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ON CARATHÉODORY-LASALLE'S THEOREMS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS AND THEIR APPLICATION

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In the paper we consider the following Cauchy problem for the systems of the ordinary differential equations

$$\begin{cases} \psi''(t) + K(t,\psi(t),\psi'(t)) = S(t), \quad t \in (0,T), \\ \psi(0) = \psi^0, \quad \psi'(0) = \psi^1, \end{cases} \qquad (*)$$

where K and S are some vector-valued functions, $\psi^0, \psi^1 \in \mathbb{R}^d$ are some fixed vectors. Such systems appear in various classical mechanics problems because they follow from Newton's second law of the dynamics. If we use the Faedo-Galerkin method for the seeking of the weak solution to the initial boundary value problem for the linear or nonlinear hyperbolic equations, then we also use the problems of type (*) for the construction of the approximation functions. In practice, the Cauchy problems of type (*) often are reduced to the following Cauchy problem of the higher dimension:

$$\begin{cases} \psi'(t) = \theta(t), \quad t \in (0,T), \\ \theta'(t) = S(t) - K(t,\psi(t),\theta(t)), \quad t \in (0,T), \\ \psi(0) = \psi^0, \quad \theta(0) = \psi^1. \end{cases} \qquad (**)$$

Then the standard Peano/Carathéodory types existence theorems are used. But these are local theorems and they should be complemented by some extension theorems. On the other hand we can seek the global solutions of problems (*) or (**) directly. In the present paper we find the conditions of the global solvability of the Cauchy problem (*). Corresponding results for problem (**) we got in our previous paper. These conditions contain the Carathéodory existence conditions and the Lasalle extension conditions for (*). Finally, we use these facts to proving the existence theorem for the initial boundary value problem for some nonlinear hyperbolic equations with the variable exponent of the nonlinearity.

Key words: ordinary differential equations, Cauchy problem, global weak solution.

1. Introduction

Let $d \in \mathbb{N}$ and $T > 0$ be fixed numbers, $\mathcal{O} := (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$. We seek a weak solution $\psi : (0,T) \to \mathbb{R}^d$ of the following problem:

$$\psi''(t) + K(t,\psi(t),\psi'(t)) = S(t), \quad t \in (0,T), \qquad (1)$$

$$\psi(0) = \psi^0, \tag{2}$$

$$\psi'(0) = \psi^1, \tag{3}$$

where $K : \mathcal{O} \to \mathbb{R}^d$ and $S : (0,T) \to \mathbb{R}^d$ are some functions (for the sake of convenience we have assumed that $K(t,0,0) = 0$ for every $t \in (0,T)$), $\psi^0 := (\psi^0_1, \dots, \psi^0_d) \in \mathbb{R}^d$, and $\psi^1 := (\psi^1_1, \dots, \psi^1_d) \in \mathbb{R}^d$.

The systems of type (1) have been widely used in many applications. For example, let us consider one classical problem: the stone with the mass m is thrown at an angle α to the horizon and it moves in a medium whose resistance is proportional to the velocity v (see Fig. 1 below). We need to find trajectory of the motion of the stone. Then at any point of the trajectory $N(x,y)$ on the stone there are two forces: the gravity $P = mg$ and the strength of the medium's resistance $F = kv$.

On the basis of the second law of dynamics, equations of the motion of the stone after certain transformations will have a form (see [1, p. 500-501])

$$\begin{cases} x''(t) + \dfrac{k}{m}\, x'(t) = 0, \\ y''(t) + \dfrac{k}{m}\, y'(t) = -g, \end{cases} \qquad t \in [0,T].$$

This is a system of type (1).

Studying many other problems of the classical mechanics results in systems of the ordinary differential equations containing derivatives of the second order of the unknown functions. In addition, the systems of such type appear in researching via Faedo-Galerkin's method of initial-boundary value mixed problems for the hyperbolic equations with the variable exponents of nonlinearity (see Section 4 below).

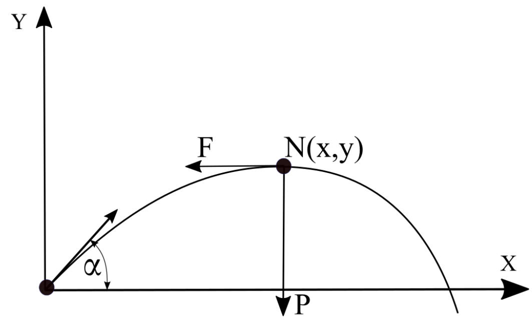

Fig. 1. Classical mechanics problem: stone is thrown upward from the ground at an angle

This paper is organized as follows. In Section 2, we formulate main result of our article. In Section 3, we prove main theorem. The results obtained here are generalizations and complements of the results from [2], where the another type of the systems is considered. The fourth Section involves the applying obtained results to the proving of the existence theorem of the global solutions to the initial-boundary value problem for some hyperbolic equations with the variable exponent of the nonlinearity.

2. Notation and statement of main result

Suppose that $p \in [1,\infty]$, X is Banach space, $C([0,T];X)$ and $C^m([0,T];X)$ are the spaces of the X-valued smooth functions defined on $[0,T]$ (see [3, p. 147]), $L^p(0,T;X)$ is the Lebesgue-Bochner space (see [3, p. 155]), and $W^{m,p}(0,T;X)$ is the Sobolev-Bochner space (see [4, p. 286]).

Definition 1. *A real-valued vector-function* $\psi \in W^{2,1}(0,T;\mathbb{R}^d)$ *is called a weak solution of the Cauchy problem (1)–(3) if* ψ *satisfies initial value condition (2)–(3) and satisfies system (1) almost everywhere.*

First, let us recall some auxiliary facts. Let $\ell \in \mathbb{N}$, $Q := (0,T)\times\mathbb{R}^\ell$, $(\cdot,\cdot)_{\mathbb{R}^\ell}$ be a scalar product in $\mathbb{R}^\ell$, and $|\cdot|_{\mathbb{R}^\ell} := \sqrt{(\cdot,\cdot)_{\mathbb{R}^\ell}}$.

Definition 2. *We shall say that the function* $F : Q \to \mathbb{R}^\ell$ *satisfies the Carathéodory condition if for every* $\zeta \in \mathbb{R}^\ell$ *the function* $(0,T) \ni t \mapsto F(t,\zeta) \in \mathbb{R}^\ell$ *is measurable and if for a.e.* $t \in (0,T)$ *the function* $\mathbb{R}^\ell \ni \zeta \mapsto F(t,\zeta) \in \mathbb{R}^\ell$ *is continuous.*

Definition 3. *(see [5, p. 241]). We shall say that the function* $F : Q \to \mathbb{R}^\ell$ *satisfies the* L^p*-Carathéodory condition if* F *satisfies the Carathéodory condition and for every* $R > 0$ *there exists a function* $h_R \in L^p(0,T)$ *such that*

$$|F(t,\zeta)|_{\mathbb{R}^\ell} \le h_R(t) \tag{4}$$

for a.e. $t \in (0,T)$ *and for every* $\zeta \in \overline{D_R} := \{y \in \mathbb{R}^\ell \mid |y| \le R\}$.

Further, let us consider the problem

$$\varphi'(t) + L(t,\varphi(t)) = M(t), \quad t \in (0,T), \tag{5}$$

$$\varphi(0) = \varphi^0, \tag{6}$$

where $M : (0,T) \to \mathbb{R}^\ell$ and $L : Q \to \mathbb{R}^\ell$ are some functions (for the sake of convenience we have assumed that $L(t,0) = 0$ for every $t \in (0,T)$), $Q := (0,T)\times\mathbb{R}^\ell$, $\ell \in \mathbb{N}$, and $\varphi^0 = (\varphi^0_1,\dots,\varphi^0_\ell) \in \mathbb{R}^\ell$.

Definition 4. *A real-valued function* $\varphi \in W^{1,1}(0,T;\mathbb{R}^\ell)$ *is called a weak solution of problem (5)–(6) if* u *satisfies (6) and satisfies system (5) almost everywhere.*

Proposition 5. (the first Carathéodory-LaSalle Theorem, see Theorem 3.24, [2], p. 872) *Suppose that* $p \ge 2$, $L : Q \to \mathbb{R}^\ell$ *satisfies* L^p*-Carathéodory condition,* $M \in L^p(0,T;\mathbb{R}^\ell)$*, and* $\varphi^0 \in \mathbb{R}^\ell$*. If there exists a nonnegative functions* $\alpha,\beta \in L^1(0,T)$ *such that for every* $\xi \in \mathbb{R}^\ell$ *and for a.e.* $t \in (0,T)$ *the inequality*

$$(L(t,\xi),\xi)_{\mathbb{R}^\ell} \ge -\alpha(t)|\xi|^2 - \beta(t) \tag{7}$$

holds, then problem (5)–(6) has a global weak solution $\varphi \in W^{1,p}(0,T;\mathbb{R}^\ell)$.

The main result of the present paper is the following Theorem.

Theorem 6. (the second Carathéodory-LaSalle Theorem) *Suppose that* $p \ge 2$*, the function* $K : \mathcal{O} \to \mathbb{R}^d$ *satisfies* L^p*-Carathéodory condition,* $S \in L^p(0,T;\mathbb{R}^d)$*, and* $\psi^0,\psi^1 \in \mathbb{R}^d$*. Then if there exist nonnegative functions* $\gamma,\sigma,\omega \in L^1(0,T)$ *such that for every* $\xi,\eta \in \mathbb{R}^d$ *and for a.e.* $t \in (0,T)$ *the inequality*

$$(K(t,\xi,\eta),\eta)_{\mathbb{R}^d} \ge -\gamma(t)|\xi|^2_{\mathbb{R}^d} - \sigma(t)|\eta|^2_{\mathbb{R}^d} - \omega(t) \tag{8}$$

holds, then problem (1)–(3) has a global weak solution $\psi \in W^{2,p}(0,T;\mathbb{R}^d)$.

3. Proof of main Theorem

Let's reduce problem (1)–(3) to the problem of type (5)–(6). We introduce a new function θ by the rule $\theta = \psi'$. Using (1)–(3), we obtain

$$\begin{cases} \psi' - \theta = 0, \\ \theta' + K(t,\psi(t),\theta(t)) = S(t), \end{cases} \quad t \in (0,T), \qquad \begin{cases} \psi(0) = \psi^0, \\ \theta(0) = \psi^1. \end{cases} \tag{9}$$

If we put

$$\varphi(t) := \begin{pmatrix} \psi(t) \\ \theta(t) \end{pmatrix},\ M(t) := \begin{pmatrix} 0 \\ S(t) \end{pmatrix},\ L(t,\varphi(t)) := \begin{pmatrix} -\theta(t) \\ K(t,\psi(t),\theta(t)) \end{pmatrix},\ \varphi_0 := \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix},$$

then instead of (9) we obtain problem (5)–(6). Let's show that the conditions of Proposition 5 are satisfied.

Clearly, if $\xi,\eta \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^{2d}$, and $\zeta = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, then

$$|\zeta|_{\mathbb{R}^{2d}} = \sqrt{|\xi|^2_{\mathbb{R}^d} + |\eta|^2_{\mathbb{R}^d}} \le \sqrt{2\max\{|\xi|^2_{\mathbb{R}^d}, |\eta|^2_{\mathbb{R}^d}\}} \le \sqrt{2}(|\xi|_{\mathbb{R}^d} + |\eta|_{\mathbb{R}^d}), \tag{10}$$

$$|\xi|_{\mathbb{R}^d} \le |\zeta|_{\mathbb{R}^{2d}}, \qquad |\eta|_{\mathbb{R}^d} \le |\zeta|_{\mathbb{R}^{2d}}, \tag{11}$$

and, in addition,

$$L(t,\zeta) = \begin{pmatrix} -\eta \\ K(t,\xi,\eta) \end{pmatrix}. \tag{12}$$

Let us check that $L : \mathcal{O} \to \mathbb{R}^{2d}$ satisfies the L^p-Carathéodory condition. Indeed, for a.e. $t \in (0,T)$ and all $\zeta \in \{y \in \mathbb{R}^{2d} \mid |y| \le R\}$, from (10)–(12), we have the estimate

$$|L(t,\zeta)|_{\mathbb{R}^{2d}} \le \sqrt{2}(|\eta|_{\mathbb{R}^d} + |K(t,\xi,\eta)|_{\mathbb{R}^d}) \le \sqrt{2}(R + h_R(t)) \in L^p(0,T),$$

where the function h_R is taken from the definition that the function $K : \mathcal{O} \to \mathbb{R}^d$ satisfies the corresponding L^p-Carathéodory condition (see (4)). In addition, the standard Carathéodory condition is also fulfilled.

Let us check that $M \in L^p(0,T;\mathbb{R}^{2d})$. Since $S \in L^p(0,T;\mathbb{R}^d)$,

$$|M(t)|_{\mathbb{R}^{2d}} = \left|\begin{pmatrix} 0 \\ S(t) \end{pmatrix}\right|_{\mathbb{R}^{2d}} = \sqrt{0^2 + |S(t)|^2_{\mathbb{R}^d}} = |S(t)|_{\mathbb{R}^d} \in L^p(0,T).$$

Let us check that the analog of condition (7) is executed. Since

$$(\xi,\eta)_{\mathbb{R}^d} \le |\xi|_{\mathbb{R}^d} \cdot |\eta|_{\mathbb{R}^d} \le \frac{1}{2}|\xi|^2_{\mathbb{R}^d} + \frac{1}{2}|\eta|^2_{\mathbb{R}^d}, \quad \xi,\eta \in \mathbb{R}^d,$$

from (8) and (12) it follows that

$$(L(t,\zeta),\zeta)_{\mathbb{R}^{2d}} = \begin{pmatrix} -\eta \\ K(t,\xi,\eta) \end{pmatrix} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} = -\eta_1\xi_1 - \eta_2\xi_2 - \ldots - \eta_d\xi_d +$$

$$+K_1\eta_1 + K_2\eta_2 + \ldots + K_d\eta_d = -(\xi,\eta)_{\mathbb{R}^d} + (K(t,\xi,\eta),\eta)_{\mathbb{R}^d} \ge$$

$$\ge -\frac{1}{2}|\xi|^2_{\mathbb{R}^d} - \frac{1}{2}|\eta|^2_{\mathbb{R}^d} - \gamma(t)|\xi|^2_{\mathbb{R}^d} - \sigma(t)|\eta|^2_{\mathbb{R}^d} - \omega(t) \ge$$

$$\ge -\max\Big\{\frac{1}{2} + \gamma(t),\ \frac{1}{2} + \sigma(t)\Big\}|\zeta|^2_{\mathbb{R}^{2d}} - \omega(t).$$

Hence, the analog of estimate (4) is true. Then first Carathéodory-LaSalle Theorem (Proposition 5) implies that problem (5)–(6) has a global weak solution $\varphi \in W^{1,p}(0,T;\mathbb{R}^{2d})$. By (9) we get that $\psi' = \theta \in W^{1,p}(0,T;\mathbb{R}^d)$. Consequently, $\psi \in W^{2,p}(0,T;\mathbb{R}^d)$ and ψ is a global weak solution to problem (1)–(3). Theorem 6 is proved. □

4. Application of main result

Let $n \in \mathbb{N}$ be a fixed number, $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, $Q_{0,T} := \Omega \times (0,T)$, either $G = \Omega$ or $G = Q_{0,T}$, $\mathcal{M}(G)$ is a set of all measurable functions $v : G \to \mathbb{R}$, $L^p(G)$ is the Lebesgue space (see [6, p. 22, 24]), $W^{m,p}(G)$ and $W_0^{m,p}(G)$ are the Sobolev spaces (see [6, p. 45]), $H^m(G) := W^{m,2}(G)$, $H_0^m(G) := W_0^{m,2}(G)$, $H^{-m}(G) := [H_0^m(G)]^*$, and

$$\mathcal{B}_+(G) := \{q \in L^\infty(G) \mid \operatorname*{ess\,inf}_{y \in G} q(y) > 0\}.$$

If $q \in \mathcal{B}_+(G)$, then by definition, put

$$q_0 := \operatorname*{ess\,inf}_{y\in G} q(y), \quad q^0 := \operatorname*{ess\,sup}_{y\in G} q(y), \qquad \rho_q(v;G) := \int_G |v(y)|^{q(y)}\,dy, \quad v \in \mathcal{M}(G).$$

The set $L^{q(y)}(G) := \{v \in \mathcal{M}(G) \mid \rho_q(v;G) < +\infty\}$ with respect to the Luxemburg norm

$$||v; L^{q(y)}(G)|| := \inf\{\lambda > 0 \mid \rho_q(v/\lambda; G) \le 1\}$$

is called the generalized Lebesgue space. It is well known that $L^{q(y)}(G)$ is the Banach space which is reflexive and separable (see [7, p. 599, 600, 604]). The properties of this space is widely studied in [8]. Various equations with variable exponents of the nonlinearities are considered in [8]- [15]. Here we consider the problem

$$u_{tt} - \Delta u + g(x,t)|u_t|^{q(x)-2}u_t = f(x,t), \quad (x,t) \in Q_{0,T}, \tag{13}$$

$$u|_{\partial\Omega\times[0,T]} = 0, \tag{14}$$

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \quad x \in \Omega, \tag{15}$$

where Δ is the Laplacian, $q \in \mathcal{B}_+(\Omega)$, and $q_0 > 1$.

Let $V := H_0^1(\Omega) \cap L^{q(x)}(\Omega)$, $(u,v)_\Omega := \int_\Omega u(x)\,v(x)\,dx$, $u, v : \Omega \to \mathbb{R}$, and let us define the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ by the rule

$$\langle Az, w\rangle_{H_0^1(\Omega)} := \int_\Omega \sum_{i=1}^n z_{x_i}(x)\,w_{x_i}(x)\,dx, \qquad z, w \in H_0^1(\Omega), \;\; t \in (0,T).$$

We will need the following assumptions:

(G): $g \in \mathcal{B}_+(Q_{0,T})$, $g_t, g_{tt} \in L^\infty(Q_T)$, $q \in \mathcal{B}_+(\Omega)$, and $q_0 > 1$;

(F): $f, f_t \in L^2(Q_T)$;

(U): $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega) \cap L^{q(x)}(\Omega)$.

Definition 7. *A real-valued function u is called a weak solution to problem (13)–(15), if*

$$u \in L^\infty(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega)), \quad u_t \in L^\infty(0,T;V) \cap C([0,T]; L^2(\Omega)),$$

$u_{tt} \in L^\infty(0,T; L^2(\Omega))$, *$u$ satisfies initial value conditions (15), and for all $v \in V$ and for a.e. $t \in (0,T)$ the following equality is true:*

$$(u_{tt}(t), v)_\Omega + \langle A(t)u(t), v\rangle_{H_0^1(\Omega)} + (g|u_t(t)|^{q(x)-2}u_t(t), v)_\Omega = (f(t), v)_\Omega.$$

The following Lemma and Proposition are needed for the sequel.

Lemma 8. *Suppose that condition **(G)** hold,* $\{w^j\}_{j\in\mathbb{N}} \subset V$, $m \in \mathbb{N}$,

$$K(t,\xi,\eta) := (K_1(t,\xi,\eta),\dots,K_m(t,\xi,\eta)),$$

where

$$K_\mu(t,\xi,\eta) := \langle Az^m, w^\mu\rangle_{H_0^1(\Omega)} + (g(t)|\zeta^m|^{q(x)-2}\zeta^m, w^\mu)_\Omega,$$

$$\mu = \overline{1,m},\ t\in(0,T),\ \xi=(\xi_1,\xi_2,...,\xi_m)\in\mathbb{R}^m,\ \eta=(\eta_1,\eta_2,...,\eta_m)\in\mathbb{R}^m,$$

$$z^m(x) := \sum_{\mu=1}^m \xi_\mu w^\mu(x), \quad \zeta^m(x) := \sum_{\mu=1}^m \eta_\mu w^\mu(x), \quad x\in\Omega.$$

Then there exists a constant $C_1(m) > 0$ *such that*

$$(K(t,\xi,\eta),\eta)_{\mathbb{R}^m} \ge -C_1(m)(|\xi|^2+|\zeta|^2), \quad t\in(0,T). \tag{16}$$

Proof. It is clear that

$$(K(t,\xi,\eta),\eta)_{\mathbb{R}^m} = \langle A(t)z^m,\zeta^m\rangle_{H_0^1(\Omega)} + (g(x,t)|\zeta^m|^{q(x)-2}\zeta^m,\zeta^m)_\Omega =$$

$$= \int_\Omega \Big[\sum_{i=1}^n z^m_{x_i}(x)\cdot\zeta^m_{x_i}(x) + g(x,t)|\zeta^m(x)|^{q(x)}\Big]\,dx \ge -\Big|\int_\Omega \sum_{i=1}^n z^m_{x_i}(x)\cdot\zeta^m_{x_i}(x)\,dx\Big|. \tag{17}$$

Using Young's inequality, we get

$$\Big|\int_\Omega \sum_{i=1}^n z^m_{x_i}(x)\cdot\zeta^m_{x_i}(x)\,dx\Big| \le C_2\int_\Omega\sum_{i=1}^n\Big[|z^m_{x_i}(x)|^2+|\zeta^m_{x_i}(x)|^2\Big]\,dx \le$$

$$\le C_3\int_\Omega\sum_{i=1}^n\Big[\sum_{\mu=1}^m|\xi_\mu|^2\cdot|\omega^\mu_{x_i}(x)|^2+\sum_{\mu=1}^m|\eta_\mu|^2\cdot|\omega^\mu_{x_i}(x)|^2\Big]\,dx \le$$

$$\le C_4\sum_{\mu=1}^m(|\eta_\mu|^2+|\zeta_\mu|^2) = C_4(|\eta|^2+|\zeta|^2). \tag{18}$$

Using (18), from (17) we get that estimate (16) holds, and Lemma 8 is proved. □

Proposition 9. (see Lemma 3.25, [2], p. 874) *Suppose that* $q\in\mathcal{B}_+(\Omega)$, $q_0>1$, $g\in L^\infty(Q_{0,T})$, $z\in L^{q(x)}(\Omega)$, $m\in\mathbb{N}$, $\xi=(\xi_1,\dots,\xi_m)\in\mathbb{R}^m$, $w^1,\dots,w^m\in L^{q(x)}(\Omega)$, *and* $w(x,\xi) := \sum\limits_{l=1}^m \xi_l w^l(x)$. *Then the function*

$$I(t,\xi) := \int_\Omega g(x,t)|w(x,\xi)|^{q(x)-2}w(x,\xi)z(x)\,dx, \quad t\in(0,T), \quad \xi\in\mathbb{R}^m,$$

satisfies the L^∞*-Carathéodory condition.*

Theorem 10. *Suppose that conditions **(G)**-**(U)** are satisfied. Then problem (13)–(15) has a weak solution.*

Sketch of Proof. The solution will be constructed via Faedo-Galerkin's method. As a V is a separable space, there exists a linearly independent dense everywhere in V set of the functions $\{w^j\}_{j\in N}$ which is orthonormal in $L^2(\Omega)$. By definition, put

$$u^m(x,t) := \sum_{j=1}^{m} \psi_j^m(t) w^j(x), \quad (x,t)\in Q_{0,T}, \quad m\in\mathbb{N}, \tag{19}$$

where $\psi^m := (\psi_1^m, ..., \psi_m^m)$ is a solution to the problem

$$(u_{tt}^m(t), w^\mu)_\Omega + \langle Au^m(t), w^\mu\rangle_{H_0^1(\Omega)} + (g(t)|u_t^m(t)|^{q(x)-2}u_t^m(t), w^\mu)_\Omega =$$

$$= (f(t), w^\mu)_\Omega, \ \ t\in(0,T), \ \ \mu=\overline{1,m}, \tag{20}$$

$$\psi_1^m(0)=\alpha_1^m, \quad \ldots, \quad \psi_m^m(0)=\alpha_m^m, \tag{21}$$

$$(\psi_1^m)_t(0)=\beta_1^m, \quad \ldots, \quad (\psi_m^m)_t(0)=\beta_m^m. \tag{22}$$

Here $\alpha_1^m,\ldots,\alpha_m^m\in\mathbb{R}^1$, $\beta_1^m,\ldots,\beta_m^m\in\mathbb{R}^1$ we choose in a such way that the functions

$$u_0^m(x) := \sum_{j=1}^{m} \alpha_j^m w^j(x), \quad u_1^m(x) := \sum_{j=1}^{m} \beta_j^m w^j(x), \quad x\in\Omega,$$

satisfy the conditions

$$u_0^m \underset{m\to\infty}{\longrightarrow} u_0 \ \text{ in } \ H^2(\Omega)\cap H_0^1(\Omega), \quad u_1^m \underset{m\to\infty}{\longrightarrow} u_1 \ \text{ in } \ H_0^1(\Omega)\cap L^{q(x)}(\Omega).$$

Notice, that $u^m|_{t=0}=u_0^m$, $u_t^m|_{t=0}=u_1^m$.

Let us show that the mentioned function $\psi^m=(\psi_1^m,...,\psi_m^m)$ exists. Let K be the vector-function from Lemma 8. Then we reduce problem (20)–(22) to problem (1)–(3) if

$$S(t)=\Big((f(t),w^1)_\Omega, \ \ldots, \ (f(t),w^m)_\Omega\Big).$$

Taking into account Proposition 9, we see that K satisfies the L^∞-Carathéodory condition. From the estimate (16) of Lemma 8 we get:

$$\Big(K(t,\psi^m,\psi_t^m),\psi_t^m\Big)_{\mathbb{R}^m} \geq -C_5(m)\Big(|\psi^m|^2+|\psi_t^m|^2\Big),$$

where $C_5>0$ is independent of t,ψ^m. Then Theorem 6 implies that there exists a global weak solution $\psi^m\in W^{2,2}(0,T;\mathbb{R}^m)$ to problem (20)–(22).

Finally, using conditions **(G)**-**(U)**, similarly as in [16], we prove that the sequence $\{u^m\}_{m\in\mathbb{N}}$ defined in (19) has a subsequence $\{u^{m_k}\}_{m_k\in\mathbb{N}}$ which tends as $m_k\to+\infty$ in a certain sense to a solution u of problem (13)–(15), and so Theorem 10 is proved. □

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ТЕОРЕМИ КАРАТЕОДОРІ-ЛАСАЛЛЯ ДЛЯ СИСТЕМ ЗВИЧАЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ТА ЇХНЄ ЗАСТОСУВАННЯ

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Розглянули таку задачу Коші для систем звичайних диференціальних рівнянь:

$$\begin{cases} \psi''(t) + K(t, \psi(t), \psi'(t)) = S(t), \quad t \in (0,T), \\ \psi(0) = \psi^0, \quad \psi'(0) = \psi^1, \end{cases} \qquad (*)$$

де K та S – деякі вектор-функції; $\psi^0, \psi^1 \in \mathbb{R}^d$ – деякі фіксовані вектори. Системи такого типу виникають у багатьох задачах класичної механіки, бо вони випливають з другого динамічного закону Ньютона. Якщо за допомогою методу Фаедо-Ґальоркіна шукати розв'язок мішаних задач для лінійних чи нелінійних рівнянь гіперболічного типу, то при побудові ґальоркінських наближень ми теж отримаємо задачі типу (*). На практиці для того, щоб розв'язати задачі Коші типу (*), їх часто зводять до такої задачі Коші для системи вищої розмірності:

$$\begin{cases} \psi'(t) = \theta(t), \quad t \in (0,T), \\ \theta'(t) = S(t) - K(t, \psi(t), \theta(t)), \quad t \in (0,T), \\ \psi(0) = \psi^0, \quad \theta(0) = \psi^1. \end{cases} \qquad (**)$$

Потім використовують стандартні теореми існування розв'язку, типу теорем Пеано чи Каратеодорі. Але ці теореми дають існування лише локальних розв'язків задач і для існування глобальних розв'язків треба додатково використовувати певні теореми про продовження знайдених локальних розв'язків. З іншого боку, можна зразу доводити існування глобальних розв'язків задач (*) чи (**). Ми знайшли умови глобальної розв'язності задачі Коші (*). Відповідні результати для задачі (**) отримали в нашій попередній статті. Отримані умови поєднують в собі умови Каратеодорі існування розв'язку задачі (*) з умовами Ласалля про продовження цього розв'язку. Наприкінці ми використали отримані результати для з'ясування існування розв'язку мішаної задачі для нелінійних гіперболічних рівнянь зі змінними показниками нелінійності.

Ключові слова: звичайні диференціальні рівняння, Задача Коші, глобальний узагальнений розв'язок.