ПРИКЛАДНА МАТЕМАТИКА

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ON METHODS WITH SUCCESSIVE APPROXIMATION OF THE INVERSE OPERATOR FOR NONLINEAR EQUATIONS WITH DECOMPOSITION OF THE OPERATOR

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We study the problem of finding an approximate solution of a nonlinear equation with a decomposition of the operator. In particular, we consider a class of problems with a differentiable and nondifferentiable part in the nonlinear operator. Typically, difference or differential-difference methods are used for the numerical solving of these equations. We develop new one-step and two-step differential-difference methods, which contain the sum of the derivative of the differentiable part and the divided difference of the nondifferentiable part of the nonlinear operator. The one-step method is constructed on the basis of the Newton method and the Steffensen type method, and the two-step method is developed on the basis of the method with derivatives and the difference method, which have the third order of convergence. The proposed iterative processes do not require finding the inverse operator. Instead of inverting the operator, its approximations are used. For each method, a type of successive approximation of the inverse operator, which provides the convergence order as in the basis methods, is chosen. The study of local convergence of the methods under the Lipschitz condition for the divided differences of the first order and the restriction of the second derivative is carried out. Error estimates are obtained, which indicate the second and third convergence orders for one-step and two-step methods, respectively. It is shown that it is possible to obtain tighter error estimates and a wider convergence domain by introducing additional but weaker conditions. A practical study of the methods is conducted. These methods are applied for solving a large scale nonlinear system and a system with a nondifferentiable operator. A comparison with the basic methods by the number of iterations is done. The values of absolute errors are also given at each iteration. The results of numerical experiments are consistent with the theoretical results and confirm the effectiveness of the proposed methods.

Key words: nonlinear equation, differential-difference method, equation with a decomposition of operator, approximation of inverse operator, local convergence, convergence order, Lipschitz conditions.

1. Introduction

To find the approximate solution x_* of the nonlinear equation

$$F(x) = 0, \tag{1}$$

where the operator F is defined on the convex set D of a Banach space X with values in a Banach space Y, Newton's method is often used [1,4,8] for $x_0 \in D$

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad k = 0,\ 1, \ldots, \tag{2}$$

where $F'(x)$ is a Fréchet derivative. If the derivative does not exist or it is difficult to calculate, one can use methods that do not use derivatives [1,4]. For example, Steffensen type method is described by the formula

$$x_{k+1} = x_k - F(x_k, u_k)^{-1}F(x_k), \quad k = 0,\ 1, \ldots, \tag{3}$$

where $F(x, y)$ is a first-order divided difference, $u_k = x_k - \beta_k F(x_k)$, and β_k is a real parameter.

The main goal of building new methods for solving (1) is to reduce the number of calculations or simplify them, increase the order of convergence and expand their applicability. Such iterative processes include methods of third-order convergence

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - F'(x_k)^{-1}F(y_k), \quad k = 0, 1, \ldots \end{aligned} \tag{4}$$

and

$$\begin{aligned} y_k &= x_k - F(x_k, u_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - F(x_k, u_k)^{-1}F(y_k), \quad k = 0, 1, \ldots. \end{aligned} \tag{5}$$

They are two-step modifications of methods (2) and (3). The number of calculations increases insignificantly compared to one-step methods, since the inverse operator in (4) and (5) is calculated once on two steps. The two-step modification (4) was investigated in [6].

It is necessary to find one or more inverse operators at each iteration of most methods for numerical solution (1). This is not always easy to achieve. Therefore, methods can be used to approximate the inverse operator. Iterative formulas of these methods consist of several branches. Some branches intend to construct approximations to the solution of the nonlinear equation, and others to construct approximations of the inverse operator. There are two approaches to the inverse operator approximation: successive and parallel. In methods with successive approximation, calculations in separate branches are performed alternately. In methods with a parallel approximation of the inverse operator, the calculations in the separate branches of the method are performed in parallel.

Methods with approximation of the inverse operator have been studied by many authors [2,3,5,10,11,13,16–21]. Ulm [17] constructed methods with successive approximation based on Newton's method

$$\begin{aligned} y_k &= x_k - A_k F(x_k), \\ A_{k+1} &= A_k(2E - F'(x_{k+1})A_k), \quad k = 0, 1, \ldots \end{aligned} \tag{6}$$

and based on Steffensen's method

$$\begin{aligned} y_k &= x_k - A_k F(x_k), \\ A_{k+1} &= A_k(2E - F(x_{k+1}, \Phi(x_{k+1}))A_k), \quad k = 0, 1, \ldots \end{aligned} \tag{7}$$

and established a quadratic order of their convergence. Here $F(x) \equiv x - \Phi(x)$ for method (7). The third-order methods with successive approximation of the inverse operator are investigated in the works [3, 20]. Their iterative formulas have the form

$$\begin{aligned} y_k &= x_k - A_k F(x_k), \\ x_{k+1} &= y_k - A_k F(y_k), \\ B_k &= A_k(2E - F'(x_{k+1})A_k), \\ A_{k+1} &= B_k(2E - F'(x_{k+1})B_k), \quad k = 0, 1, \ldots \end{aligned} \tag{8}$$

and

$$\begin{aligned} &y_k = x_k - A_k F(x_k), \\ &x_{k+1} = y_k - A_k F(y_k), \\ &B_k = A_k(2E - F(x_{k+1}, u_{k+1})A_k), \\ &A_{k+1} = B_k(2E - F(x_{k+1}, u_{k+1})B_k), \quad k = 0, 1, \ldots \end{aligned} \tag{9}$$

where E is the identity operator, $u_{k+1} = x_{k+1} - \beta_{k+1} F(x_{k+1})$, and β_{k+1} is a real parameter.

The paper is organized as follows: in Section 2, we develop combined methods for solving nonlinear equations with decomposition of operator; in Section 3 and 4, we give local convergence theorems for the proposed methods; in Section 5, we present numerical experiments.

2. Methods for solving nonlinear equations with decomposition of operator

Consider a nonlinear equation with decomposition operator

$$H(x) \equiv F(x) + G(x) = 0. \tag{10}$$

Here F and G are defined on a convex set D of a Banach space X with values in a Banach space Y, and F is a continuously differentiable operator, G is a continuous operator. Many iterative processes [1, 7, 12, 14, 15] have been proposed and investigated for solving (10). Among them, the best are the combined differential-difference methods, which use the sum of the derivative of the differentiable part and the divided difference of the nondifferentiable part of the nonlinear operator.

In this paper, to solve (10) we propose the following combined methods with successive approximation of the inverse operator

$$\begin{aligned} &x_{k+1} = x_k - A_k H(x_k), \\ &A_{k+1} = A_k(2E - [F'(x_{k+1}) + G(x_{k+1}, u_{k+1})]A_k), \quad k = 0, 1, \ldots \end{aligned} \tag{11}$$

and

$$\begin{aligned} &y_k = x_k - A_k H(x_k), \\ &x_{k+1} = y_k - A_k H(y_k), \\ &B_k = A_k(2E - [F'(x_{k+1}) + G(x_{k+1}, u_{k+1})]A_k), \\ &A_{k+1} = B_k(2E - [F'(x_{k+1}) + G(x_{k+1}, u_{k+1})]B_k), \quad k = 0, 1, \ldots, \end{aligned} \tag{12}$$

where $u_{k+1} = x_{k+1} - \beta_{k+1} H(x_{k+1})$, and β_{k+1} is a real parameter.

In this paper, we investigate the local convergence and the convergence order of the proposed iterative processes (11) and (12). It is assumed that G is differentiable at the point x_* and we use the notation $H'(x_*) = F'(x_*) + G(x_*, x_*)$.

3. Local convergence and convergence order of the method (11)

Theorem 1. *Let F and G be nonlinear operators defined on a convex set D of a Banach space X with values in a Banach space Y, F be a continuously differentiable operator, and G be a continuous operator. Assume that:*

1) equation (10) has a solution $x_* \in D$, operator $A_* = [H'(x_*)]^{-1}$ exists and

$$\|A_*\| \le B; \tag{13}$$

2) in set $U = \{x \in D : \|x - x_*\| \le R_0\}$, the following conditions are satisfied

$$\|F'(x) + G(x, y)\| \le C, \tag{14}$$

$$\|F''(x)\| \le L, \quad \|G(x, y) - G(v, z)\| \le M(\|x - v\| + \|y - z\|), \tag{15}$$

where $R_0 = \max\{r_0, a_1(\beta C + 1)r_0^2\}$, $r_0 = \max\{\|x_0 - x_*\|, \|A_0 - A_*\|\}$, $|\beta_k| \le \beta$, $a_1 = C + (\frac{3}{2}L + M)(B + r_0)$;

3) initial approximations x_0, A_0 are such that

$$qr_0 < 1, \tag{16}$$

where $q = \max\{a_1, a_2\}$, $a_2 = C + (L + M(\beta C + 2))(B + r_0)^2 a_1$. Then, sequences $\{x_k\}$, $\{A_k\}$, $k \ge 0$, generated by method (11), converge to x_*, A_*, respectively, and the following estimate is satisfied

$$r_k = \max\{\|A_k - A_*\|, \|x_k - x_*\|\} \le (qr_0)^{2^k - 1} r_0, \quad k \ge 0. \tag{17}$$

Proof. The proof is performed by mathematical induction. It follows from

$$\|x_0 - x_*\| \le r_0 = (qr_0)^{2^0 - 1} r_0,$$

that $x_0 \in U$, and (17) is true for $k = 0$. Suppose that $x_k \in U$ and the estimate (17) is true $k \ge 0$. It follows that $r_k \le r_0$, since $qr_0 < 1$ by (16). Taking into account (13) and the definition of r_0, we get

$$\|A_k\| \le \|A_*\| + \|A_k - A_*\| \le B + r_k \le B + r_0. \tag{18}$$

We obtain from the first equality of (11) and Taylor's formula

$$\begin{aligned} x_* - x_{k+1} &= x_* - x_k + A_k(H(x_k) - H(x_*)) = x_* - x_k + A_k F'(x_k)(x_k - x_*) - \\ &\quad - A_k \int_0^1 F''(x_k + t(x_* - x_k))(x_* - x_k)^2 (1 - t) dt + \\ &\quad + A_k G(x_k, x_*)(x_k - x_*) = \\ &= -A_k \int_0^1 F''(x_k + t(x_* - x_k))(x_* - x_k)^2 (1 - t) dt + \\ &\quad + (E - A_k[F'(x_k) + G(x_k, x_*)])(x_* - x_k). \end{aligned} \tag{19}$$

By the conditions (14), (15), the estimate (18) and since

$$\begin{aligned} E - A_k[F'(x_k) + G(x_k, x_*)] &= A_k(F'(x_*) - F'(x_k)) + A_k(G(x_*, x_*) - G(x_k, x_*)) + \\ &\quad + (A_* - A_k)H'(x_*), \end{aligned}$$

we have that

$$\begin{aligned} \|E - A_k[F'(x_k) + G(x_k, x_*)]\| &\le \|A_k\|(\|F'(x_*) - F'(x_k)\| + \\ &\quad + \|G(x_*, x_*) - G(x_k, x_*)\|) + \\ &\quad + \|H'(x_*)\|\|A_* - A_k\| \le \\ &\le C\|A_* - A_k\| + (B + r_0)(L + M)\|x_* - x_k\| \le \\ &\le r_k(C + (L + M)(B + r_0)) = ar_k, \end{aligned} \tag{20}$$

where $a = C + (L + M)(B + r_0)$.

It follows from (19) and (20) that

$$\begin{aligned}
\|x_* - x_k\| &\leq \|A_k\|\|\int_0^1 F''(x_k + t(x_* - x_k))(x_* - x_k)^2(1-t)dt\| + \\
&\quad + \|E - A_k[F'(x_k) + G(x_k, x_*)]\|\|x_* - x_k\| \leq \\
&\leq (B + r_0)\frac{L}{2}\|x_* - x_k\|^2 + (C + (L + M)(B + r_0))r_k\|x_* - x_k\| \leq \\
&\leq (B + r_0)\frac{L}{2}r_k^2 + (C + (L + M)(B + r_0))r_k^2 = \\
&= (C + (\frac{3}{2}L + M)(B + r_0))r_k^2 = a_1 r_k^2. \qquad (21)
\end{aligned}$$

Thus, $\|x_* - x_{k+1}\| \leq a_1 r_0^2 \leq R_0$ and $x_{k+1} \in U$.

We obtain from the second equality (11)

$$A_* - A_{k+1} = (A_* - A_k)H'(x_*)(A_* - A_k) - A_k(H'(x_*) - F'(x_{k+1}) - G(x_{k+1}, u_{k+1}))A_k.$$

From this relationship, based on the conditions (14), (15), (21) and

$$\begin{aligned}
\|x_* - u_{k+1}\| &\leq \|x_* - x_{k+1}\| + \|x_{k+1} - u_{k+1}\| = \\
&= \|x_* - x_{k+1}\| + \|x_{k+1} - x_{k+1} + \beta_{k+1}H(x_{k+1})\| = \\
&= \|x_* - x_{k+1}\| + \|\beta_{k+1}H(x_{k+1})\|, \\
\|\beta_{k+1}H(x_{k+1})\| &\leq \beta\|[\int_0^1 F'(x_* + \theta(x_{k+1} - x_*))d\theta + G(x_{k+1}, x_*)](x_{k+1} - x_*)\| \leq \\
&\leq \beta C\|x_* - x_{k+1}\|, \\
\|x_* - u_{k+1}\| &\leq (\beta C + 1)\|x_* - x_{k+1}\|, \qquad (22)
\end{aligned}$$

we get in turn

$$\begin{aligned}
\|A_* - A_{k+1}\| &\leq \|A_k\|^2[\|F'(x_*) - F'(x_{k+1})\| + \|G(x_*, x_*) - G(x_{k+1}, u_{k+1})\|] + \\
&\quad + \|H'(x_*)\|\|A_* - A_k\|^2 \leq \\
&\leq Cr_k^2 + \|A_k\|^2[(L + M)\|x_{k+1} - x_*\| + M\|x_* - u_{k+1}\|] \leq \\
&\leq (C + (L + M(\beta C + 2))(B + r_0)^2 a_1)r_k^2 = a_2 r_k^2. \qquad (23)
\end{aligned}$$

By induction assumptions, estimates (21) and (23), we obtain

$$\begin{aligned}
r_{k+1} &= \max\{\|x_{k+1} - x_*\|, \|A_{k+1} - A_*\|\} = \max\{a_1, a_2\}r_k^2 = \\
&= qr_k^2 = q((qr_0)^{2^k - 1}r_0)^2 = (qr_0)^{2^{k+1}-1}r_0.
\end{aligned}$$

That is, (17) is fulfilled for $k + 1$.

The convergence of sequences $\{x_k\}$ and $\{A_k\}$ follows from the estimate (17) for $k \to \infty$. □

Remark 1. *It turns out that the results of Theorem 1 can be extended, and under weaker conditions. Indeed, let us consider instead of the second assumption in (15) the weaker ones*

$$\|G(x_*, x_*) - G(x, x_*)\| \leq M_0\|x - x_*\|,$$

$$\|G(x_*, x_*) - G(x, y)\| \le M_1(\|x - x_*\| + \|y - x_*\|)$$

as well as

$$\|F'(x_*) - F'(x)\| \le L_0\|x - x_*\|$$

(implied by the first assumption in (15)).

Then, we have

$$M_0 \le M_1 \le M$$

and

$$L_0 \le L.$$

Examples where the preceding two inequalities are strict can be found in [1–3].

If the proof of Theorem 1 is followed carefully we see that M_0, M_1 can replace M in the definition of a, a_1, a_2, q as

$$\begin{aligned}
\bar{a} &= C + (L_0 + M_0)(B + r_0),\\
\bar{a}_1 &= C + (\frac{3}{2}L_0 + M_0)(B + r_0),\\
\bar{a}_2 &= C + (L_0 + M_1(\beta C + 2))(B + r_0)^2\bar{a}_1,\\
\bar{q} &= \max\{\bar{a}_1,\ \bar{a}_2\}.
\end{aligned}$$

Notice also that

$$\begin{aligned}
\bar{a} &\le a,\\
\bar{a}_1 &\le a_1,\\
\bar{a}_2 &\le a_2,\\
\bar{q} &\le q,\\
\bar{r}_k &\le r_k
\end{aligned}$$

and $qr_0 < 1$ implies $\bar{q}\bar{r}_0 < 1$ but not necessarily vice versa.

Hence, the bar parameters can be used to provide a wider convergence domain (i.e. more initial points become available) and tighter error bounds (i.e. fewer iterates are needed to obtain a desired error tolerance).

4. Local convergence and convergence order of the method (12)

Theorem 2. *Let F and G be nonlinear operators defined on a convex set D of a Banach space X with values in a Banach space Y, F be a continuously differentiable operator, and G be a continuous operator. Assume that:*

1) equation (10) has a solution $x_* \in D$, operator $A_* = [H'(x_*)]^{-1}$ exists and

$$\|A_*\| \le B; \tag{24}$$

2) in set $U = \{x \in D : \|x - x_*\| \le R_0\}$, the following conditions are satisfied

$$\|F'(x) + G(x, y)\| \le C, \tag{25}$$

$$\|F''(x)\| \le L, \quad \|G(x, y) - G(v, z)\| \le M(\|x - v\| + \|y - z\|), \tag{26}$$

where $R_0 = \max\{r_0, a_1r_0^2, a_2(\beta C + 1)r_0^3\}$, $r_0 = \max\{\|x_0 - x_*\|,\ \|A_0 - A_*\|\}$, $|\beta_k| \le \beta$,

$a_1 = C + (\frac{3}{2}L + M)(B + r_0);$
$a_2 = (C + (\frac{3}{2}L + M)Ba_1r_0 + (\frac{3}{2}L + M)a_1r_0^2)a_1;$
3) initial approximations x_0, A_0 are such that

$$qr_0 < 1, \tag{27}$$

where $q = (\max\{a_2, a_3\})^{\frac{1}{2}}$, $a_3 = C\gamma^2 r_0 + (L + M(\beta C + 2))(B + \gamma r_0^2)^2 a_2$, $\gamma = C + (L + M(\beta C + 2))(B + r_0)^2 a_2 r_0$. Then, sequences $\{x_k\}$, $\{A_k\}$, $k \geq 0$, generated by method (12), converge to x_*, A_*, respectively, and the following estimate is satisfied

$$r_k = \max\{\|A_k - A_*\|, \|x_k - x_*\|\} \leq (qr_0)^{3^k - 1} r_0, \quad k \geq 0. \tag{28}$$

Proof. The proof is carried out in the similar way as for Theorem 1 but with some differences. It follows from

$$\|x_0 - x_*\| \leq r_0 = (qr_0)^{3^0 - 1} r_0,$$

that $x_0 \in U$, and (28) cs true for $k = 0$. Suppose that $x_k \in U$ and the estimate (28) is true $k \geq 0$. Hence, we have that $r_k \leq r_0$, since $qr_0 < 1$ by (27).

We obtain from the first equality of (12) and Taylor's formula

$$\begin{aligned} x_* - y_k &= x_* - x_k + A_k(H(x_k) - H(x_*)) = \\ &= -A_k \int_0^1 F''(x_k + t(x_* - x_k))(x_* - x_k)^2 (1 - t)dt + \\ &\quad + (E - A_k[F'(x_k) + G(x_k, x_*)])(x_* - x_k). \end{aligned} \tag{29}$$

It follows from (29), the conditions (25), (26) and the estimates (18) and (20) that

$$\begin{aligned} \|x_* - y_k\| &\leq (B + r_0)\frac{L}{2}\|x_* - x_k\|^2 + (C + (L + M)(B + r_0))r_k\|x_* - x_k\| \leq \\ &\leq (B + r_0)\frac{L}{2}r_k^2 + (C + (L + M)(B + r_0))r_k^2 = \\ &= (C + (\frac{3}{2}L + M)(B + r_0))r_k^2 = a_1 r_k^2. \end{aligned} \tag{30}$$

Thus, we get $\|x_* - y_k\| \leq a_1 r_0^2 \leq R_0$ and $y_k \in U$.

We obtain from the second equality of (12) and Taylor's formula

$$\begin{aligned} x_{k+1} - x_* &= y_k - x_* - A_k(H(y_k) - H(x_*)) = \\ &= A_k \int_0^1 F''(y_k + t(x_* - y_k))(x_* - y_k)^2 (1 - t)dt + \\ &\quad + (E - A_k[F'(y_k) + G(y_k, x_*)])(y_k - x_*). \end{aligned} \tag{31}$$

Hence, we have, given

$$\begin{aligned} E - A_k[F'(y_k) + G(y_k, x_*)] &= A_k(F'(x_*) - F'(y_k)) + \\ &\quad + A_k(G(x_*, x_*) - G(y_k, x_*)) + (A_* - A_k)H'(x_*), \end{aligned}$$

the conditions (24)–(26) and the estimate (30),

$$\begin{aligned}\|x_{k+1}-x_*\| &\le (B+\|A_*-A_k\|)\frac{L}{2}\|x_*-y_k\|^2+\\ &+\big(C\|A_*-A_k\|+(B+\|A_*-A_k\|)(L+M)\|x_*-y_k\|\big)\|x_*-y_k\|\le\\ &\le C\|A_*-A_k\|\|x_*-y_k\|+\frac{3}{2}BL\|x_*-y_k\|^2+\\ &+\frac{3}{2}L\|x_*-y_k\|^2\|A_*-A_k\|+\\ &+BM\|x_*-y_k\|^2+M\|x_*-y_k\|^2\|A_*-A_k\|\le\\ &\le \|x_*-y_k\|(Cr_k+(\frac{3}{2}L+M)B\|x_*-y_k\|+(\frac{3}{2}L+M)r_k\|x_*-y_k\|)\le\\ &\le (C+(\frac{3}{2}L+M)Ba_1r_0+(\frac{3}{2}L+M)a_1r_0^2)a_1r_k^3=a_2r_k^3. \end{aligned}\tag{32}$$

Thus, we derive $x_{k+1}\in U$, since $\|x_{k+1}-x_*\|\le a_2r_0^3\le R_0$.

On the other hand, based on the third formula of (12)

$$A_*-B_k=(A_*-A_k)H'(x_*)(A_*-A_k)-A_k(H'(x_*)-F'(x_{k+1})-G(x_{k+1},u_{k+1}))A_k.$$

Hence, we have, given (18), (22), (25), (26) and (32),

$$\begin{aligned}\|A_*-B_k\| &\le \|A_k\|^2[\|F'(x_*)-F'(x_{k+1})\|+\|G(x_*,x_*)-G(x_{k+1},u_{k+1})\|]+\\ &+\|H'(x_*)\|\|A_*-A_k\|^2\le\\ &\le Cr_k^2+\|A_k\|^2[(L+M)\|x_{k+1}-x_*\|+M\|x_*-u_{k+1}\|]\le\\ &\le (C+(L+M(\beta C+2))(B+r_0)^2a_2r_0)r_k^2=\gamma r_k^2.\end{aligned}\tag{33}$$

Given (24) and the estimate (33), we have

$$\|B_k\|\le\|A_*\|+\|A_*-B_k\|\le B+\gamma r_k^2.$$

In accordance with the fourth equality of (12)

$$A_*-A_{k+1}=-B_k(H'(x_*)-F'(x_{k+1})-G(x_{k+1},u_{k+1}))B_k+(A_*-B_k)H'(x_*)(A_*-B_k).$$

We get in turn from this relationship, based on the conditions (25), (26) and estimates (32), (33),

$$\begin{aligned}\|A_*-A_{k+1}\| &\le \|B_k\|^2[\|F'(x_*)-F'(x_{k+1})\|+\|G(x_*,x_*)-G(x_{k+1},u_{k+1})\|]+\\ &+\|H'(x_*)\|\|A_*-B_k\|^2\le\\ &\le C\gamma^2r_k^4+\|B_k\|^2[(L+M)\|x_{k+1}-x_*\|+M\|x_*-u_{k+1}\|]\le\\ &\le (C\gamma^2r_0+(L+M(\beta C+2))(B+\gamma r_0^2)^2a_2)r_k^3=a_3r_k^3.\end{aligned}\tag{34}$$

Given induction assumptions, based on the estimates (32) and (34), we obtain

$$\begin{aligned}r_{k+1} &= \max\{\|x_{k+1}-x_*\|,\|A_{k+1}-A_*\|\}=\max\{a_2,a_3\}r_k^3=\\ &= q^2r_k^3=q^2((qr_0)^{3^k-1}r_0)^3=(qr_0)^{3^{k+1}-1}r_0.\end{aligned}$$

That is, (28) is fulfilled for $k+1$. It follows from the estimate (28) for $k\to\infty$ the convergence of sequences $\{x_k\}$ and $\{A_k\}$. □

Comments similar to the ones for method (11) can follow but holding for method (12).

5. Numerical experiments

We present here the results obtained by differential-difference methods (11) and (12). First of all, we compare these iterative processes with Newton's method. For this, let us consider a system of nonlinear equations from [9], where

$$F(x) = \begin{cases} 3x_1^3 + 2x_2 - 5, \\ 3x_i^3 + 4x_i + 2x_{i+1} - 8, \quad 1 < i < n, \\ 4x_n - 3, \end{cases}$$

$$G(x) = \begin{cases} \sin(x_1 - x_2)\sin(x_1 + x_2), \\ \sin(x_i - x_{i+1})\sin(x_i + x_{i+1}) - x_{i-1}\exp(x_{i-1} - x_i), \quad 1 < i < n, \\ -x_{n-1}\exp(x_{n-1} - x_n). \end{cases}$$

Since H is differentiable, Newton's method can be applied. The iterative processes were stopped when the conditions are verified

$$\|x_{k+1} - x_k\| \le 10^{-10} \quad \text{and} \quad \|H(x_{k+1})\| \le 10^{-10}.$$

We will compare the methods by the number of iterations required to obtain an approximate solution and the error values. The initial approximations were calculated by the formulas $x_0 = 2s$, $s \in \mathbb{R}$, $A_0 = [F'(x_0) + G(x_0, x_0 - \beta_0 H(x_0))]^{-1}$.

Tables 1 and 2 show the results for $n = 20$ and $\beta_k = 0.0001$.

Table 1

The number of iterations

s	**0.45**	**1**	**2**	**5**	**10**
Newton's method	5	7	8	11	12
Steffensen's method	5	7	8	11	12
Method (11)	5	8	11	15	18
Method (12)	4	5	7	9	10

Table 2

The value of error $\|x_k - x_*\|$ at each iteration for $s = 0.53$

k	**Newton's**	**Steffensen's**	**(11)**	**(12)**
1	2.8317e-03	2.8285e-03	2.8316e-03	2.5960e-04
2	6.7350e-06	6.7129e-06	2.9429e-05	4.0289e-11
3	3.8629e-11	3.8261e-11	5.5721e-09	0
4	0	1.1102e-16	1.1546e-14	
5			3.3307e-15	

Next, consider the problem with a nondifferentiable operator. Let

$$F(x) = \begin{cases} x_1^3 - x_2 + 1, \\ x_1 + x_2^2 - 7, \end{cases} \qquad G(x) = \begin{cases} \frac{|x_1^2 - 1|}{9}, \\ \frac{|x_1 x_2 - 2|}{9}. \end{cases}$$

Initial approximations were calculated by formulas $x_0 = (1, 2.5)^T s$, $s \in \mathbb{R}$, $A_0 = [F'(x_0) + G(x_0, x_0 - \beta_0 H(x_0))]^{-1}$. Table 3 gives the results for $\beta_k = 0.01$.

Table 3

The number of iterations

s	**1**	**2**	**5**	**10**	**20**
Steffensen's method	5	7	8	8	10
Method (11)	6	8	12	15	18
Method (12)	4	5	7	9	10

We see that the two-step methods converge faster than the corresponding one-step iterative processes and Newton's method. In addition, the errors $\|x_k - x_*\|$ decrease faster, which is consistent with the theoretical results. The methods with a successive approximation of the inverse operator are somewhat inferior to the basic methods, but their advantage is the simplicity of calculations in iterative formulas.

6. Conclusions

In this paper, we propose two methods with successive approximation of the inverse operator for solving nonlinear operator equations. The local convergence of methods is investigated, the convergence orders are determined and numerical results are given.

References

1. *Argyros I. K.* Convergence and Applications of Newton's-type Iterations / I. K. Argyros. – New-York: Springer-Verlag, 2008. – 506 p.
2. *Argyros I. K.* On Ulm's method for Fréchet differentiable operators / I. K. Argyros // J. Appl. Math. Comput. – 2005. – Vol. 31. – P. 97–111.
3. *Argyros I. K.* Method of Third-Order Convergence with Approximation of Inverse Operator for Large Scale Systems / I. K. Argyros, S. Shakhno, H. Yarmola // Symmetry. – 2020. – Vol. 12, 978.
4. Dennis J. E. Numerical Methods for Unconstrained Optimization and Nonlinear Equations / J. E. Dennis, R. B. Schnabel. – Philadelphia: SIAM, 1996.
5. *Esquerro J. A.* An Ulm-type method with R-order of convergence three / J. A. Esquerro, M. A. Hernández // Nonlinear Analysis: Real World Applications. – 2012. – Vol. 12. – P. 14–26.
6. *Esquerro J. A.* A discretization scheme for some conservative problems / J. A. Ezquerro, M. A. Hernńdez, M. A. Salanova // J. Comp. Appl. Math. – 2000. – Vol. 115. – P. 181–192.
7. *Hernandez M. A.* A uniparametric family of iterative processes for solving nondifferentiable operators / M. A. Hernandez, M. J. Rubio // J. Math. Anal. Appl. – 2002. – Vol. 275. – P. 821–834.
8. *Kantorovich L. V.* Functional Analysis / L. V. Kantorovich, G. P. Akilov. – Moscow: Nauka, 1977. – 744 p. (In Russian).
9. *Lukšan L.* Inexact Trust Region Method for Large Sparse Systems of Nonlinear Equations / L. Lukšan // Journal of Optimization Theory and Applications. – 1994. – Vol. 81. – P. 569–590.
10. *Rooze A. F.* An iterative method for solving nonlinear equations using parallel inverse operator approximation / A. F. Rooze // Izv. Acad. Nauk Est. SSR. Fiz. Mat. – 1982. – Vol. 31, № 1. – P. 32–37. (In Russian).

11. *Rooze A. F.* An asynchronous iteration method of solving non-linear equations using parallel approximation of an inverse operator / A. F. Rooze // USSR Comput. Math. Math. Phys. – 1986. – V 26, № 8. – P. 1250–1254.
12. *Shakhno S. M.* Combined Newton's-Potra method for solving nonlinear operator equations / S. M. Shakhno, A.-V. I. Babjak, H. P. Yarmola // Journal of Computational and Applied Mathematics. – 2015. – № 3 (120). – P. 170–178. (In Ukrainian).
13. *Shakhno S. M.* Convergence of one iterative method with successive approximation of the inverse operator / S. M. Shakhno // Visnyk of Lviv. Univ. Ser. mech.-math. Problems of applied mathematics and mechanics. – 1989. – V. 31. – P. 62–66. (In Ukrainian).
14. *Shakhno S. M.* On the two-step method for solving nonlinear equations with nondifferentiable operator / S. M. Shakhno, H. P. Yarmola // PAMM – Proc. Appl. Math. Mech. – 2012. – Vol. 12. – P. 617–618. – doi 10.1002/pamm.201210297.
15. *Shakhno S. M.* Two-point method for solving nonlinear equation with nondifferentiable operator / S. M. Shakhno, H. P. Yarmola // Matematchni Studii. – 2011. – Vol. 36, №2. – P. 213–220. (In Ukrainian).
16. *Shakhno S. M.* Two-step secant type method with approximation of the inverse operator / S. M. Shakhno, H. P. Yarmola // Carpathian Math. Publ. – 2013. – Vol. 5, № 1. – P. 150–155. (In Ukrainian).
17. *Ulm S.* On iterative methods with successive approximation of the inverse operator / S. Ulm // Izv. Acad. Nauk Est. SSR. Fiz. Mat. – 1967. – Vol. 6., № 4. – P. 403–411. (In Russian).
18. *Vaarmann O.* On some iterative methods with successive approximation of the inverse operator I / O. Vaarmann // Izv. Acad. Nauk Est. SSR. Fiz. Mat. – 1968. – Vol. 17, № 4. – P. 379–387. (In Russian).
19. *Vaarmann O.* On some iterative methods with successive approximation of the inverse operator II / O. Vaarmann // Izv. Acad. Nauk Est. SSR. Fiz. Mat. – 1969. – Vol. 18, № 1. – P. 14–21. (In Russian).
20. *Vasilyev F. P.* On high-order methods for solving operator equations / F. P. Vasilyev, L. M. Khromova // Dokl. Nauk SSSR. – 1983. – Vol. 270. – P. 28-31. (In Russian).
21. *Verzhbitskiy V. M.* Iterative methods with successive approximation of inverse operators / V. M. Verzhbitskiy // Izv. IMI UdGU. – 2006. – № 2 (36). – P. 129–138. (In Russian).

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ПРО МЕТОДИ З ПОСЛІДОВНОЮ АПРОКСИМАЦІЄЮ ОБЕРНЕНОГО ОПЕРАТОРА ДЛЯ НЕЛІНІЙНИХ РІВНЯНЬ З ДЕКОМПОЗИЦІЄЮ ОПЕРАТОРА

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Розглянуто задачу знаходження наближеного розв'язку нелінійного рівняння з декомпозицією оператора. Зокрема, розглядаємо задачі, для яких в нелінійному операторі можна виділити диференційовну та недиференційовну частини. Зазвичай для чисельного розв'язування цих рівнянь застосовують різницеві або диференціально-різницеві методи. Побудовано нові однокроковий і двокроковий диференціально-різницеві методи, які містять суму похідної від диференційовної частини та поділеної різниці від недиференційовної частини нелінійного оператора. Однокроковий метод розроблено на підставі методу Ньютона та методу типу Стеффенсена, а двокроковий – на підставі методу з похідними та різницевого методу, які мають третій порядок збіжності. Також запропоновані ітераційні процеси не потребують знаходження оберненого оператора. Замість обертання оператора використовується його апроксимація. Для кожного методу обрано такий тип послідовної апроксимації оберненого оператора, який забезпечує порядок збіжності як у базових методів. Проведено дослідження локальної збіжності методів за умови Ліпшиця для поділених різниць першого порядку й обмеженості другої похідної. Отримано оцінки похибок, які вказують на другий і третій порядки збіжності для однокрокового та двокрокового методів, відповідно. Доведено, що можна отримати точніші оцінки похибок та більшу область збіжності, ввівши додаткові, але слабші умови. Проведено практичне дослідження методів. Їх застосовано до розв'язування нелінійної системи великої розмірності та системи з недиференційовним оператором. Виконано порівняння з базовими методами за кількістю ітерацій. Також наведено значення абсолютних похибок на кожній ітерації. Результати чисельних експериментів узгоджуються з теоретичними результатами та підтверджують ефективність запропонованих методів.

Ключові слова: нелінійне рівняння, диференціально-різницевий метод, рівняння з декомпозицією оператора, апроксимація оберненого оператора, локальна збіжність, порядок збіжності, умови Ліпшиця.