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**AN EXACT FINITE ELEMENT SCHEME OF THE  
BOUNDARY VALUE PROBLEM FOR AN ORDINARY  
DIFFERENTIAL EQUATION****V. Verbitskyi<sup>1</sup>, A. Loktev<sup>2</sup>***Odessa, I. I. Mechnikov National University,**Dvoryanskaya str., 2, Odessa, 65082,**e-mail:*<sup>1</sup>[v.verbitskyi@onu.edu.ua](mailto:v.verbitskyi@onu.edu.ua), <sup>2</sup>[anton.loktev@stud.onu.edu.ua](mailto:anton.loktev@stud.onu.edu.ua)

A scheme of the finite element method for solving a boundary value problem on a finite interval for an ordinary differential equation of the second order is proposed. The differential equation contains a nonnegative piecewise constant coefficient of the unknown function. For the boundary value problem, the corresponding continuous and finite-dimensional variational problems are constructed. Since the bilinear form of these problems is continuous and positive definite, the variational problems are uniquely solvable. Basic functions of a finite-dimensional subspace of a finite-dimensional variational problem are constructed using solutions of the Dirichlet problem for the corresponding homogeneous differential equation on a finite element. If all the points of discontinuity of the coefficient of a differential equation are contained among the mesh nodes, such solutions are easy to construct, because the differential equation will have constant coefficients on a finite element. Trial and test functions of the finite element method are selected from one finite-dimensional subspace. The matrix of the finite element approximation of the boundary value problem is tridiagonal symmetric and positive definite. The scheme is exact in the sense that the finite element and exact solutions coincide at the mesh nodes. The result can be regarded as an illustration of the phenomenon of superconvergence of the finite element method, which consists in the fact that in a given area there can be points for which the difference between the exact and finite element solutions (or their derivatives) converges to zero faster than some norm of the difference of these solutions. As an example, it is shown that the proposed scheme of the finite element method allows one to construct an analytical solution to a boundary value problem on a finite interval for an ordinary differential equation with constant coefficients if the solution of the Dirichlet problem for the corresponding homogeneous differential equation is known.

*Key words:* finite element method, superconvergence, exact finite element scheme.

**1. INTRODUCTION**

The research on superconvergence for finite element methods has been conducted extensively by many researchers over the last four decades. This phenomenon was first addressed in [17], and the term “superconvergence” was first used in [11]. Since then, superconvergence has become to be an active research topic in finite element methods for partial differential equations; see [1, 2, 6–10, 12–14, 19–21] and the references cited therein for an incomplete list of publications.

The phenomenon of superconvergence is that in the considered region there can be points  $x_i$  for which the difference  $u(x_i) - u_h(x_i)$  of exact  $u(x)$  and finite element  $u_h(x)$  solutions (or  $\nabla u(x_i) - \nabla u_h(x_i)$ ) tends to zero faster than norm of this difference.

At the same time, of great interest are cases where at some points the equality of the approximate and exact solutions is observed. For example, the finite element solution of the Dirichlet problem for the equation  $-u'' = f$  constructed using continuous linear splines coincides at the grid nodes with the values of the exact solution. If we use cubic

basic splines, then this phenomenon is no longer observed. On the contrary, it is possible to change not the space of finite elements, but the equation. For example, for the one-dimensional Helmholtz equation, the indicated values can differ quite significantly [5]. The so-called generalized FEM (or GFEM) was proposed in [3, 4, 15], the idea of which is to take into account the structure of a particular equation when constructing a finite-dimensional space. Regarding superconvergence for this method, some fairly general results were obtained in [2], where a theoretical study of the inhomogeneous Neumann problem for the Poisson equation in the two-dimensional domain is given, and the results are illustrated by numerical experiments on the one-dimensional problem. In [18] for one class of the generalized finite element method it is proved that the approximate solution of the boundary value problem for an ordinary differential equation is coincided with the exact solution at the grid nodes. In the proposed finite element scheme the spaces of test and verification functions are different.

We proposed to choose test and verification functions from one finite-dimensional space. For a boundary value problem with piecewise constant coefficients, the existence of a finite element solution and its coincidence at the grid nodes with an exact solution is proved.

## 2. FINITE-ELEMENT APPROXIMATION OF A BOUNDARY VALUE PROBLEM

Let's consider the Dirichlet problem,

$$\begin{aligned} Lu \equiv -u'' + q(x)u &= f, & x \in (0, 1), \\ u(0) = u(1) &= 0, \end{aligned} \quad (1)$$

where  $f \in L^2(0; 1)$ ,  $q(x) \geq 0$  is a piecewise constant function. The weak form for the problem (1) seeks  $u \in H_0^1(0; 1)$  such that

$$a(u, v) \equiv \int_0^1 [u'v' + q(x)uv] dx = \int_0^1 f v dx \quad \forall v \in H_0^1(0; 1). \quad (2)$$

On the interval  $(0; 1)$  we define a grid  $\omega_h$  with nodes

$$0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1.$$

We assume that among the grid nodes there are all discontinuity points of the function  $q(x)$ . On the interval  $(x_{i-1}; x_i)$  ( $i = \overline{1, n}$ ) we consider boundary value problems

$$L\varphi_i^{(1)}(x) = 0, \quad \varphi_i^{(1)}(x_{i-1}) = 1, \quad \varphi_i^{(1)}(x_i) = 0, \quad (3)$$

$$L\varphi_i^{(2)}(x) = 0, \quad \varphi_i^{(2)}(x_{i-1}) = 0, \quad \varphi_i^{(2)}(x_i) = 1. \quad (4)$$

Since the function  $q(x)$  is a constant on the interval  $(x_{i-1}; x_i)$ , it is easy to find solutions of problems (3), (4):

$$\varphi_i^{(1)}(x) = \frac{sh(k_i(x_i - x))}{sh(k_i h_i)}, \quad \varphi_i^{(2)}(x) = \frac{sh(k_i(x - x_{i-1}))}{sh(k_i h_i)},$$

where  $q(x) \equiv k_i^2$  on  $(x_{i-1}, x_i)$ ,  $h_i = x_i - x_{i-1}$  and  $sh(x) = (e^x - e^{-x})/2$  is hyperbolic sinus.

With the node  $x_i$  we associate a continuous function

$$\varphi_i(x) = \begin{cases} \varphi_i^{(2)}(x), & x \in (x_{i-1}; x_i), \\ \varphi_{i+1}^{(1)}(x), & x \in (x_i; x_{i+1}), \\ 0, & x \in (0; 1) \setminus (x_{i-1}; x_{i+1}). \end{cases}$$

We define a finite-dimensional subspace  $V_h \subset H_0^1(0; 1)$  with basis  $\{\varphi_i(x)\}_{i=1}^{n-1}$ . With variational problem (2) we associate a finite-dimensional variational problem which searches for a function  $u_h \in V_h$  such that

$$a(u_h, v_h) = \int_0^1 f v_h dx \quad \forall v_h \in V_h. \tag{5}$$

Since the bilinear form  $a(\cdot, \cdot)$  is continuous on  $H_0^1(0; 1)$  and  $H_0^1(0; 1)$ -elliptical, it follows from the Lax–Milgram lemma (see, for example, [16, p. 536]) that problems (2) and (5) are uniquely solvable.

The solution of problem (5) will be sought in the form

$$u_h(x) = \sum_{i=1}^{n-1} \alpha_i \varphi_i(x),$$

where  $\alpha_i = u_h(x_i)$ . Then problem (5) is a system of linear algebraic equations (SLAE) with unknowns  $\alpha_i$ :

$$\sum_{i=1}^{n-1} \alpha_i a(\varphi_i, \varphi_j) = \int_0^1 f \varphi_j(x) dx, \quad j = \overline{1, n-1}. \tag{6}$$

Performing integration by parts and taking into account (3) and (4), we obtain

$$\begin{aligned} & \sum_{i=1}^{n-1} \alpha_i a(\varphi_i, \varphi_j) = \\ & = \alpha_{j-1} a(\varphi_{j-1}, \varphi_j) + \alpha_j a(\varphi_j, \varphi_j) + \alpha_{j+1} a(\varphi_{j+1}, \varphi_j) = \\ & = \alpha_{j-1} \int_{x_{j-1}}^{x_j} \varphi_j^{(1)'}(x) \varphi_j^{(2)'}(x) + q(x) \varphi_j^{(1)}(x) \varphi_j^{(2)}(x) dx + \\ & + \alpha_j \left\{ \int_{x_{j-1}}^{x_j} \varphi_j^{(2)'}(x) \varphi_j^{(2)'}(x) + q(x) \varphi_j^{(2)}(x) \varphi_j^{(2)}(x) dx + \right. \\ & \left. + \int_{x_j}^{x_{j+1}} \varphi_{j+1}^{(1)'}(x) \varphi_{j+1}^{(1)'}(x) + q(x) \varphi_{j+1}^{(1)}(x) \varphi_{j+1}^{(1)}(x) dx \right\} + \\ & + \alpha_{j+1} \int_{x_j}^{x_{j+1}} \varphi_{j+1}^{(2)'}(x) \varphi_{j+1}^{(1)'}(x) + q(x) \varphi_{j+1}^{(2)}(x) \varphi_{j+1}^{(1)}(x) dx = \\ & = \alpha_{j-1} \varphi_j^{(1)}(x) \varphi_j^{(2)'}(x) \Big|_{x_{j-1}}^{x_j} + \\ & + \alpha_j \left\{ \varphi_j^{(2)}(x) \varphi_j^{(2)'}(x) \Big|_{x_{j-1}}^{x_j} + \varphi_{j+1}^{(1)}(x) \varphi_{j+1}^{(1)'}(x) \Big|_{x_j}^{x_{j+1}} \right\} + \\ & + \alpha_{j+1} \varphi_{j+1}^{(2)}(x) \varphi_{j+1}^{(1)'}(x) \Big|_{x_j}^{x_{j+1}} \end{aligned}$$

Thus, SLAE (6) takes the form:

$$\frac{-k_j}{sh(k_j h_j)} \alpha_{j-1} + \left\{ \frac{k_j ch(k_j h_j)}{sh(k_j h_j)} + \frac{k_{j+1} ch(k_{j+1} h_{j+1})}{sh(k_{j+1} h_{j+1})} \right\} \alpha_j + \frac{-k_{j+1}}{sh(k_{j+1} h_{j+1})} \alpha_{j+1} = f_j, \quad j = \overline{1, n-1}, \quad (7)$$

where  $\alpha_0 = \alpha_n = 0$ ,  $f_j = \int_{x_{j-1}}^{x_{j+1}} f \varphi_j(x) dx$ .

**Theorem 1.** *The finite element solution  $u_h(x)$  coincides with the solution  $u(x)$  of problem (2) at the nodes of the grid  $\omega_h$ .*

*Proof.* Note that  $u_h(x_0) = u(x_0) = 0$ ,  $u_h(x_n) = u(x_n) = 0$  by the construction of the finite element scheme. It follows from (2) and (5) that

$$a(u - u_h, \varphi_j) = 0, \quad j = \overline{1, n-1}. \quad (8)$$

Integrating by parts, we obtain

$$\begin{aligned} a(u - u_h, \varphi_j) &= \int_{x_{j-1}}^{x_j} (u(x) - u_h(x))' \varphi_j^{(2)'}(x) + q(x)(u(x) - u_h(x))' \varphi_j^{(2)'}(x) dx + \\ &+ \int_{x_j}^{x_{j+1}} (u(x) - u_h(x)) \varphi_{j+1}^{(1)'}(x) + q(x)(u(x) - u_h(x))' \varphi_{j+1}^{(1)'}(x) dx = \\ &= (u(x) - u_h(x)) \varphi_j^{(2)'}(x) \Big|_{x_{j-1}}^{x_j} + (u(x) - u_h(x)) \varphi_{j+1}^{(1)'}(x) \Big|_{x_j}^{x_{j+1}} = \\ &= -(u(x_{j-1}) - u_h(x_{j-1})) \varphi_j^{(2)'}(x_{j-1}) + \\ &+ (u(x_j) - u_h(x_j)) (\varphi_j^{(2)'}(x_j) - \varphi_{j+1}^{(1)'}(x_j)) + \\ &+ (u(x_{j+1}) - u_h(x_{j+1})) \varphi_{j+1}^{(1)'}(x_{j+1}) = \\ &= \frac{-k_j}{sh(k_j h_j)} (u(x_{j-1}) - u_h(x_{j-1})) + \\ &+ \left\{ \frac{k_j ch(k_j h_j)}{sh(k_j h_j)} + \frac{k_{j+1} ch(k_{j+1} h_{j+1})}{sh(k_{j+1} h_{j+1})} \right\} (u(x_j) - u_h(x_j)) + \\ &+ \frac{-k_{j+1}}{sh(k_{j+1} h_{j+1})} (u(x_{j+1}) - u_h(x_{j+1})). \end{aligned}$$

Therefore, system (8) is homogeneous SLAE with the matrix of system (6). Thus, system (8) has the unique trivial solution  $u(x_j) - u_h(x_j) = 0$  ( $j = \overline{1, n-1}$ ).  $\square$

### 3. EXAMPLE

Let's consider the boundary value problem,

$$\begin{aligned} -u''(x) + k^2 u(x) &= f(x), \quad x \in (0; 1), \\ u(0) = u(1) &= 0, \end{aligned} \quad (9)$$

where  $f(x) \in L^2(0; 1)$ . If  $f(x) \equiv 1$ , then the solution of the boundary value problem is the function

$$u(x) = \frac{(e^{-k} - 1)e^{kx} + (1 - e^k)e^{-kx} + e^k - e^{-k}}{k^2(e^k - e^{-k})}. \quad (10)$$

We solve problem (9) by the finite element method using the grid  $\omega_h = \{0, x, 1\}$ , where  $x$  is an arbitrary point in the interval  $(0; 1)$ . Then

$$u_h(t) = \alpha_1 \varphi_1(t),$$

where

$$\varphi_1(t) = \begin{cases} \frac{sh(kt)}{sh(kx)}, & t \in (0; x), \\ \frac{sh(k-kt)}{sh(k-kx)}, & t \in (x; 1), \end{cases}$$

and the coefficient  $\alpha_1$  is determined from the equation

$$\alpha_1 a_{11} = f_1,$$

where, according to (7),

$$a_{11} = \frac{k ch(kx)}{sh(kx)} + \frac{k ch(k-kx)}{sh(k-kx)},$$

$$f_1 = \int_0^x f(t) \frac{sh(kt)}{sh(kx)} dt + \int_x^1 f(t) \frac{sh(k-kt)}{sh(k-kx)} dt.$$

Moreover,  $\alpha_1 = f_1/a_{11}$  is the value of the exact solution of the boundary value problem at an arbitrary point  $x$ . Thus, we have established the formula for calculating the exact solution of the boundary value problem (9):

$$u(x) = \frac{f_1}{a_{11}}. \quad (11)$$

Indeed, it is easy to verify that if  $f(x) \equiv 1$ , then

$$f_1 = \frac{ch(kx) - 1}{k sh(kx)} + \frac{ch(k-kx) - 1}{k sh(k-kx)},$$

and solution (11) coincides with (10).

#### 4. CONCLUSION

A scheme of the finite element method for solving a boundary value problem for an ordinary differential equation with piecewise constant coefficients is proposed. The basic functions of finite-dimensional subspaces are constructed using solutions of the Dirichlet problem for a homogeneous differential equation on a finite element. Those solutions are easy to construct if all the points of discontinuity of the equation coefficients are contained among the grid nodes. The scheme is exact in the sense that the finite element solution coincides with the exact solution at the nodes of the grid used to construct the finite element subspace. The result can be considered as an illustration of the phenomenon of superconvergence of the finite element method.

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## ТОЧНА СКІНЧЕННО–ЕЛЕМЕНТНА АПРОКСИМАЦІЯ КРАЙОВОЇ ЗАДАЧІ ДЛЯ ЗВИЧАЙНОГО ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ

В. Вербіцький<sup>1</sup>, А. Локтев<sup>2</sup>

*Одеський національний університет імені І. І. Мечникова,  
вул. Дворянська, 2, Одеса, 65082,  
e-mail:<sup>1</sup>[v.verbitskyi@onu.edu.ua](mailto:v.verbitskyi@onu.edu.ua), <sup>2</sup>[anton.loktev@stud.onu.edu.ua](mailto:anton.loktev@stud.onu.edu.ua)*

Запропоновано схему методу скінченних елементів розв'язання крайової задачі на скінченному проміжку для звичайного диференціального рівняння другого порядку. Диференціальне рівняння містить невід'ємний кусково-постійний коефіцієнт у невідомій функції. Для крайової задачі побудовано відповідні неперервна та скінченновимірна варіаційні задачі. Оскільки білінійна форма цих задач є неперервною та додатно визначеною, то варіаційні задачі однозначно розв'язні. Базисні функції скінченновимірного підпростору скінченновимірної варіаційної задачі будуються з використанням рішень задачі Діріхле для відповідного однорідного диференціального рівняння на скінченному елементі. Якщо серед вузлів сітки містяться всі точки розриву коефіцієнта диференціального рівняння, то такі рішення легко побудувати, бо на скінченному елементі диференціальне рівняння матиме постійні коефіцієнти. Пробні та перевірочні функції методу скінченних елементів вибирають з одного скінченновимірного підпростору. Матриця скінченно-елементної апроксимації крайової задачі є тридіагональною симетричною та додатно визначеною. Схема точна в тому сенсі, що скінченно-елементний і точний розв'язки збігаються в вузлах сітки. Результат можна розглядати як ілюстрацію явища суперзбіжності методу скінченних елементів, яке полягає в тому, що в цій області можуть знайтися точки, для яких різниця точного і скінченно-елементного розв'язків, (або їхніх похідних) збігаються до нуля швидше, ніж деяка норма різниці цих рішень. Як приклад наведено, що запропонована схема методу скінченних елементів дає змогу побудувати аналітичний розв'язок крайової задачі на скінченному проміжку для звичайного диференціального рівняння з постійними коефіцієнтами, якщо відомо рішення задачі Діріхле для відповідного однорідного диференціального рівняння.

*Ключові слова:* метод скінченних елементів, суперзбіжність, точна скінченно-елементна схема.