МАТЕМАТИЧНЕ МОДЕЛЮВАННЯ

UDC 517.925; 519.6

DIRECT METHOD OF LIE-ALGEBRAIC DISCRETE APPROXIMATIONS FOR SOLVING HEAT EQUATION

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A direct method of Lie-algebraic discrete approximation for numerical solving the Cauchy problem for heat transfer equation is proposed in this paper. The key idea of direct method of Lie-algebraic discrete approximations is using analytical approaches, in particular the method of small parameter or Taylor series expansion, to construct analytical approximation of the solution for the problem in the form of power series with respect to the time variable.

The conditions for convergence of analytical series are studied in particular. By means of small parameter method the recurrence relation for evaluation of each member of a sequence is provided. This approach enables fast computation and significant reduction of computational cost in compare to Generalized method of Lie-algebraic discrete approximations which performs complete discretization by all variables.

Thereafter, the discrete match of recurrence relation is built using quasi-representations of the Lie-algebra basis elements, which means, that each differential operator is replaced by its analogue matrix which is quasi-representation of differential operator in finite dimensional space. It is proved that computational scheme has a factorial rate of convergence.

The proposed approach is applied to model case and obtained results are compared with finite difference method, classical method of Lie-algebraic discrete approximations and Generalized method of Lie-algebraic discrete approximation. The convergence rates for all of these methods are compared in different functional spaces. In addition, we study the count of arithmetical operations for equal set of nodes.

Key words: direct method of Lie-algebraic discrete approximations, heat equation, finite dimensional quasi representation, Lagrange polynomial, small parameter method, factorial convergence.

1. Introduction

The heat equation is the partial differential equation that describes how the distribution of some quantity (such as heat) evolves over time in a solid medium, as it spontaneously flows from places where it is higher towards places where it is lower. It has many applications in the diverse scientific fields: physics, engineering and earth sciences, probability theory, financial mathematics [18, 19, 27, 28]. Hence effective numerical solution is an actual problem besides the existing of various approaches [1, 9, 19].

The Direct method of Lie-algebraic discrete approximations was firstly proposed for advection equation in [24] and has been approbated on conference [25]. Further this method was extended on nonlinear equation, namely Burger's nonviscous equation and was discussed on conference [5]. This method is one of the whole family of methods which use Lie-algebraic discrete approximations [1–4, 7, 8, 10–18, 21–23, 26, 28].

The main prerequisite of these method is that differential operator should be the element of universe enveloping Heisenberg-Weyl's algebra with basis elements from the

Lie algebra $\{1, x, d/dx\}$, i.e. differential operator for the problem must be superposition and/or linear combination of these base elements of Lie algebra. As a next step we introduce the finite dimensional discrete quasi representations of $\{1, x, d/dx\}$ as matrices $\{I, X, Z\}$.

Further, if we reduce partial differential equation to system of ordinary differential equations we get the (classic) Method of Lie-algebraic discrete approximations [1, 7, 8, 10, 12–17, 26, 28]; if reduce partial differential equation to the system of algebraic equation (either linear or nonlinear) we get the (classic) Method of Lie-algebraic discrete approximations [3, 4, 21–23].

Let us explain the idea of Direct method of Lie-algebraic discrete approximation on the model problem investigated in [24]. Considering a bounded domain $\Omega := (a, b) \subset \mathbb{R}$, time limit $T < +\infty$, cylinder $Q_T = \Omega \times (0, T]$ we take the Banach space $V = W^{\infty,\infty}\left(\overline{Q_T}\right)$ and formulate the Cauchy problem

$$\begin{cases} \text{given advection coefficient } c \in \mathbb{R}, \\ \text{distribution at initial moment of time } \varphi = \varphi(x); \\ \text{find function } u = u(x,t) \in V \text{ such, that:} \\ \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0, \ \forall (x,t) \in Q_T, \\ u|_{t=0} = \varphi, \varphi \in W^{\infty,\infty}\left((-|c|T, |c|T) \cup \overline{Q}\right), \end{cases} \tag{1}$$

where space $V = W^{\infty,\infty}\left(\overline{Q_T}\right)$ denotes the functional space in which all functions and its derivatives up to arbitrary order are bounded in the domain $\overline{Q_T}$, i.e.:

$$W^{\infty,\infty}\left(\overline{Q_T}\right) = \{u : Q_T \to \mathbb{R} : D^\alpha u \in L^\infty(Q_T), \forall \alpha \in \mathbb{N}\}.$$

The idea of a direct method of Lie-algebraic discrete approximations consists in the use of analytical approaches, in particular the method of a small parameter, to construct an approximate analytic solution of a problem (1) in the form of a power series:

$$u_n(x,t) = \sum_{k=0}^{n} \left(\tilde{u}_k \frac{t^k}{k!}\right) = \varphi - c\varphi' t + c^2 \varphi'' \frac{t^2}{2!} + ... + (-1)^n c^n \varphi^{(n)} \frac{t^n}{n!}. \tag{2}$$

After this, the corresponding discrete series was constructed for (2) using the finite dimensional quasi-representations of elements of the Lie algebra:

$$u_{n,h}(t) = \sum_{k=0}^{n} \left(\tilde{u}_{k,h} \frac{t^k}{k!}\right) = \varphi_h - cZ\varphi_h t + c^2 Z^2 \varphi_h \frac{t^2}{2!} + ... + (-1)^n c^n Z^n \varphi_h \frac{t^n}{n!}, \tag{3}$$

where the matrix Z approximates the differential operator d/dx. Moreover, the series (3) is finite, since the matrix Z is nilpotent [15].

It was proved in [24] that the computational scheme is convergent with error rate:

$$\|u - u_h\|_{V_h} \leq \frac{|c|^{n+1} T^{n+1} + \left(2max\left\{|c|T, diam\Omega\right\}\right)^{n+1}}{(n+1)!} \|\varphi^{(n+1)}\|_\infty.$$

Computational experiments showed that with the same accuracy and convergence indicators that are characteristic for the generalized method of Lie-algebraic discrete approximations, we succeeded in significantly reducing the number of arithmetic operations using approach from [24].

This paper is constructed in the following way: we formulate the model problem to which we apply the proposed numerical scheme in second chapter, analytical foundations for the proposed numerical approach are discussed in the third chapter and its Lie-algebraic discretization of the recurrence relation is investigated in the fourth chapter. Numerical results with arithmetic operations count for the model problem are provided in the fifth chapter.

2. Problem formulation

Considering a bounded domain $\Omega = (a,\, b) \subset \mathbb{R}$, time limit $T < +\infty$, cylinder $Q_T = \Omega \times (0, T]$ we take the Banach space $V = W^{\infty,\infty}\left(\overline{Q_T}\right)$ and formulate the Cauchy problem

$$\begin{cases} \text{given heat conduction coefficient } a \in \mathbb{R}, a > 0, \\ \text{temperature at initial moment of time } \varphi = \varphi(x); \\ \text{find function } u = u(x,t) \in V \text{ such, that} \\ \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2},\ (x,t) \in Q_T, \\ u|_{t=0} = \varphi(x),\ \ \varphi(x) \in C_x^{\infty}(\Omega)\,. \end{cases} \tag{4}$$

The solution of problem (4) we seek using iterative approach method via Lie-algebraic discrete approximations, i.e. by means of Direct method of Lie-algebraic discrete approximations.

3. Iterative approach and its convergence

3.1. Taylor series expansion

The main idea of the direct method of Lie-algebraic discrete approximation is to approximate the solution directly. First of all we make the analytical setup for the proposed approach.

Lemma 1. *(An integro-differential representation of the solution). The function $u = u(x,t)$ in the integro-differential expression*

$$u(x,t) = \phi(x) + a\int_0^t \left(\frac{\partial^2 u(x,\tau)}{\partial x^2}\right) d\,\tau, \tag{5}$$

is the solution of Cauchy problem (4).

Proof. Integration of the equation in (4) on the interval $(0,t)$

$$\int_0^t \frac{\partial u(x,\tau)}{\partial \tau} d\,\tau = a\int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2} d\,\tau$$

yields

$$u(x,t) - u(x,0) = a\int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2} d\,\tau.$$

By taking into account the initial condition from (4), i.e. $u|_{t=0} = \phi(x)$ there is obtained integro-differential representation of the solution (5).

It can be shown that such defined function is the solution of the problem (4): if integro-differential expression (5) is differentiated with respect to time variable, there is retrieved the equation we deal with, namely

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}.$$

Next, we evaluate the expression (5) at the initial moment of time $u|_{t=0}$ and get the initial condition:

$$u(x,0) = \phi(x) + a\int_0^0 \left(\frac{\partial^2 u(x,\tau)}{\partial x^2}\right) d\,\tau = \phi(x).$$

Thus all requirements hold and the lemma has been proven. □

For further purposes let us denote the derivative of the function as $\varphi^{(k)} = d^k\varphi/dx^k$.

Lemma 2. *(The identity of series expansions). The solution expansion*

$$u_I = \sum_{k=0}^{+\infty} \tilde{u}_k \frac{t^k}{k!}, \tag{6}$$

where $\tilde{u}_k = a^k\varphi^{(2k)}$, can be derived by means of iterative approach and provided here expansion is a Taylor series expansion with respect to time variable.

Proof. At first we show that the series (6) can be derived by means of iterative approach. To accomplish this we recall the integro-differential representation of the solution (5) and set up an iterative process in the following form:

$$\begin{cases} u_{k+1}(x,t) = \phi(x) + a\displaystyle\int_0^t \left(\frac{\partial^2 u_k(x,\tau)}{\partial x^2}\right) d\tau, \\ u_0(x,t) = \varphi(x). \end{cases} \tag{7}$$

According to [11] the starting element in recurrence sequence $u_0(x,t) = \varphi(x)$ can be obtained by setting $u_{-1}(x,t) = 0$ in recurrence relation (7). Let us evaluate $u_k(x,t)$ for $k = 1, 2, 3$:

$$u_1 = \varphi(x) + a\int_0^t \left(\frac{d^2\varphi(x)}{d\,x^2}\right) d\,\tau = \varphi(x) + a\varphi''(x)\,t,$$

$$u_2 = \varphi(x) + a\int_0^t \frac{\partial^2}{\partial x^2}\left(\varphi(x) + a\varphi''(x)\,t\right) d\,\tau = \varphi(x) + a\varphi''(x)\,t + a^2\varphi^{(4)}\frac{t^2}{2},$$

$$u_3 = \varphi(x) + a\int_0^t \frac{\partial^2}{\partial x^2}\left(\varphi(x) + a\varphi''(x)\,t + a^2\varphi^{(4)}\frac{t^2}{2}\right) d\,\tau =$$
$$= \varphi(x) + a\varphi''(x)\,t + a^2\varphi^{(4)}\frac{t^2}{2} + a^3\varphi^{(6)}\frac{t^3}{3!}.$$

Let us show that $u_n(x,t) = \displaystyle\sum_{k=0}^{n}\left(a^k\varphi^{(2k)}\frac{t^k}{k!}\right)$. We assume that this statement holds for $u_n(x,t)$ and we check whether it holds for $u_{n+1}(x,t)$, i.e. $u_{n+1}(x,t) = \displaystyle\sum_{k=0}^{n+1}\left(a^k\varphi^{(2k)}\frac{t^k}{k!}\right)$

Hence we consider the term $u_{n+1}(x,t)$:

$$u_{n+1}(x,t) = \varphi(x) + a\int\limits_0^t \frac{\partial^2}{\partial x^2}\left(\sum_{k=0}^{n}\left(a^k\varphi^{(2k)}\frac{\tau^k}{k!}\right)\right)d\tau =$$
$$= \varphi(x) + \int\limits_0^t\left(\sum_{k=0}^{n}\left(a^{k+1}\varphi^{(2k+2)}\frac{\tau^k}{k!}\right)\right)d\tau = \varphi(x) + \sum_{k=0}^{n}\left(a^{k+1}\varphi^{(2k+2)}\int\limits_0^t\frac{\tau^k}{k!}d\tau\right) =$$
$$= \varphi + \sum_{k=0}^{n}a^{k+1}\varphi^{(2k+2)}\frac{t^{k+1}}{(k+1)!} = \varphi + \sum_{k=1}^{n+1}a^k\varphi^{(2k)}\frac{t^k}{(k)!} = \sum_{k=0}^{n+1}a^k\varphi^{(2k)}\frac{t^k}{(k)!}.$$

At the current moment we have shown that expansion (6) can be derived by means of the iterative approach (7). As a second step we derive the Taylor series for the problem (4) with respect to time variable and verify that obtained expression is identical to the expression stated in (6). Let us consider Taylor expansion at the initial moment of time:

$$u_n(x,t) = u(x,0) + \left(\frac{\partial u}{\partial t}\bigg|_{t=0}\right)t + \left(\frac{\partial^2 u}{\partial t^2}\bigg|_{t=0}\right)\frac{t^2}{2!} + \left(\frac{\partial^3 u}{\partial t^3}\bigg|_{t=0}\right)\frac{t^3}{3!} + ... + \left(\frac{\partial^n u}{\partial t^n}\bigg|_{t=0}\right)\frac{t^n}{n!}$$

with the remainder $R_n = \left(\frac{\partial^{n+1}u}{\partial t^{n+1}}\Big|_{t=\tau}\right)\frac{t^{n+1}}{(n+1)!}$ having the following property

$$u(x,t) - u_n(x,t) = R_n(x,t).$$

Although we have not explicit formulas for the $\left(\frac{\partial^k u}{\partial t^k}\Big|_{t=0}\right), k \geq 1$, nevertheless we can evaluate them with respect to initial condition and along the equation from (4). For instance, one can evaluate $\left(\frac{\partial u}{\partial t}\Big|_{t=0}\right)$ and $\left(\frac{\partial^2 u}{\partial t^2}\Big|_{t=0}\right)$:

$$\left(\frac{\partial u}{\partial t}\bigg|_{t=0}\right) = a\left(\frac{\partial^2 u}{\partial x^2}\bigg|_{t=0}\right) = a\frac{\partial^2}{\partial x^2}\left(u|_{t=0}\right) = a\varphi''(x),$$
$$\left(\frac{\partial^2 u}{\partial t^2}\bigg|_{t=0}\right) = a\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\bigg|_{t=0}\right) = a^2\frac{\partial^4}{\partial x^4}\left(u|_{t=0}\right) = a^4\varphi^{(4)}(x).$$

To show that

$$\left(\frac{\partial^k u}{\partial t^k}\bigg|_{t=0}\right) = a^k\varphi^{(2k)}(x), k \geq 1,$$

we assume that expression of the above holds, and retrieve the following

$$\left(\frac{\partial^{k+1} u}{\partial t^{k+1}}\bigg|_{t=0}\right) = a\frac{\partial^2}{\partial x^2}\left(\frac{\partial^k u}{\partial t^k}\bigg|_{t=0}\right) = a\frac{d^2}{dx^2}\left(a^k\varphi^{(2k)}\right) = a^{k+1}\varphi^{(2k+2)}(x).$$

We see that expression holds and we have derived the expansion (6) using the symbolic computation tools.

As a third step we show the connection between these approaches. Let us consider

$u_n(x,t) = \sum_{k=0}^{n} \left(a^k \varphi^{(2k)} \frac{t^k}{k!} \right)$. The calculation of the $\left.\frac{\partial^m u_n}{\partial t^m}\right|_{t=0}$ yields:

$$\left.\frac{\partial u_n}{\partial t}\right|_{t=0} = \left.\left(a\varphi''(x) + \sum_{k=2}^{n} \left(a^k \varphi^{(2k)}(x) \frac{t^{k-1}}{(k-1)!} \right) \right)\right|_{t=0} = a\varphi''(x),$$
$$\left.\frac{\partial^2 u_n}{\partial t^2}\right|_{t=0} = \left.\left(a^2\varphi^{(4)}(x) + \sum_{k=3}^{n} \left(a^k \varphi^{(2k)}(x) \frac{t^{k-2}}{(k-2)!} \right) \right)\right|_{t=0} = a^2\varphi^{(4)}(x),$$
$$\left.\frac{\partial^m u_n}{\partial t^m}\right|_{t=0} = \left.\left(a^m\varphi^{(2m)}(x) + \sum_{k=m+1}^{n} \left(a^k \varphi^{(2k)}(x) \frac{t^{k-m}}{(k-m)!} \right) \right)\right|_{t=0} = a^m\varphi^{(2m)}(x).$$

Thus we have proven the expansion (6). □

Lemma 3. *(Convergence of the iterative approach). The sequence $\{u_k(x,t)\}$ defined in (6) converges uniformly to the exact solution, i.e.:*

$$\lim_{n\to\infty} u_n(x,t) = u(x,t),$$

where $u(x,t)$ is the solution of the problem (4).

Proof. First of all let us show that series defined in (6) is formal solution of (4). Algebraic calculation yields the following:

$$\frac{\partial u_I}{\partial t} = \sum_{k=0}^{\infty} a^{k+1} \varphi^{(2k+2)} \frac{t^k}{k!}, \quad \frac{\partial^2 u_I}{\partial x^2} = \sum_{k=0}^{\infty} a^{k} \varphi^{(2k+2)} \frac{t^k}{k!},$$

and finally we get

$$\frac{\partial u_I}{\partial t} - a\frac{\partial^2 u_I}{\partial x^2} = \sum_{k=0}^{\infty} a^{k+1} \varphi^{(2k+2)} \frac{t^k}{k!} - a\left(\sum_{k=0}^{\infty} a^{k} \varphi^{(2k+2)} \frac{t^k}{k!} \right) \equiv 0,$$

which proves that infinite series is formal solution of the problem (4).

As a next step we consider the finite series $u_n(x,t) = \sum_{k=0}^{n} \left(a^k \varphi^{(2k)} \frac{t^k}{k!} \right)$. Conducted algebraic calculations

$$\frac{\partial u_n}{\partial t} = \sum_{k=0}^{n-1} a^{k+1} \varphi^{(2k+2)} \frac{t^k}{k!}, \quad \frac{\partial^2 u_n}{\partial x^2} = \sum_{k=0}^{n} a^{k} \varphi^{(2k+2)} \frac{t^k}{k!},$$

we substitute in the equation from (4):

$$\left| \frac{\partial u_n}{\partial t} - a\frac{\partial^2 u_n}{\partial x^2} \right| = \left| \sum_{k=0}^{n-1} a^{k+1} \varphi^{(2k+2)} \frac{t^k}{k!} - \sum_{k=0}^{n-1} a^{k+1} \varphi^{(2k+2)} \frac{t^k}{k!} - a^{n+1}\varphi^{(2n+2)} \frac{t^{n+1}}{(n+1)!} \right| =$$
$$= |a|^{n+1} \left| \varphi^{(2n+2)} \right| \frac{t^{n+1}}{(n+1)!} \leq |a|^{n+1} M \frac{T^{n+1}}{(n+1)!},$$

since $\varphi \in W^{\infty,\infty}(\Omega)$, i.e $\exists M > 0, \forall n \in \mathbb{N} : \|\varphi^{(n)}\|_\infty < M$. Finally we get

$$\lim_{n\to\infty} \left| \frac{\partial u_n}{\partial t} - a\frac{\partial^2 u_n}{\partial x^2} \right| \leq M \lim_{n\to\infty} \left(\frac{(|a|T)^{n+1}}{(n+1)!} \right) = 0.$$

Let us prove the uniform convergence of a series (6). According do Weierstrass' majorant theorem we have to show that sequence of positive constants is convergent, thus we consider majorant series

$$|u_I| \le \sum_{k=0}^{\infty} |a|^k \|\varphi^{(2k)}\|_\infty \frac{T^k}{k!} \le M \sum_{k=0}^{\infty} |a|^k \frac{T^k}{k!} = M \sum_{k=0}^{\infty} b_k,$$

where $b_k = |a|^k \dfrac{T^k}{k!}$. Since ratio test

$$\lim_{n\to\infty} \frac{b_{k+1}}{b_k} = \lim_{n\to\infty} \frac{|a|^{k+1} T^{k+1}}{(k+1)!} \frac{k!}{|a|^k T^k} = |a| T \lim_{n\to\infty} \frac{1}{k+1} = 0,$$

yields that majorant series is convergent, therefore series (6) converges uniformly on Q_T.

Since series (6) is Taylor's series of the solution (4) the following error estimation holds:

$$\|u - u_n\| \le \left\| \frac{\partial^{n+1} u}{\partial t^{n+1}} \right\|_\infty \frac{T^{n+1}}{(n+1)!}.$$

Thus, lemma has been proven. □

3.2. Small parameter method

Expansion provided in (6) from the previous section can derived using various approach. Nevertheless some of them appear to be more or less effective from the different perspectives. It is obvious that iterative approach is computationally expensive, since at each step one should integrate an increasing expression. On the other hand. Taylor expansion requires symbolic computations and, thus, it cannot be used for the finite dimensional calculations we are targeting to. Next section is intended to provide approach which can be used for the finite dimensional calculations and simultaneously be effective even for symbolic computations.

Lemma 4. *(Recurrence relation for the expansion terms). Terms* $\{\tilde{u}_k(x)\}_{k=0}^{n}$ *in expression (6) can be computed by means of the following recurrence relation:*

$$\begin{cases} \tilde{u}_{k+1} = a \dfrac{d^2}{dx^2} (\tilde{u}_k), \\ \tilde{u}_0 = \varphi. \end{cases} \tag{8}$$

Proof. To prove this proposition we will use a Small Parameter Method. We seek solution as a formal expansion by small parameter:

$$u_\varepsilon(x,t) = \sum_{k=0}^{\infty} \hat{u}_k(x,t) \varepsilon^k$$

for the parameterized problem:

$$\begin{cases} \text{given heat conduction coefficient } a \in \mathbb{R}, a > 0, \\ \text{temperature at initial moment of time } \varphi = \varphi(x); \\ \text{find function } u_\varepsilon = u_\varepsilon(x,t) \in V \text{ such, that} \\ \dfrac{\partial u_\varepsilon}{\partial t} = \varepsilon \left(a \dfrac{\partial^2 u_\varepsilon}{\partial x^2} \right), \ (x,t) \in Q_T, \\ u_\varepsilon|_{t=0} = \varphi(x), \ \ \varphi(x) \in C_x^\infty(\Omega). \end{cases}$$

After substitution of formal series into parameterized problem, performing auxiliary algebraic calculations and taking into account initial condition as a conclusion we obtain:

$$\frac{\partial \hat{u}_0}{\partial t}+\sum_{k=1}^{\infty}\left(\frac{\partial \hat{u}_k}{\partial t}-a\frac{\partial^2 \hat{u}_{k-1}}{\partial x^2}\right)\varepsilon=0.$$

and the set of Cauchy condition for each term in the formal expansion by small parameter

$$\begin{cases} \hat{u}_0|_{t=0}=\varphi(x), \\ \hat{u}_k|_{t=0}=0, k\geq 1. \end{cases}$$

As a next step we introduce the set Cauchy problems:

$$\begin{cases} \text{find function } \hat{u}_k=\hat{u}_k(x,t)\in V \text{ such, that} \\ \dfrac{\partial \hat{u}_k}{\partial t}=\left(a\dfrac{\partial^2 \hat{u}_{k-1}}{\partial x^2}\right), \ (x,t)\in Q_T, \ k\geq 1 \\ \hat{u}_k|_{t=0}=0, \end{cases} \tag{9}$$

and

$$\begin{cases} \text{find function } \hat{u}_0=\hat{u}_0(x,t)\in V \text{ such, that} \\ \dfrac{\partial \hat{u}_0}{\partial t}=0, \ (x,t)\in Q_T, \\ \hat{u}_0|_{t=0}=\varphi(x), \ \ \varphi(x)\in C_x^{\infty}(\Omega)\,. \end{cases} \tag{10}$$

The solution of the first problem (10) is $\hat{u}_0=\varphi(x)$. In fact, equation $\frac{\partial \hat{u}_0}{\partial t}=0$ shows that there are no changes during the evolution and $\hat{u}_0|_{t=0}=\varphi(x)$ will not change during the all process, thus $\hat{u}_0=\varphi(x)$. On the other hand, the equation $\frac{\partial \hat{u}_0}{\partial t}=0$ leads that solution of this equation might be an arbitrary function that doesn't depend on time variable and since that $\hat{u}_0$ at initial moment of time has a constraint as $\hat{u}_0|_{t=0}=\varphi(x)$ then the solution is $\hat{u}_0=\varphi(x)$.

Similar approach we use for the next problems. Evaluation of $\hat{u}_1$ yields the following expressions:

$$\frac{\partial \hat{u}_1}{\partial t}=\left(a\frac{\partial^2 \hat{u}_0}{\partial x^2}\right)\Rightarrow\frac{\partial \hat{u}_1}{\partial t}=a\varphi''(x)\Rightarrow\hat{u}_1=a\int\varphi''(x)dt=a\varphi''(x)t+C_1t,$$

where $C_1(t)$ is an arbitrary function. Taking into account the initial condition $\hat{u}_1|_{t=0}=0$ we obtain $C_1\equiv 0$ and $\hat{u}_1=a\varphi''(x)t=\tilde{u}_1(x)t$.

Let us assume that $\hat{u}_k=a^k\varphi^{(2k)}(x)\frac{t^k}{k!}=\tilde{u}_k(x)\frac{t^k}{k!}$ thus, we should prove that this expression holds for $k+1$. We consider the equation from (9) taking into account our assumption:

$$\frac{\partial \hat{u}_{k+1}}{\partial t}=a^{k+1}\varphi^{(2k+2)}(x)\frac{t^k}{k!}=\tilde{u}_k(x)\frac{t^k}{k!}.$$

Performing similar calculations as of above we prove our assumption, namely we obtain

$$\hat{u}_{k+1}=a^{k+1}\varphi^{(2k+2)}(x)\frac{t^{k+1}}{(k+1)!}=\tilde{u}_{k+1}(x)\frac{t^{k+1}}{(k+1)!}.$$

Since $\tilde{u}_k(x) = a^k \varphi^{2k}(x)$ and $\tilde{u}_{k+1}(x) = a^{k+1}\varphi^{2k+2}(x)$ it implies that

$$\tilde{u}_{k+1}(x) = a\frac{d^2}{dx^2}\left(\tilde{u}_k(x)\right)$$

and it proves lemma. □

Approach based on *Small Parameter Method* allows fast symbolic computation in order to obtain the analytical solution and it is a good basis to make of use the Lie-algebraic discrete approximations. Next chapter is devoted to constructing the numerical scheme on top of recurrence relation via finite dimensional quasi representations. This is an essence of *Direct method of Lie-algebraic discrete approximations.*

4. Numerical scheme

The main idea of numerical scheme construction for the direct method of Lie-algebraic discrete approximation is to replace the elements from Lie-algebra $G = \{1, x, \partial/\partial x\}$ in recurrence relation (8) by theirs finite dimensional quasi representations $G = \{I, X, Z\}$ respectively. Lagrange polynomials have been chosen as a tool for finite dimensional quasi representations construction.

We examine numerical scheme construction, recall approximation properties and prove convergence of proposed numerical scheme in this section.

4.1. Lie-algebraic discretization

Let n_x denotes the count of nodes in domain Ω and n_t denotes count of nodes in interval $[0, T]$ and $Q_{T,h}$ denotes the mesh of nodes built upon nodes $\{x_i\}_{i=0}^{n_x}$ and $\{t_i\}_{i=0}^{n_t}$. Lagrange polynomials $l_j(x)$ built at the nodes $\{x_i\}_{i=0}^{n_x}$ form the basis in finite dimensional space V_h.

Let us denote the matrix Z as finite dimensional quasi representation of the differential operator d/dx. The matrix Z is built upon the rule $Z_{ij} = l'_j(x_i)$ [16]. The key property of this matrix is such, that matrix $Z^k = (Z)^k$ approximates differential operator d^k/dx^k and matrix Z is nilpotent [15], i.e. there is some number n that all further multiplications give nil matrix: $\forall k \geq n : Z^k = \mathbf{0}$.

Having built all required quasi-representations we provide the following lemma as a key finding of this paper, namely the discrete recurrence relation as a Lie-algebraic discrete approximation of the recurrence relation.

Lemma 5. *(Finite dimensional recurrence relation for the expansion terms). Terms* $\{\tilde{u}_{k,h}\}_{k=0}^{n}$ *in expression*

$$u_{n,h} = \sum_{k=0}^{n} \tilde{u}_{k,h}\frac{t^k}{k!}, \tag{11}$$

can be computed by means of the following recurrence relation:

$$\begin{cases} \tilde{u}_{k+1,h} = aZ^2\left(\tilde{u}_{k,h}\right), \\ \tilde{u}_{0,h} = \varphi_h\,. \end{cases} \tag{12}$$

which is the Lie-algebraic discretization of the recurrence relation (8).

Proof. Since $\tilde{u}_{k+1,h}$ from discrete expansion (11) is the finite dimensional quasi representation of infinite series expansion (6), matrix Z^2 is the finite dimensional quasi representation of differential operator d^2/dx^2, then expression

$$\tilde{u}_{k+1}(x) = a\frac{d^2}{dx^2}\left(\tilde{u}_k(x)\right)$$

from (8) may be rewritten as a finite dimensional quasi representation

$$\tilde{u}_{k+1,h} = aZ^2\left(\tilde{u}_{k,h}\right).$$

Since the matrix Z is nilpotent, the length of a sequence of $\tilde{u}_{k,h}$ in (11) is $\lfloor\frac{n+1}{2}\rfloor$. To prove that property one can rewrite the recurrence relation into the following form

$$\tilde{u}_{k,h} = a^k Z^{2k}\left(\varphi_h\right).$$

When the index k reaches the number $\dfrac{n_x+1}{2}$ due to nilpotent property $Z^{n_x+1}=0$ we have $\tilde{u}_{k,h}=0$ for all further $k \geq \frac{n_x+1}{2}$. □

4.2. Approximation properties

Since the numerical scheme uses Lagrange interpolation and Lagrange polynomials, we need to discuss some issues regarding the approximation properties of Lagrange polynomials in the context of constructed numerical scheme.

The tooling of Lagrange interpolation has not been changed since previously discussed model in [24], so we will recall main approximation lemmas proved in [24] in current section.

Lemma 6. *(Derivative error bounds for Lagrange interpolation). Let $v(x) \in W^{\infty,\infty}(\Omega)$ and v_I denotes the Lagrange interpolation of function $v(x)$ built at nodes $\{x_i\}_{i=0}^n$. Then the following estimation of the error bounds for Lagrange interpolation holds:*

$$\left\|v^{(k)} - v_I^{(k)}\right\|_\infty \leq \frac{(\operatorname{diam}\Omega)^{n-k+1}}{(n-k+1)!}\|v^{(n+1)}\|_\infty.$$

Proof. Proved in [24], please see the lemma *"Derivative error bounds for Lagrange interpolation"*. □

Let us consider the cylinder norm for the function $v = v(x) : \mathbb{R} \to \mathbb{R}$: as a following functional: $\|v\|^2_{V_h} = \dfrac{1}{n+1}\displaystyle\sum_{i=0}^{n} v^2(x_i)$, being a norm in the finite dimensional space V_h. One can verify that the following inequality $\|v\|_{V_h} \leq \|v\|_\infty$ holds [24].

Lemma 7. *(Derivative error bounds for quasi representation). Let $v(x) \in W^{\infty,\infty}(\Omega)$ and v_I denotes the Lagrange interpolation of function $v(x)$ built at nodes $\{x_i\}_{i=0}^n$, matrix Z^k as finite dimensional quasi representation of the differential operator d^k/dx^k, then the following estimation of the error bounds for finite dimensional quasi representations holds:*

$$\|v^{(k)} - Z^k v\|_{V_h} \leq \|v^{(k)} - v_I^{(k)}\|_\infty.$$

Proof. Proved in [24], please see the lemma *"Derivative error bounds for quasi representation"*. □

4.3. Convergence and error estimation

The key finding of this paper is the proposition of method which has almost the same properties regarding the convergence but has more comprehensive way in the constructing and implementation of the numerical scheme. The discussion concerning the convergence of numerical scheme we start from proof of some auxiliary lemmas which finally allow us to formulate the key finding of current article.

Lemma 8. *The series*

$$\sum_{k=0}^{n/2} \frac{1}{(n-2k)!k!} \tag{13}$$

can be estimated as the following expression:

$$\sum_{k=0}^{n/2} \frac{1}{(n-2k)!k!} \le \frac{2^{n-1}}{\left(\frac{n}{2}-1\right)!} \tag{14}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{n/2} \frac{1}{(n-2k)!k!} = 0.$$

Proof. We can recall that series

$$\sum_{k=0}^{n/2} \frac{1}{(n-2k)!(2k)!}$$

has the property, so that:

$$\sum_{k=0}^{n/2} \frac{1}{(n-2k)!(2k)!} = \frac{2^{n-1}}{n!}$$

Let us consider a series

$$\sum_{k=0}^{n/2} \frac{(2k)!}{k!}$$

which has the following rough estimation:

$$\sum_{k=0}^{n/2} \frac{(2k)!}{k!} \le \frac{n}{2} \frac{n!}{\left(\frac{n}{2}\right)!} = \frac{n!}{\left(\frac{n}{2}-1\right)!}.$$

At the current moment we can proceed to the estimation of (13):

$$\sum_{k=0}^{n/2} \frac{1}{(n-2k)!k!} = \sum_{k=0}^{n/2} \frac{1}{(n-2k)!(2k)!} \cdot \frac{(2k)!}{k!} \le$$
$$\le \left(\sum_{k=0}^{n/2} \frac{1}{(n-2k)!(2k)!}\right) \left(\sum_{k=0}^{n/2} \frac{(2k)!}{k!}\right) \le \frac{2^{n-1}}{n!} \frac{n!}{\left(\frac{n}{2}-1\right)!} = \frac{2^{n-1}}{\left(\frac{n}{2}-1\right)!},$$

which proves the estimation (14).

One can verify that

$$\lim_{n\to\infty} \frac{2^{n-1}}{\left(\frac{n}{2}-1\right)!} = 0.$$

In fact, by making a substitution $m = n/2 - 1$ which yields: $n - 1 = 2m + 1$ we obtain

$$\lim_{n\to\infty} \frac{2^{n-1}}{\left(\frac{n}{2}-1\right)!} = \lim_{m\to\infty} \frac{2^{2m+1}}{m!} = 2\lim_{m\to\infty} \frac{2^{2m}}{m!} = 2\lim_{m\to\infty} \frac{4^{m}}{m!} = 0.$$

Set us consider

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n/2} \frac{1}{(n-2k)!k!}\right) \leq \lim_{n\to\infty}\left(\frac{2^{n-1}}{\left(\frac{n}{2}-1\right)!}\right) = 0,$$

since $\sum\limits_{k=0}^{n/2} \dfrac{1}{(n-2k)!k!} > 0$ the lemma is proven. □

Theorem 9. *(Convergence of the direct Lie-algebraic numerical scheme). Let $u = u(x,t)$ be the solution of the problem (4), $u_n = \sum\limits_{k=0}^{n/2}\left(a^k\varphi^{(2k)}\dfrac{t^k}{k!}\right)$ be the Taylor expansion of the solution and $u_h = \sum\limits_{j=0}^{n}\left[\left(\sum\limits_{k=0}^{n/2}\left(a^k Z^{2k}\varphi_h\dfrac{t^k}{k!}\right)\right)l_j(x)\right]$ be the finite dimensional solution. Then built numerical scheme (12) is convergent having the factorial rate of convergence:*

$$\|u-u_h\|_{V_h} \leq \frac{T^{n/2+1}}{\left(\frac{n}{2}+1\right)!}\left\|\frac{\partial^{n+1}u}{\partial t^{n+1}}\right\|_{\infty} + \frac{(2max\,\{a, diam\Omega, T\})^{n+1}}{4\,(n/2-1)!}\left\|\varphi^{(n+1)}\right\|_{\infty}. \tag{15}$$

Proof. Triangle inequality shows the natural way to split the norm $\|u-u_h\|_{V_h}$ in the following way:

$$\|u-u_h\|_{V_h} \leq \|u-u_{n/2}\|_{V_h} + \|u_{n/2}-u_h\|_{V_h}$$

where the first norm $\|u-u_{n/2}\|_{V_h}$ represents the accuracy of approximation of the solution by means Taylor expansion and second form represents the error of Taylor series approximation by means of Lie-algebraic finite dimensional quasi representations. Using the property of error estimation of Taylor series we obtain the estimation for the first norm:

$$\|u-u_{n/2}\|_{V_h} \leq \|u-u_{n/2}\|_{\infty} \leq \frac{T^{n/2+1}}{\left(\frac{n}{2}+1\right)!}\left\|\frac{\partial^{n+1}u}{\partial t^{n+1}}\right\|_{\infty}.$$

Decomposition of the $\|u_{n/2}-u_h\|_{V_h}$ implies yields the following calculations:

$$\|u_{n/2}-u_h\|_{V_h} = \left\|\sum_{k=0}^{n/2} a^k\varphi^{(2k)}\frac{t^k}{k!} - \sum_{k=0}^{n/2} a^k Z^{2k}\varphi_h\frac{t^k}{k!}\right\|_{V_h} =$$

$$= \left\|\sum_{k=0}^{n/2} a^k\left(\varphi^{(2k)} - Z^{2k}\varphi_h\right)\frac{t^k}{k!}\right\|_{V_h} \leq \sum_{k=0}^{n/2} a^k\left\|\varphi^{(2k)} - Z^{2k}\varphi_h\right\|_{V_h}\frac{t^k}{k!} \leq$$

$$\leq \sum_{k=0}^{n/2} a^k\left\|\varphi^{(2k)} - \varphi_I^{(2k)}\right\|_{\infty}\frac{t^k}{k!} \leq \sum_{k=0}^{n/2} a^k\left(\frac{(diam\Omega)^{n+1-2k}}{(n+1-2k)!}\right)\frac{T^k}{k!}\left\|\varphi^{(n+1)}\right\|_{\infty}.$$

Let us denote $M = \max\{a, diam\Omega, T\}$ then we derive the estimation for $\|u_{n/2} - u_h\|_{V_h}$:

$$\|u_{n/2} - u_h\|_{V_h} \leq \sum_{k=0}^{n/2} M^k \left(\frac{M^{n+1-2k}}{(n+1-2k)!} \right) \frac{M^k}{k!} \left\| \varphi^{(n+1)} \right\|_\infty =$$

$$= \left\| \varphi^{(n+1)} \right\|_\infty \cdot \sum_{k=0}^{n/2} \frac{M^{n+1}}{(n+1-2k)!} \leq \frac{M^{n+1} 2^{n-1}}{\left(\frac{n}{2}-1\right)!} \left\| \varphi^{(n+1)} \right\|_\infty = \frac{(2M)^{n+1}}{4\left(\frac{n}{2}-1\right)!} \left\| \varphi^{(n+1)} \right\|_\infty .$$

As a conclusion of the above findings we can verify that $\lim\limits_{n\to\infty} \|u_{n/2} - u_h\|_{V_h} = 0$, in fact:

$$\lim_{n\to\infty} \|u_{n/2} - u_h\|_{V_h} \leq \left\| \varphi^{(n+1)} \right\|_\infty \lim_{n\to\infty} \left(\frac{(2M)^{n+1}}{4\left(\frac{n}{2}-1\right)!} \right) = 0.$$

Finally we have the estimation (15) which implies the convergence of the proposed in (11) numerical scheme, namely $\lim\limits_{n\to\infty} \|u - u_h\|_{V_h} = 0$, since

$$\lim_{n\to\infty} \|u - u_h\|_{V_h} \leq \left(\lim_{n\to\infty} \|u - u_{n/2}\|_{V_h} + \lim_{n\to\infty} \|u_{n/2} - u_h\|_{V_h} \right) = 0.$$

□

5. Numerical example

Let us proceed to the analysis of numerical results. For that purpose, we consider a cylindric domain $Q_T := (0,1) \times (0,1)$, i.e $x \in (0,1)$, $t \in (0,1)$. and a model problem from [14]:

$$\begin{cases} \text{find function } u = u(x,t) \text{ such, that:} \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \ (x,t) \in Q_T, \\ u|_{t=0} = \sin x, \end{cases} \tag{16}$$

having the exact solution $u(x,t) = e^{-t}\sin(x)$.

The norm of the error of approximating the exact solution $u - u_h = u(x,t) - u_h(x,t)$ in the functional space $L^2(Q_T)$ is calculated by the formula

$$\|u - u_h\|^2_{L^2(Q_T)} = \int\limits_{Q_T} (u - u_h)^2 \, dxdt,$$

in the functional space $L^\infty(Q_{T,h})$ is calculated at the discretization nodes:

$$\|u - u_h\|_{L^\infty(Q_{T,h})} = \sup_{(x,t)\in Q_{T,h}} |u(x,t) - u_h(x,t)|,$$

and the norm in the Sobolev's space $W^{1,2}(Q_T)$ [6] is calculated according to

$$\|u - u_h\|^2_{W^{1,2}(Q_T)} = \int\limits_{Q_T} \left[(u - u_h)^2 + \left(\frac{\partial u}{\partial x} - \frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial t} - \frac{\partial u_h}{\partial t} \right)^2 \right] dxdt.$$

The exact solution is known for the problem (16), thus we use the following rule for evaluating the rate of convergence: $p_h = \log_2\left(\dfrac{\|u - u_h\|}{\|u - u_{h/2}\|}\right)$. If we get value $\|u-u_h\| = 0$ and $\|u - u_{h/2}\| = 0$, thus the value 0/0 is shown as NaN (*not a number*).

The model problem is investigated by explicit scheme of finite differences method (FDM), the method of Lie-algebraic discrete approximations (MLADA), Generalized method of Lie-algebraic discrete approximations (GMLADA) and Direct method of Lie-algebraic discrete approximations (DMLADA). The solution of Cauchy problem with the system of differential equations was performed using Mathematica. Let us denote the step of discretization by space variable by $\Delta x = \frac{1}{(n_x-1)}$, and $\Delta t = \frac{1}{(n_t-1)}$ as the step of discretization by time variable. If discretization steps by both variables are equal then we use $h = \Delta x = \Delta t$ for FDM and GMLADA. Nevertheless h denotes the step of discretization by space variable for MLADA, because time step is chosen automatically while solving the Cauchy problem with the system of differential equation by means of Wolfram Mathematica software.

Table 1

Error estimations in $L^2(Q_T)$ space

Step h	FDM	MLADA	GMLADA	DMLADA
$h = 1/2$	0.0419804	0.129574	0.0479767	0.0507986
$h = 1/4$	0.0199765	0.051718	0.0146769	0.0146827
$h = 1/8$	0.00965197	0.00343672	0.000637523	0.000637523
$h = 1/16$	$2.05974 \cdot 10^{10}$	3019.86	$1.83044 \cdot 10^{-7}$	$1.83044 \cdot 10^{-7}$

Table 2

Error estimations in $L^\infty(Q_{T,h})$ space

Step h	FDM	MLADA	GMLADA	DMLADA
$h = 1/2$	0.0904097	0.46952	0.23476	0.23476
$h = 1/4$	0.0414633	0.249107	0.085016	0.085016
$h = 1/8$	0.0200078	0.0226241	0.0046676	0.0046676
$h = 1/16$	$4.19664 \cdot 10^{11}$	18151.3	$1.7980 \cdot 10^{-6}$	$1.7980 \cdot 10^{-6}$

Table 3

Error estimations in $W^{1,2}(Q_T)$ space

Step h	FDM	MLADA	GMLADA	DMLADA
$h = 1/2$	0.119842	0.452594	0.202487	0.203796
$h = 1/4$	0.0594391	0.2167	0.0727922	0.0727891
$h = 1/8$	0.0294888	0.0197359	0.00419756	0.00419756
$h = 1/16$	$1.24976 \cdot 10^{12}$	24988.7	$1.85966 \cdot 10^{-6}$	$1.85966 \cdot 10^{-6}$

Table 4

Rates of convergence in $L^2(Q_T)$ space

Step h	FDM	MLADA	GMLADA	DMLADA
$h = 1/2$	1.07141	1.32504	1.70879	1.79067
$h = 1/4$	1.04941	3.91156	4.52493	4.5255
$h = 1/8$	−40.9567	−19.745	11.7661	11.7661

Table 5

Rates of convergence in $L^\infty(Q_T)$ space

Step h	FDM	MLADA	GMLADA	DMLADA
$h = 1/2$	1.12464	0.91442	1.46539	1.46539
$h = 1/4$	1.05128	3.46084	4.18698	4.18698
$h = 1/8$	−44.2537	−19.6138	11.3421	11.3421

Table 6

Rates of convergence in $W^{1,2}(Q_T)$ space

Step h	FDM	MLADA	GMLADA	DMLADA
$h = 1/2$	1.01165	1.06252	1.47598	1.48533
$h = 1/4$	1.01124	3.45681	4.11616	4.11616
$h = 1/8$	−45.2685	−20.272	11.1403	11.1403

From the above tables we can see the increase of errors in MLADA. This is caused by the stiff system of ordinary differential equations to which the partial differential equation was reduced to. Such systems need either increasing the count of nodes or usage of some special numerical techniques.

Also we can observe that proposed numerical method has the same accuracy as Generalized method of Lie-algebraic discrete approximations. The norm of the error of approximating the exact solution was evaluated as

$$\|u - u_h\|_{BL} = \max_{i=\overline{1,n_t}} \sqrt{\sum_{j=1}^{n_x} (u(x_j, t_i) - u_h(x_j, t_i))^2}. \tag{17}$$

in the [14]. For the case $n_x = 10, n_t = 10$ by means of MLADA there was obtained the following error in [14]: $\|u - u_h\|_{BL} = 7,75 \cdot 10^{-3}$. With the same count of nodes and with respect to the same norm (17) using GMLADA and DMLADA we achieved the error $\|u - u_h\|_{BL} = 2,69 \cdot 10^{-3}$ which is almost three times $(2,88104)$ more precise than the result obtained by means of classic approach.

These tables highlight the main benefit of using the proposed numerical scheme is reduced count of arithmetic operations maintaining the same computational properties as a generalized method of Lie-algebraic discrete approximations.

Table 7

Count of arithmetic operations for $n_x = n_t = 3$

Step $h = 1/16$	FDM	MLADA	GMLADA	DMLADA
Error in $L^2(\Omega_T)$ space	0.0419804	0.129574	0.0479767	0.0507986
Additions, substractions	46	145	1093	44
Multiplications	50	157	1179	54
Divisions	18	3	42	2
Time (ms)	8	3	24	4

Table 8

Count of arithmetic operations for $n_x = n_t = 5$

Step $h = 1/16$	FDM	MLADA	GMLADA	DMLADA
Error in $L^2(\Omega_T)$ space	0.0199765	0.051718	0.0146769	0.0146827
Additions, substractions	163	615	92937	158
Multiplications	152	663	93805	180
Divisions	66	5	420	4
Time (ms)	8	3	79	4

Table 9

Count of arithmetic operations for $n_x = n_t = 9$

Step $h = 1/16$	FDM	MLADA	GMLADA	DMLADA
Error in $L^2(\Omega_T)$ space	0.00965197	0.00343672	0.000637523	0.000637523
Additions, substractions	613	3323	14007761	786
Multiplications	524	3499	14018265	864
Divisions	258	9	5256	8
Time (ms)	8	3	571	11

Table 10

Count of arithmetic operations for $n_x = n_t = 17$

Step $h = 1/16$	FDM	MLADA	GMLADA	DMLADA
Error in $L^2(\Omega_T)$ space	$4.19664 \cdot 10^{11}$	18151.3	$1.85966 \cdot 10^{-6}$	$1.85966 \cdot 10^{-6}$
Additions, substractions	2313	21252	2767151635	5008
Multiplications	1940	21843	2767300273	5304
Divisions	1026	17	74256	16
Time (ms)	8	3	21389	20

6. Conclusions

We have applied the direct method of Lie-algebraic discrete approximations for solving the Cauchy problem for heat equation in this paper. There were compared different numerical schemes (finite difference method, classical method of Lie-algebraic discrete approximations, generalized method of Lie-algebraic discrete approximations and direct method of Lie-algebraic discrete approximations) for solving the Cauchy problem for advection equation. One can obtain numerical result with the same high precision and with significantly less computational costs in compare to the generalized method of Lie-algebraic discrete approximations because that method approximates the solution instead of the differential operator of the equation.

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Article: received 04.09.2019
revised 30.10.2019
printing adoption 06.11.2019

Прямий метод Лі-алгебричних дискретних апроксимацій для розв'язування рівняння теплопровідності

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Запропоновано й обґрунтовано прямий метод Лі-алгебричних дискретних апроксимацій для чисельного розв'язування задачі Коші для рівняння теплопровідності. Ідея прямого методу Лі-алгебричних апроксимацій полягає в тому, що з використанням аналітичних підходів, зокрема методу малого параметра, або розкладу у ряд Тейлора, побудовано наближений аналітичний розв'язок задачі у вигляді степеневого ряду за часовою змінною. Після цього побудовано його дискретний відповідник з використанням квазізображень елементів алгебри Лі. Доведено, що обчислювальна схема має факторіальний порядок збіжності.

Ключові слова: прямий метод Лі-алгебричних дискретних апроксимацій, рівняння теплопровідності, скінченновимірне квазізображення, поліном Лагранжа, метод малого параметра, факторіальна збіжність.