = -\frac{c_1}{2} (R_{|i-j|} + R_{i+j}) I + \frac{1}{n} \tilde{H}_{11}(s_i, s_j)$$

for $i = \overline{0,n}$, $j = \overline{1, n-1}$,

$$A^{12}_{ij} = \frac{1}{2n} \tilde{H}_{12}(s_i, s_j), \ i = \overline{0,n}, \ j = \overline{0, 2n-1},$$

$$A^{21}_{i\ell} = \frac{1}{2n} \tilde{H}_{21}(s_i, s_\ell), \quad \ell = 0, n, \quad i = \overline{0, 2n-1},$$

$$A^{21}_{ij} = \frac{1}{n} \tilde{H}_{21}(s_i, s_j), \quad i = \overline{0, 2n-1}, \ j = \overline{1, n-1},$$

$$A^{22}_{ij} = -c_1 R_{|i-j|} I + \frac{1}{2n} \tilde{H}_{22}(s_i, s_j), \ i, j = \overline{0, 2n-1}.$$

Here we have used notation $R_k = R_k(0)$.

The convergence analysis and error estimate of this method can be carried out via a collective compact operators theory (see [4]) or relied on some estimate for trigonometric interpolation in Hölder spaces (as it shown in [5]).

Lemma 4. *For $\tilde{f}_1 \in C^{m,\alpha}_e[0, 2\pi]$, $\tilde{f}_2 \in C^{m,\alpha}[0, 2\pi]$ and a sufficiently large n the system (8) is uniquely solvable for $\tilde{\mu}_{kn} \in T_n$. For exact solutions $\tilde{\mu}_k$ of (6), $k = 1, 2$ the following error estimates are holds*

$$\|\tilde{\mu}_k - \tilde{\mu}_{kn}\|_{m,\alpha} \le C_k \frac{\ln n}{n^{q-m+\beta-\alpha}} \|\tilde{\mu}_k\|_{q,\beta},$$

where $0 \le m \le q$, $0 < \alpha \le \beta < 1$ and $C_k > 0$ are some constants that depend on α, β, m, q.

Note that for analytic boundaries and cracks and for analytic boundary functions we receive the error estimate

$$\|\tilde{\mu}_k - \tilde{\mu}_{kn}\|_{m,\alpha} \le C_k e^{-c_k n}, \quad k = 1, 2$$

for some constants $c_k > 0$ (see [11]). We also remark that the error analysis for this numerical method can also be carried out in a Sobolev space setting (see [11, Section 13.4]).

The numerical solution of the boundary value problem (1), (2) can be calculated as

$$u_n(x) = \frac{\pi}{n} \left[\sum_{j=1}^{n-1} \Phi(x, \tilde{x}_1(s_j)) \tilde{\mu}_{1j} + \frac{1}{2} \sum_{j=0,n} \Phi(x, \tilde{x}_1(s_j)) \tilde{\mu}_{1j} + \sum_{j=0}^{2n-1} \Phi(x, x_2(s_j)) \tilde{\mu}_{2j} \right]$$

for $x \in D \setminus \Gamma_1$.

The proposed approach can be applied with some modification to the case with the boundary conditions on the crack:

$$u^{\pm} = f_1^{\pm} \quad \text{on } \Gamma_1^{\pm},$$

where Γ_1^- and Γ_1^+ mean the left-hand and right-hand sides of Γ_1, respectively. Based on the results in [7] the solution of the elastostatics problem can be presented as follows

$$u(x) = \sum_{\ell=1}^{2} \int_{\Gamma_\ell} \Phi(x,y)\phi_\ell(y)ds(y) + \int_{\Gamma_1} \left[T_y\Phi(x,y)\right]^\top [f_1](y)ds(y),$$

where $x \in D \setminus \Gamma_1^\pm$ and $[f_1] = f_1^+ - f_1^-$. By ν we denote the unit normal vector to Γ_1 directed towards Γ_1^+ and T is the stress tensor acting on vector function ω as

$$T\omega = \lambda \mathrm{div}\, \omega\, \nu + 2\mu(\nu \cdot \mathrm{grad})\omega + \mu \mathrm{div}(Q\omega)Q\nu,$$

where Q is a known rotational matrix $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As a result we reduce the corresponding elastostatics boundary value problem to the system

$$\sum_{\ell=1}^{2} \int_{\Gamma_\ell} \Phi(x,y)\phi_\ell(y)\, ds(y) = (\ell-1)f_2(x) + \frac{2-\ell}{2}(f_1^+(x) + f_1^-(x)) - \int_{\Gamma_1} \left[T_y\Phi(x,y)\right]^\top [f_1](y)\, ds(y) \tag{9}$$

for $x \in \Gamma_\ell$, $\ell = 1, 2$.
Here we received the integral equations with singularities analogously to the above considered case. Note that the system (9) is equivalent to (4) in the case $f_1^+ = f_1^-$.

The numerical solution of (9) can be obtained by the considered quadrature method. The convergence analysis and error estimate are analogously to Lemma 4. We need to take into account the integrals contained in the right-hand side of the system (9).

4. Numerical experiments

We shall present numerical results for two different configurations.

Example 1. As the exact solution to compare our numerical approximation with, we take $u_{ex}(x) = \Phi_1(x, y^*)$, $x \in D$, where Φ_1 is the first column of the matrix constituting the fundamental solution, and y^* is an arbitrary point which does not belong to the domain D. Let coefficients be $\lambda = 2$, $\mu = 3$ and $y^* = (5, 4)$.
Consider the domain of Fig. 2 having boundaries

$$\Gamma_1 = \{x_1(t) = (t, -\sin(0.5\pi(t+1)) + 0.5) : t \in [-1, 1]\},$$

$$\Gamma_2 = \{x_2(t) = (2.5\sin t, 2.5\cos t - \cos 2t + 1) : t \in [0, 2\pi]\}.$$

In Table 1 and Table 2 presented results of numerical approximation for vector function $u = (u_1, u_2)$ at point $x = (1, -1)$. Calculations are done for different discretization parameter n.

Example 2. For second example take a domain of Fig. 3 with boundary curves

$$\Gamma_1 = \{x_1(t) = (t, 0) : t \in [-1, 1]\},$$

$$\Gamma_2 = \{x_2(t) = (3\cos t, r(t)\sin t) : t \in [0, 2\pi]\},$$

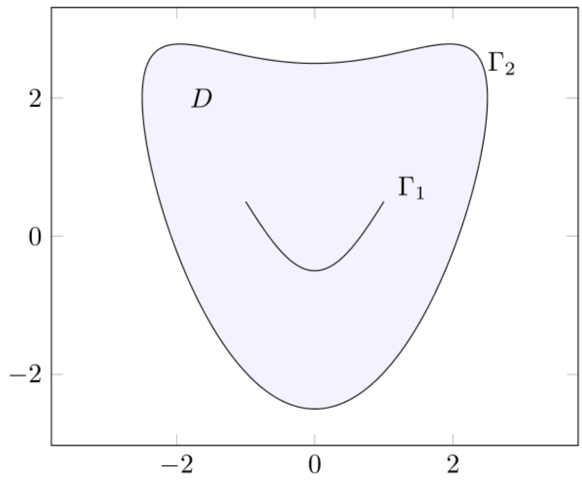

Fig. 2. Domain in Example 1

Table 1

Results for $u_1(x)$ at $x = (1, -1)$

n	Exact $u_1(x)$	Approximated $u_1(x)$
8		-0.122846939155
16		-0.122610186785
32	-0.122505716657	-0.122505933896
64		-0.122505716676
128		-0.122505716657

Table 2

Results for $u_2(x)$ at $x = (1, -1)$

n	Exact $u_2(x)$	Approximated $u_2(x)$
8		0.017280992360
16		0.016138037202
32	0.016174282834	0.016175069648
64		0.016174282791
128		0.016174282834

where

$$r(t) = 4\sqrt{0.4\sin^2 t + \cos^2 t}.$$

Let $f_l(x) = (x_1 + x_2, x_2)^\top$ for $x = (x_1, x_2) \in \Gamma_l$, $l = 1, 2$. Results of approximation at $x = (1, 2)$ are shown in Table 3.

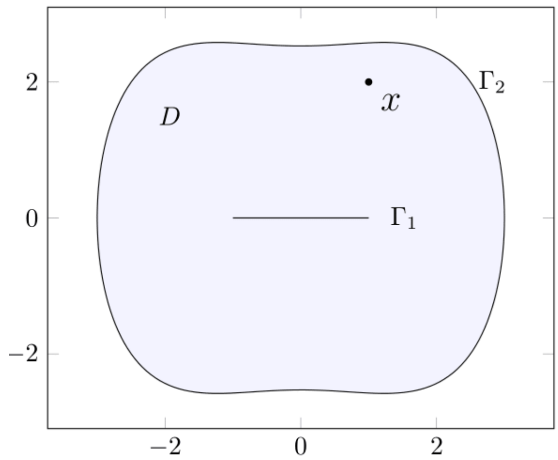

Fig. 3. Domain in Example 2

Table 3

Results for $u_1(x)$ and $u_2(x)$ at $x = (1, 2)$

n	Approximated $u_1(x)$	Approximated $u_2(x)$
8	2.791083688820	-0.087320441270
16	2.785901332784	-0.089108848765
32	2.785759500615	-0.088740333728
64	2.785759884313	-0.088740602493
128	2.785759884314	-0.088740602492

The suggested approach performs well as verified by both numerical examples.

5. Conclussion

An integral equation method based on the elastostatic single layer potential was presented for the Dirichlet problem for the case of bounded domain with a crack inside. As result, a system of boundary integral equations to be solve for two unknown vector valued densities is obtained by matching against the input data. The system has unique solution in corresponding Hölder spaces. Special care was taken to handle the singularities in the kernels and in the density. The full discretization is realized by a trigonometrical quadrature method, which belong to the methods with spectral properties. Numerical examples for two different solution domains confirmed the theoretical convergence result. The outlined method is a lightweight and flexible approach for planar elastostatic problems with a crack.

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ЧИСЕЛЬНЕ РОЗВ'ЯЗУВАННЯ ЗАДАЧІ ДІРІХЛЕ ДЛЯ РІВНЯННЯ ЕЛАСТОСТАТИКИ В ОБМЕЖЕНІЙ ОБЛАСТІ З ТРІЩИНОЮ МЕТОДОМ ІНТЕГРАЛЬНИХ РІВНЯНЬ

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Розглядаємо задачу Діріхле рівняння еластостатики в обмеженій області з тріщиною всередині. Використовуючи непрямий метод інтегральних рівнянь, цю задачу можна звести до системи інтегральних рівнянь першого роду з невідомими густинами. Густини є векторними значеннями, що визначені на границі області та на тріщині.

Цей спосіб ґрунтується на використанні еластостатичного потенціалу простого шару, де ядром є матриця фундаментального розв'язку рівняння. В подальшому, використовуючи параметричне подання граничної кривої та тріщини, отриману систему інтегральних рівнянь можна подати у параметризованому вигляді. Параметричне подання тріщини можна неперервно продовжити так, щоб область визначення збігалась із областю визначення границі області, заданої параметрично. Деякі ядра інтегралів системи місять логарифмічні особливості. Ці сингулярності можна виділити явно, у вигляді окремих доданків, застосовуючи спеціальні вагові функцій. Невідома густина, що визначена на тріщині, теж містить сингулярність, якої можна позбутися за допомогою так званої косинус-заміни. Відомо, що задача коректна у відповідних просторах Ґьольдера. Часткову дискретизацію задачі можна виконати методом квадратур з використанням тригонометричних квадратурних формул. Щоб отримати повністю дискретну систему, колокуємо отримані співвідношення у вузлах квадратурних формул. Однією з основних переваг цього методу дискретизації є те, що він має спектральні властивості. Проведено аналіз збіжності й оцінку похибки методу. Наведено приклади реалізації алгоритму для двох різних конфігурацій області та вхідних даних, із відомим і невідомим точним розв'язком, відповідно. Чисельні експерименти демонструють застосовність й ефективність запропонованого методу та підтверджують наведені апріорні оцінки похибки.

Ключові слова: еластостатика, задача Діріхле, метод інтегральних рівнянь, тригонометричні квадратури, область з тріщиною.