UDC 519.21

CONVERGENCE OF THE STOCHASTIC APPROXIMATION PROCEDURE TO THE ORNSTEIN-UHLENBECK PROCESS

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The main asymptotic property of the continuous stochastic approximation procedure, namely the convergence to the Ornstein-Uhlenbeck process, is considered. The regression function of the procedure depends on a uniformly ergonomic Markov process, which describes the external influence in the form of switches. To obtain sufficient conditions for the convergence of the procedure, we use ergodic properties of the generating Markov process generator, the existence of a stationary distribution of this process, and the existence of the potential for a generating Markov process generator. The stochastic approximation procedure, as a random process, is constructed in the form of a differential equation in the fixed state of a Markov process with a corresponding generator of the equation. Asymptoticity over time is achieved by using a small parameter that normalizes time. This made it possible to obtain a normalized stochastic approximation procedure and its differential representation. The generator of the obtained differential equation is used to construct a generator of a two-component Markov process, which consists of a procedure and a switching process. The singular representation of the last generator by a small parameter makes it possible to solve the singular perturbation problem and determine the form of the limited generator. Such form defines the representation of a limited process as the Ornstein-Uhlenbeck diffusion process. Note that the convergence to the limited process is weak, which follows from the Koroliuk's theorem. The conditions for the existence of the Lyapunov function for the dynamic procedure, which is averaged by the stationary distribution of the Markov process, are important. Additional conditions on the Lyapunov function make it possible to establish the boundedness of the residual terms of the solution of a singular perturbation. The one-dimensional case of the procedure can be expanded to multidimensional with the corresponding complication of calculations of the components of the limited generator. The work summarizes the studies of Nevelson and Khasminsky in the case of a direct influence of the Markov process on the regression function.

Key words: Stochastic approximation procedure, Markov process, Ornstein-Uhlenbeck process, Weak convergence.

1. Introduction

Dynamic random evolutions are discussed in [1], as a dynamic system with the action of random forces which asymptotically transform into "white noise". In this case limit theorem about the convergence of differential equation solutions with a random right side to the diffusion process was proved for the first time. Problems of studying the asymptotic properties of random evolutions are the research subject of works [10], [11], [4] and works of other mathematicians. In particular, in the works [6] and [7] cases of using small series parameter $\varepsilon \to 0, (\varepsilon > 0)$ and solutions of singular perturbation problem in the schemes of averaging and diffusion approximation for stochastic evolutions with Markov and semi-Markov switching are classified. Small parameters were used in studying of large deviations asymptotic in the work [3] or estimation of exponentially small probabilities.

Approaches, which are given in works [6] and [7], are used for the generalization of results given in [9] in the case of continuous stochastic approximation procedure of direct influence of regression function on uniformly ergodic Markov process, which is considered in the [2].

Along with the main problem of the convergence of stochastic approximation procedure to the root of the regression equation, an important issue is to estimate the procedure convergence rate which is determined by the asymptotic normality, which is given in works [9] and [8].

The main object of the asymptotic analysis of stochastic evolutions including stochastic approximation procedure is generating generator of the corresponding two-component Markov process ([2, 6, 7]).

In this article the asymptotic normality for continuous stochastic approximation procedure with Markov perturbations of regression function is obtained by method of singular perturbation problem solutions for the ergodic Markov process generator. As in the work [2] here the second method of Lyapunov functions is used.

In section 2 problem statement is regarded and basic notations are given. In section 3 the main result is formulated, namely, the theorem of asymptotic normality of stochastic approximation procedure. Section 4 displays the properties of the normed stochastic approximation procedure from which the proof of the theorem follows.

2. Problem formulation

Continuous stochastic approximation procedure with Markov perturbation in the series scheme is given by evolution equation [2]

$$\frac{du^\varepsilon(t)}{dt} = a(t)C(u^\varepsilon(t), x(t/\varepsilon)). \tag{1}$$

Under appropriate conditions on regression function $C(u,x) = (C_k(u,x), k = \overline{1,d})$, $u \in R^d$, $x \in X$ and on normalizing scalar function $a(t), t \geq 0$, provided uniform ergodicity of Markov process $x(t), t \geq 0$, there is a convergence with probability one

$$u^\varepsilon(t) \Rightarrow u_0, t \to \infty \tag{2}$$

to the equilibrium point u_0 of averaged system

$$\begin{gathered} \frac{du(t)}{dt} = C(u(t)), \\ C(u) = \int_X \pi(dx)C(u,x), \end{gathered} \tag{3}$$

which, without reducing generality, further considered zero, namely

$$C(0) = 0. \tag{4}$$

From the research of stochastic approximation procedure with additive perturbation of Brownian motion (see [9])it is established, that convergence rate (2) is characterized by asymptotic normality of the normed stochastic approximation procedure at $t \to \infty$.

In the article asymptotic normality for stochastic approximation procedure (1) on the real axis $u \in R$ with ergodic Markov perturbation in stability conditions of averaged system (3) at $a(t) = \frac{a}{t}, a > 0$ is studied.

For stochastic approximation procedure, which is defined by equation

$$\frac{du^\varepsilon(t)}{dt} = \frac{a}{t} C\left(u^\varepsilon(t), x\left(\frac{t}{\varepsilon^2}\right)\right), \tag{5}$$

normed stochastic approximation procedure is specified as

$$v^\varepsilon(t) = \frac{\sqrt{t}}{\varepsilon} u^\varepsilon(t), t \geq t_0 > 0. \tag{6}$$

Remark 2.1. Convergence (2), taking into account the assumption (4), means, that estimate of convergence rate can be realized with the additional multiplier, for example

$$u_1^\varepsilon(t) = \sqrt{t} u^\varepsilon(t). \tag{7}$$

Analysis of limiting transition at $t \to \infty$ replaces by the analysis of normed stochastic approximation procedure convergence (7) at $\varepsilon \to 0$ with substitution of ε to ε^2, that is $v^\varepsilon(t) = \sqrt{\frac{t}{\varepsilon^2}} u^\varepsilon(t)$, namely (6).

Uniformly ergodic Markov process $x(t), t \geq 0$, in dimensional phase space $(X, \mathbf{X})$ is given by the generator

$$\mathbf{Q}\varphi(x) = q(x) \int_X P(x, dy)[\varphi(y) - \varphi(x)], \varphi(x) \in \mathfrak{B}(X), \tag{8}$$

where $\mathfrak{B}(X)$ – Banach space of real-valued bounded functions with supremum norm.

Stationary distribution $\pi(B), B \in \mathbf{X}$ of Markov process $x(t), t \geq 0$, satisfies the conditions

$$\pi(dx)q(x) = q\rho(dx), q = \int_X \pi(dx)q(x),$$

where stationary distribution $\rho(dx)$ of embedded Markov chain x_n, $n \geq 0$, is determined by the ratio

$$\rho(B) = \int_X \rho(dx)P(x, B), B \in \mathbf{X}, \rho(X) = 1.$$

In Banach space $\mathfrak{B}(X)$ projector

$$\Pi\varphi(x) = \int_X \pi(dx)\varphi(x)\mathbf{1}(x) = \varphi\mathbf{1}(x), \varphi = \int_X \pi(dx)\varphi(x), \mathbf{1}(x) \equiv 1, x \in X,$$

is considered and also potential $\mathbf{R}_0$ for generator $\mathbf{Q}$, which is defined by the equation $\mathbf{R}_0\mathbf{Q} = \mathbf{Q}\mathbf{R}_0 = \mathrm{I} - \Pi$.

Regression function $C(u, x) \in C^2(R)$ on u. Second derivative satisfies a global Lipschitz condition on u. Let introduce the notation

$$C_0(x) = C(0, x); C_1(x) = C'_u(0, x); b(x) = aC_1(x) + 1/2;$$
$$\tilde{C}(u, x) = C(u) - C(u, x). \tag{9}$$

3. Asymptotic normality

Theorem 1. *(Asymptotic normality). Let the conditions of stochastic approximation procedure* (6) *convergence holds (see [2]])*

$C1 : C(u)V'(u) \leq -c_0 V(u), c_0 > 0;$

$C2 :| \tilde{C}(u,x)V'(u) |\leq c_1(1 + V(u));$

$C3 :| C(u,x)\boldsymbol{R}_0[\tilde{C}(u,x)V'(u)]' \leq c_2(1 + V(u)).$

And there are additional conditions

$D1 : \rho^2 = -2 \int\limits_X \pi(dx)C_0(x)\boldsymbol{R}_0 C_0(x) > 0;$

$D2 : c = - \int\limits_X \pi(dx)C_1(x) > 0;$

$D3 : b = ac - 1/2 > 0.$

Then for normed stochastic approximation procedure (6) *there is weak convergence*

$$v^\varepsilon(t) \Rightarrow \zeta(t), \varepsilon \to 0, \tag{10}$$

in each finite interval $0 < t_0 \leq t \leq T$, $T > 0$, *where* $\zeta(t), t \geq 0$ – *Ornstein-Uhlenbeck process with generator* $\boldsymbol{L}_t = t^{-1}\boldsymbol{L}$,

$$\boldsymbol{L}\varphi(v) = \frac{\sigma^2}{2}\varphi''(v) - bv\varphi'(v), \tag{11}$$

and dispersion $\sigma^2 = a^2\rho^2$.

Remark 3.1. Condition D_1 provides diffusivity and condition D_3 – limited process ergodicity.

Remark 3.2. Limited Ornstein-Uhlenbeck process with generator **L** in theorem conditions is ergodic with stationary normal distribution $N(0, \sigma_0^2)$, where dispersion calculated by the formula $\sigma_0^2 = \sigma^2/2b$.

4. Normed stochastic approximation procedure properties

Lemma 2. *Normed stochastic approximation procedure* (5) *is a solution of stochastic differential equation*

$$dv^\varepsilon(t) = \frac{a}{\varepsilon\sqrt{t}}C\left(\frac{\varepsilon}{\sqrt{t}}v^\varepsilon(t), x_t^\varepsilon\right)dt + v^\varepsilon(t)\frac{dt}{t}, \tag{12}$$

where $x_t^\varepsilon = x\left(\dfrac{t}{\varepsilon^2}\right)$.

Proof. Equation (12) is obtained from stochastic approximation procedure (5) and representation (6). □

Lemma 3. *Generator of two-component Markov process* $v^\varepsilon(t)$, x_t^ε, $t \geq 0$ *has representation*

$$\boldsymbol{L}_t^\varepsilon\varphi(v,x) = \varepsilon^{-2}\boldsymbol{Q}\varphi(v,x) + \left[\varepsilon^{-1}\frac{a}{\sqrt{t}}C\left(\frac{\varepsilon}{\sqrt{t}}v, x\right) + \frac{v}{2t}\right]\varphi_v'(v,x). \tag{13}$$

Proof. According to definition of generator of Markov process $v^\varepsilon(t)$, x_t^ε, $t \geq 0$, it is necessary to calculate the conditional expectation, which looks as follows

$$E[\varphi(v + \Delta v^\varepsilon, x_{t+\Delta}^\varepsilon) - \varphi(v, x) \mid v^\varepsilon(t) = v, x_t^\varepsilon = x] = \Delta \mathbf{L}_t^\varepsilon \varphi(v, x) + o(\Delta),$$

where generator $\mathbf{L}_t^\varepsilon$ has form (13). □

In operator form (13) is written as

$$\mathbf{L}_t^\varepsilon = \varepsilon^{-2}\mathbf{Q} + \mathbf{V}_t^\varepsilon(x),$$

where generator $V_t^\varepsilon(x)$ is determined by the rate of evolution equation (12)

$$\mathbf{V}_t^\varepsilon(x)\varphi(v) = \left[\varepsilon^{-1}\frac{a}{\sqrt{t}}C\left(\frac{\varepsilon}{\sqrt{t}}v, x\right) + \frac{v}{2t}\right]\varphi'(v). \tag{14}$$

Lemma 4. *Generator* (13) *on test-functions* $\varphi(v, x) \in C^2(R)$ *on* v *has asymptotic representation*

$$\begin{aligned} \boldsymbol{L}_t^\varepsilon \varphi(v, x) = \varepsilon^{-2}\boldsymbol{Q}\varphi(v, x) + \varepsilon^{-1}\frac{1}{\sqrt{t}}\boldsymbol{C}(x)\varphi(v, x) + \frac{1}{t}\mathbf{V}(x)\varphi(v, x) + \\ + \varepsilon^2\frac{v^2}{2t}\boldsymbol{C}_2^\varepsilon(x)\varphi(v, x), \end{aligned} \tag{15}$$

where

$$\boldsymbol{C}(x)\varphi(v) = C_0(x)\varphi'(v), \tag{16}$$

$$\boldsymbol{V}(x)\varphi(v) = vb(x)\varphi'_(v), \tag{17}$$

$$\boldsymbol{C}_2^\varepsilon(x)\varphi(v) = C_2^\varepsilon(v, x)\varphi'(v), \tag{18}$$

$$C_2^\varepsilon(v, x) = C''\left(\frac{\varepsilon\theta v}{\sqrt{t}}, x\right), 0 \leq \theta \leq 1. \tag{19}$$

Proof. Using Taylor's formula for the regression function, taking into account the notations (9), we get

$$C\left(\frac{\varepsilon}{\sqrt{t}}v, x\right) = C_0(x) + \varepsilon\frac{v}{\sqrt{t}}C_1(x) + \varepsilon^2\frac{v^2}{2t}C_2^\varepsilon(v, x),$$

where residual term is defined by the formula (19). Substituting this expression into (14), we get (15)–(19). □

Lemma 5. *Solution of singular perturbation problem for generator* $\boldsymbol{L}_t^\varepsilon$ *has form*

$$\boldsymbol{L}_t^\varepsilon \varphi_t^\varepsilon(v, x) = \frac{1}{t}\boldsymbol{L}\varphi(v) + \varepsilon\theta_t^\varepsilon(x)\varphi(v).$$

Here limited generator $\boldsymbol{L}$ *determined by the formula* (11) *and residual term*

$$\theta_t^\varepsilon(x)\varphi(v) = \theta_3^\varepsilon(t, x)\varphi'''(v) + \theta_2^\varepsilon(t, x)\varphi''(v) + \theta_1^\varepsilon(t, x)\varphi'(v), \tag{20}$$

where functions $\theta_k^\varepsilon(v, x), k = 1, 2, 3,$ *can be calculated explicitly.*

Proof. In accordance with the scheme of solution of the singular perturbation problem ([7]), let us compute the value of the generator (15) on perturbed function

$$\varphi_t^\varepsilon(v, x) = \varphi(v) + \varepsilon\frac{1}{\sqrt{t}}\varphi_1(v, x) + \varepsilon^2\frac{1}{t}\varphi_2(v, x).$$

Considering the asymptotic representation of generator (15) and notation (16)–(19) we have

$$\begin{aligned}\mathbf{L}_t^{\varepsilon}\varphi_t^{\varepsilon}(v,x) = \varepsilon^{-2}\mathbf{Q}\varphi(v) + \varepsilon^{-1}\frac{1}{\sqrt{t}}[\mathbf{Q}\varphi_1(v,x) + \mathbf{C}(x)\varphi(v)] + \\ + \frac{1}{t}[\mathbf{Q}\varphi_2(v,x) + \mathbf{C}(x)\varphi_1(v,x) + \mathbf{V}(x)\varphi(v)] + \varepsilon\theta_t^{\varepsilon}(x)\varphi(v).\end{aligned} \tag{21}$$

Here, from definition, residual term has form

$$\begin{aligned}\theta_t^{\varepsilon}(x)\varphi(v) = \mathbf{C}_2^{\varepsilon}(x)\varphi(v) + \frac{1}{t}\mathbf{V}(x)\varphi_1(v,x) + \\ + \frac{1}{t^{3/2}}\left[\mathbf{C}(x) + \varepsilon\frac{1}{\sqrt{t}}\mathbf{V}(x)\right]\varphi_2(v,x).\end{aligned} \tag{22}$$

Obviously, that $\mathbf{Q}\varphi(v) = 0$. Next, for function $\varphi_1(v,x)$ there is equation

$$\mathbf{Q}\varphi_1(v,x) + \mathbf{C}(x)\varphi(v) = 0. \tag{23}$$

According to assumption (4), function $C_0(x)$ ((9)) satisfies balance condition, indeed

$$\Pi C_0(x) = \int_X \pi(dx)C_0(x) = \int_X \pi(dx)C(0,x) = C(0) = 0.$$

Thus, equation (23) has solution

$$\varphi_1(v,x) = -a\mathbf{R}_0 C_0(x)\varphi'(v). \tag{24}$$

Now we go to the equation for the function $\varphi_2(v,x)$

$$\mathbf{Q}\varphi_2(v,x) + \mathbf{C}(x)\varphi_1(v,x) + \mathbf{V}(x)\varphi(v) = \mathbf{L}\varphi(v), \tag{25}$$

where limited generator $\mathbf{L}$ is defined by condition of equation solvability (25). Using (24), equation (25) reduced to

$$\mathbf{Q}\varphi_2(v,x) + \mathbf{L}(x)\varphi(v) = \mathbf{L}\varphi(v), \tag{26}$$

in which

$$\mathbf{L}(x)\varphi(v) = \frac{a^2}{2}\sigma(x)\varphi''(v) - vb(x)\varphi'(v).$$

Here, from definition,

$$\sigma(x) = -2C_0(x)\mathbf{R}_0 C_0(x).$$

Now condition of equation solvability (26)

$$\Pi\mathbf{L}(x)\varphi(v) = \mathbf{L}\varphi(v) \tag{27}$$

defines limited generator $\mathbf{L}$ by formula (11).

Solution of equation (26), provided (27), has representation (see [7])

$$\varphi_2(v,x) = \mathbf{R}_0\tilde{\mathbf{L}}(x)\varphi(v), \tilde{\mathbf{L}}(x) = \mathbf{L} - \mathbf{L}(x). \tag{28}$$

Finally, let us calculate residual term in form (21) using formulas (22), (24) and (28). An expression (20) is obtained. □

Proof of theorem.

Proof. The use of Model limit theorem ([7]) justifies the weak convergence (10) in the theorem. □

5. Conclusion

In theorem conditions stochastic approximation procedure $v^\varepsilon(t)$ has asymptotic normal distribution $N(0, \sigma_0^2)$, namely $v^\varepsilon(t) \Rightarrow v$, $\varepsilon \to 0$, $t \to \infty$, where random variable v has distribution $N(0, \sigma_0^2)$.

The weak convergence of stochastic approximation procedure exists with renormed time in each finite interval $0 < t < T$ and limited process $\zeta(t), t \geq 0$ is Ornstein-Uhlenbeck process with generator

$$\mathbf{L}\varphi(v) = \frac{\sigma^2}{2}\varphi''(v) - bv\varphi'(v).$$

This is based on the use of martingale characterization of normed process and Model limited theorem of Korolyuk.

Similar result of asymptotic normality for stochastic approximation procedure in Euclidean space $R^d, d > 1$ can be obtained with a superior technical complications.

Obtained result can be used for the receiving the normality of the asymptotically dissipative systems with Markov switching in the asymptotically small diffusion scheme [5].

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Article: received 17.11.2019
revised 20.11.2019
printing adoption 20.11.2019

ЗБІЖНІСТЬ ПРОЦЕДУРИ СТОХАСТИЧНОЇ АПРОКСИМАЦІЇ ДО ПРОЦЕСУ ОРНШТЕЙНА–УЛЕНБЕКА

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Розглянуто основну асимптотичну властивість неперервної процедури стохастичної апроксимації, а саме збіжність до процесу Орнштейна-Уленбека. У цьому випадку функція регресії процедури залежить від рівномірно ергодичного процесу Маркова, який описує зовнішній вплив у вигляді переключень. Для отримання достатніх умов збіжності процедури використовуємо ергодичні властивості породжуючого генератора процесу Маркова, існування стаціонарного розподілу цього процесу та існування потенціалу для породжуючого генератора процесу Маркова. Процедура стохастичної апроксимації, як випадковий процес, будується в вигляді диференціального рівняння в фіксованому стані процесу Маркова з відповідним генератором рівняння. Асимптотичність за часом досягається використанням малого параметра, за яким нормується час. Це дало змогу отримати нормовану процедуру стохастичної апроксимації, та її диференціальне подання. Генератор отриманого диференціального рівняння використовують для побудови генератора двокомпонентного процесу Маркова, який складається з процедури та процесу переключень. Сингулярне подання цього генератора за малим параметром дає підстави розв'язати проблему сингулярного збурення й обчислити вигляд граничного генератора. Таке подання визначає вигляд граничного процесу як дифузійного процесу Орнштейна-Уленбека. Зауважимо, що збіжність до граничного процесу є слабкою, яка випливає з теореми Королюка. Важливими є умови існування функції Ляпунова для усередненої за стаціонарним розподілом процесу Маркова динамічної процедури. Додаткові умови на функцію Ляпунова допомагають визначити обмеженість залишкових членів розв'язку сингулярного збурення. Одновимірний випадок процедури можна розширити до багатовимірного з відповідним ускладненням обчислень складників граничного генератора. Робота узагальнює дослідження Невельсона та Хасьмінського на випадок безпосереднього впливу процесу Маркова на функцію регресії.

Ключові слова: Процедура стохастичної апроксимації, процес Маркова, процес Орнштейна–Уленбека, слабка збіжність.